Istanbul Commerce University Journal of Science
İstanbul Ticaret Üniversitesi Fen Bilimleri Dergisi, 15(30), Güz 2016
http://dergipark.gov.tr/ticaretfbd

Research Article

SOME COMMON FIXED POINT RESULTS FOR CONTRACTIVE MAPPINGS IN ORDERED G_p -METRIC SPACES*

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Abstract

In this present article, the sufficient conditions for the existence and uniqueness of fixed points and common fixed points of single and double mappings satisfying various contractive conditions within the partially ordered G_p -complete G_p -metric spaces have been obtained. Also, some examples supporting the results obtained have been given. The theorems obtained generalize some fixed point results existing in the literature.

Keywords: Common fixed point, fixed point, partially ordered G_p -complete G_p -metric spaces, Banach pairs.

Araştırma Makalesi

KISMİ SIRALI G_p -METRİK UZAYLARDA DARALMA DÖNÜŞÜMLERİ İÇİN BAZI ORTAK SABİT NOKTA SONUÇLARI

ÖZ

Bu çalışmada, kısmi sıralı G_p -tam G_p -metrik uzaylarda çeşitli daralma şartlarını sağlayan tek ve çift dönüşümlerin sabit noktalarının ve ortak sabit noktalarının varlığı ve tekliği için gerekli olan şartlar elde edilmiştir. Aynı zamanda, elde edilen sonuçları destekleyen birkaç örnek verilmiştir. Elde edilen teoremler literatürde bulunan bazı sabit nokta sonuçlarını genelleştirir.

Anahtar Kelimeler: Ortak sabit nokta, sabit nokta, kısmi sıralı G_p -tam G_p -metrik uzaylar, Banach çiftleri

* Received / Geliş tarihi: 21/06/2016 Accepted / Kabul tarihi: 30/12/2016
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1. INTRODUCTION

In 1922, the Polish mathematician Stefan Banach proved his noteworthy theorem relating to the existence and uniqueness of a fixed point under appropriate conditions for the first time (Banach, 1922). In the last decades, the Banach contraction principle has been studied and generalized considerably by several authors in different ways, for more details see (Matthews, 1994; Schellekens, 2003; Oltra and Valero, 2004; Valero, 2005; Altun et al., 2010; Altun and Erduran, 2010; Karapınar, 2011; Mustafa and Sims, 2006; Beiranvand et al., 2009; Ran and Reurings, 2003; Nieto and López, 2005; Harjani and Sadarangani, 2009; Chen and Lee, 2007).

One of the such generalizations is a G_p-metric space. The notation of G_p-metric space was defined by Zand and Nezhad as a new generalization and unification of both partial metric space and G-metric space (Zand and Nezhad, 2011). In particular, Aydi, Karapınar and Salimi introduced the notions of 0-G_p-Cauchy sequence and 0-G_p-complete G_p-metric space (Aydi et al., 2012), for more details see (Barakat and Zidan, 2015; Bilgili et al., 2013; Ciric et al., 2013; Parvaneh et al., 2013; Popa and Patriciu, 2015; Salimi and Vetro, 2014; Kaya et al., ud., Parvaneh et al. 2014).

Now, we review the necessary notations, definitions and fundamental results produced on G_p-metric spaces that we will need in this work.

Definition of a G_p-metric space was given by Zand and Nezhad as follows:

Definition 1 (Zand and Nezhad, 2011) *A* G_p*-metric on a non-empty set* X *is a function* $G_p : X \times X \times X \to [0,\infty)$, *such that for all* $x, y, z, a \in X$ *the following properties hold:*

G_{p1}. $x = y = z$ if $G_p(x,y,z) = G_p(z,z,z) = G_p(y,y,y) = G_p(x,x,x)$;

G_{p2}. $0 \leq G_p(x,x,x) \leq G_p(x,x,y) \leq G_p(x,y,z)$;

G$_{p3}$. $G_p(x,y,z) = G_p(x,z,y) = G_p(y,z,x) = \dots$, symmetry in all three variables;

G$_{p4}$. $G_p(x,y,z) \leq G_p(x,a,a) + G_p(a,y,z) - G_p(a,a,a)$.

In this case, the pair (X, G_p) is said to be a G_p-metric space.

On the other hand, instead of (G$_{p2}$), Parvaneh, Roshan and Kadelburg used the following condition (Parvaneh et al., 2013):

G$_{p2}$*. $0 \leq G_p(x,x,x) \leq G_p(x,x,y) \leq G_p(x,y,z)$ for all $x, y, z \in X$ with $z \neq y$.

Also, they stated an important remark as following:

Remark 1 *With (G$_{p2}$) assumption, it is very easy to obtain that*

$$G_p(x,x,y) = G_p(x,y,y)$$

holds for all $x, y \in X$, i.e., the respective space is symmetric.

On the other hand, there are a lot of examples of asymmetric G-metric spaces. Hence, the claim stated in (Zand and Nezhad, 2011; Aydi et al., 2012) that each G-metric space is a G_p-metric space (satisfying (G$_{p2}$)) is false. With the assumption (G$_{p2}$*) this conclusion holds true.

We will use definition of G_p-metric space given by Zand and Nezhad throughout the rest of this paper, that is, (X, G_p) is a symmetric G_p-metric space in this paper.

Example 1. (Zand and Nezhad, 2011) *Let $X = [0,\infty)$ and let $G_p : X \times X \times X \to [0,\infty)$ be a mapping defined by $G_p(x,y,z) = \max\{x,y,z\}$, for all $x, y, z \in X$. Then (X, G_p) is a symmetric G_p-metric space but not a G-metric space.*

The following proposition gives some properties of a G_p-metric space.

Proposition 1.1 (Zand and Nezhad, 2011) *Let* (X, G_p) *be a* G_p*-metric space. Then, the following statements hold:*

i. $G_p(x,y,z) \le G_p(x,x,y) + G_p(x,x,z) - G_p(x,x,x)$;

ii. $G_p(x,y,y) \le 2G_p(x,x,y) - G_p(x,x,x)$;

iii. $G_p(x,y,z) \le G_p(x,a,a) + G_p(y,a,a) + G_p(z,a,a) - 2G_p(a,a,a)$;

iv. $G_p(x,y,z) \le G_p(x,a,z) + G_p(a,y,z) - G_p(a,a,a)$;

for any x, y, z *and* $a \in X$.

The following proposition shows that to every G_p-metric space we can associate one metric.

Proposition 1.2 (Zand and Nezhad, 2011) *Every* G_p*-metric on* X *induces a metric* d_{G_p} *on* X *defined by*

$$d_{G_p}(x,y) = G_p(x,y,y) + G_p(y,x,x) - G_p(x,x,x) - G_p(y,y,y)$$

for all $x, y \in X$.

In their paper, Zand and Nezhad also introduced the basic topological concepts like G_p-convergence, G_p-Cauchy sequence and G_p-completeness in G_p-metric spaces as follows.

Definition 2 (Zand and Nezhad, 2011) *Let* (X, G_p) *be a* G_p*-metric space and let* $\{x_n\}$ *be a sequence of points of* X. *A point* $x \in X$ *is said to be the limit of the sequence* $\{x_n\}$ *and denoted by* $x_n \to x$ *if*

$$\lim_{n,m\to\infty} G_p(x, x_n, x_m) = G_p(x, x, x).$$

In this case, we say that the sequence $\{x_n\}$ is G_p-convergent to x.

Thus if $x_n \to x$ in a G_p-metric space (X, G_p), then for any $\varepsilon > 0$, there exists $l \in \mathbb{N}$ such that $|G_p(x, x_n, x_m) - G_p(x, x, x)| < \varepsilon$, for all $n, m > l$.

Using the above definition, one can easily prove the following proposition.

Proposition 1.3 (Zand and Nezhad, 2011) *Let* (X, G_p) *be a* G_p*-metric space. Then, for any sequence* $\{x_n\}$ *in* X *and a point* $x \in X$ *the following are equivalent:*

i. $\{x_n\}$ is G_p-convergent to x;
ii. $G_p(x_n, x_n, x) \to G_p(x, x, x)$ as $n \to \infty$;
iii. $G_p(x_n, x, x) \to G_p(x, x, x)$ as $n \to \infty$.

Proof. If we take $m = n$ in (i), we get that (i) implies (ii). Also, we obtain that (ii) $\Leftrightarrow$ (iii) with (G_{p2}) assumption. For the converse we have:

$$\begin{aligned} G_p(x, x_n, x_m) - G_p(x, x, x) &= G_p(x_n, x_m, x) - G_p(x, x, x) \\ &\le G_p(x_n, x, x) + G_p(x, x_m, x) - G_p(x, x, x) - G_p(x, x, x) \\ &= [G_p(x_n, x, x) - G_p(x, x, x)] + [G_p(x, x_m, x) - G_p(x, x, x)]. \end{aligned}$$

If we take the limit as $n, m \to \infty$ in the previous inequality, we get that (iii) implies (i).

The proof is completed.

Definition 3 (Zand and Nezhad, 2011) Let (X, G_p) be a G_p-metric space.

i. *A sequence* $\{x_n\}$ *is called a* G_p*-Cauchy sequence if and only if* $\lim_{n,m\to\infty} G_p(x_n, x_m, x_m)$ *exits and is finite;*

ii. *A* G_p*-metric space* (X, G_p) *is said to be* G_p*-complete if and only if every* G_p*-Cauchy sequence in* X *is* G_p*-converges to* $x \in X$ *such that*

$$G_p(x, x, x) = \lim_{n,m\to\infty} G_p(x_n, x_m, x_m).$$

The following lemma, which given by Parvaneh et al. provides the characterizations of concepts of Cauchy and completeness for G_p-metric spaces (Parvaneh et al., 2013).

Lemma 1.4

i. *A sequence* $\{x_n\}$ *is a* G_p*-Cauchy sequence in a* G_p*-metric space* (X, G_p) *if and only if it is a Cauchy sequence in the metric space* (X, d_{G_p}).

ii. *A* G_p*-metric space* (X, G_p) *is* G_p*-complete if and only if the metric space* (X, d_{G_p}) *is complete. Moreover,*

$$\lim_{n\to\infty} d_{G_p}(x, x_n) = 0$$

if and only if

$$\lim_{n\to\infty} G_p(x, x_n, x_n) = \lim_{n\to\infty} G_p(x_n, x, x) = \lim_{n,m\to\infty} G_p(x_n, x_n, x_m)$$

$$= \lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = G_p(x, x, x).$$

The following useful lemmas have a crucial role in the proof of our main results.

Lemma 1.5 (Aydi et al., 2012) *Let* (X, G_p) *be a* G_p*-metric space. Then*

i. If $G_p(x,y,z)=0$, then $x=y=z$;

ii. If $x \neq y$, then $G_p(x,y,y)>0$.

Proof. Let $G_p(x,y,z)=0$. Then, by (G_{p2}) we get

$$0 \leq G_p(z,z,z), G_p(y,y,y), G_p(x,x,x) \leq G_p(x,y,z)=0.$$

Hence, we have $G_p(z,z,z)=G_p(y,y,y)=G_p(x,x,x)=G_p(x,y,z)=0$. By ($G_{p1}$) we conclude that $x=y=z$. So, the assertion (i) is proved.

On the other hand, let $x \neq y$ and $G_p(x,y,y)=0$. Then, by (i), $x=y$ which is a contradiction. Thereby, (ii) holds.

Lemma 1.6 (Aydi et al., 2012) *Assume that* $\{x_n\} \to x$ *as* $n \to \infty$ *in a* G_p*-metric space* (X,G_p) *such that* $G_p(x,x,x)=0$. *Then, for every* $y \in X$ *,*

$$\lim_{n\to\infty} G_p(x_n,y,y)=G_p(x,y,y).$$

Proof. First note that $\lim_{n\to\infty} G_p(x_n,x,x)=G_p(x,x,x)=0$. By the rectangle inequality and (G_{p2}), we get

$$\begin{aligned} G_p(x_n,y,y) &\leq G_p(x_n,x,x)+G_p(x,y,y)-G_p(x,x,x) \\ &= G_p(x_n,x,x)+G_p(x,y,y) \end{aligned}$$

and

$$\begin{aligned} G_p(x,y,y) &\leq G_p(x,x_n,x_n)+G_p(x_n,y,y)-G_p(x_n,x_n,x_n) \\ &\leq G_p(x,x_n,x_n)+G_p(x_n,y,y) \\ &= G_p(x_n,x,x)+G_p(x_n,y,y). \end{aligned}$$

Hence, we have

$$0 \leq | G_p(x_n, y, y) - G_p(x, y, y) | \leq G_p(x_n, x, x).$$

Letting $n \to \infty$ we conclude our claim.

The following proposition of Zand and Nezhad will be required in the sequel (Zand and Nezhad, 2011).

Proposition 1.7 *Let* (X_1, G_1) *and* (X_2, G_2) *be* G_p*-metric spaces. Then a function* $f : X_1 \to X_2$ *is* G_p*-continuous at a point* $x \in X_1$ *if and only if it is* G_p*-sequentially continuous at* x*; that is, whenever* $\{x_n\}$ *is* G_p*-convergent to* x *one has* $\{f(x_n)\}$ *is* G_p*-convergent to* $f(x)$.

Kaya et al. given an important remark, which investigates relationship between the concepts of G_p-continuity and d_{G_p}-continuity, as follows (Kaya et al., ud).

Remark 2 *It is worth noting that the notions of* G_p*-continuity and* d_{G_p}*-continuity of any function in the contex of* G_p*-metric space are incomparable, in general. Indeed, if* $X = [0, +\infty)$, $G_p(x, y, z) = \max\{x, y, z\}$, $d_{G_p}(x, y) = |x - y|$, $f(0) = 1$ *and* $f(x) = x^2$ *for all* $x > 0$, $g(x) = |\sin x|$, *then* f *is a* G_p*-continuous and* d_{G_p}*-discontinuous at point* $x = 0$*; while* g *is a* G_p*-discontinuous and* d_{G_p}*-continuous at point* $x = \pi$*. Therefore, in this paper, we take that* $T : X \to X$ *is continuous if both* $T : (X, G_p) \to (X, G_p)$ *and* $T : (X, d_{G_p}) \to (X, d_{G_p})$ *are continuous.*

Also, Kaya et al. defined the concepts of sequentially convergent and subsequentially convergent (Kaya et al., ud).

Definition 4 *Let* (X, G_p) *be a* G_p*-metric space. A mapping* $T : X \to X$ *is said to be:*

i. *sequentially convergent if for any sequence* $\{y_n\}$ *in* X *such that* $\{Ty_n\}$

is convergent in (X, d_{G_p}) *implies that* $\{y_n\}$ *is convergent in* (X, d_{G_p}),

ii. *a subsequentially convergent if for any sequence* $\{y_n\}$ *in* X *such that* $\{Ty_n\}$ *is convergent in* (X, d_{G_p}) *implies that* $\{y_n\}$ *has a convergent subsequence in* (X, d_{G_p}).

The concept of Banach operator pair was introduced by Chen and Li as following (Chen and Li, 2007):

Definition 5 *Let* f *and* T *be self mappings of a nonempty set* M *of a normed linear space* X. *Then,* (f, T) *is a Banach operator pair, if any one of the following conditions is satisfied:*

i. $f[F(T)] \subseteq F(T)$,
ii. $Tfx = fx$ for each $x \in F(T)$,
iii. . $fTx = Tfx$ for each $x \in F(T)$,
iv. $\| fTx - Tx \| \leq k \| Tx - x \|$ for some $k \geq 0$.

Definition 6 (Altun and Şimşek, 2010) Let $(X, \prec)$ *be a partially ordered set. A pair* (f, g) of self maps of X is called weakly increasing if $fx \prec gfx$ and $gx \prec fgx$ for all $x \in X$.

In this work, our purpose is to obtain common fixed point theorems and their results related to f-contraction mappings in partially ordered G_p-complete G_p-metric spaces and also to illustrate the usability of our results with a number of examples.

2. MAIN RESULTS

The aim of this section is to present our findings on common fixed point theorems and their results related to f-contraction mappings in partially ordered G_p-complete G_p-metric spaces. We start by stating our first result.

Theorem 2.1 *Let* $(X, G_p, \prec)$ *be a partially ordered* G_p*-complete* G_p*-metric space and* $T : X \to X$ *be a nondecreasing self mapping. Let* $f : X \to X$ *be a continuous, injective mapping and subsequentially convergent such that*

$$G_p(fTx, fTy, fTy) \leq kM(fx, fy, fy) \tag{2.1}$$

for all comparable $x, y \in X$ *, where* $k \in [0, \frac{1}{2})$ *and*

$$M(fx, fy, fy) = \max\{G_p(fx, fTx, fy), G_p(fy, fT^2x, fTy), G_p(fTx, fT^2x, fTy),$$
$$G_p(fy, fTx, fTy), G_p(fx, fy, fy), G_p(fx, fTx, fTx), G_p(fy, fTy, fTy),$$
$$G_p(fy, fTx, fTx), G_p(fx, fTy, fTy)\}.$$

If there exists $x_0 \in X$ with $x_0 \prec Tx_0$ and one of the following conditions is satisfied:

i. T is a continuous self map on X ;
ii. for any nondecreasing sequence $\{x_n\}$ in $(X, \prec)$ with $x_n \to z$ it follows $x_n \prec z$ for all $n \in \mathbb{N}$;

then, T has a fixed point in X . Furthermore, the set of fixed points of T is well ordered if and only if fixed point of T is unique. Moreover, if (f, T) is a Banach pair, then f and T have a unique common fixed point in X .

Proof. Let $x_0 \in X$ be an arbitrary point in X and define the sequence $\{x_n\}$ in X with $x_n = Tx_{n-1} = T^n x_0$ for $1 \leq n$. As $x_0 \prec Tx_0$ and T is a nondecreasing mapping with respect to ``$\prec$'', by given assumption, we obtain the following:

$$x_0 \prec Tx_0 = x_1 \prec Tx_1 = x_2 \prec \ldots \prec x_n \prec x_{n+1} \prec \ldots$$

Notice that, if $x_n = x_{n+1}$ for any $n \in \mathbb{N}$, then obviously T has a fixed point. Thus suppose $x_n \neq x_{n+1}$ for any $n \in \mathbb{N}$. As $x_{n-1} \prec x_n$ for all $n \in \mathbb{N}$, applying the considered contraction (2.1), we get

$$G_p(fx_n, fx_{n+1}, fx_{n+1}) = G_p(fTx_{n-1}, fTx_n, fTx_n) \leq kM(fx_{n-1}, fx_n, fx_n) \tag{2.2}$$

where

$$M(fx_{n-1}, fx_n, fx_n) = \max\{G_p(fx_{n-1}, fTx_{n-1}, fx_n), G_p(fx_n, fT^2x_{n-1}, fTx_n),$$
$$G_p(fTx_{n-1}, fT^2x_{n-1}, fTx_n), G_p(fx_n, fTx_{n-1}, fTx_n), G_p(fx_{n-1}, fx_n, fx_n),$$
$$G_p(fx_{n-1}, fTx_{n-1}, fTx_{n-1}), G_p(fx_n, fTx_n, fTx_n), G_p(fx_n, fTx_{n-1}, fTx_{n-1}),$$
$$G_p(fx_n, fTx_n, fTx_n), G_p(fx_n, fTx_{n-1}, fTx_{n-1}), G_p(fx_{n-1}, fTx_n, fTx_n)\}$$
$$= \max\{G_p(fx_{n-1}, fx_n, fx_n), G_p(fx_n, fx_{n+1}, fx_{n+1}), G_p(fx_n, fx_{n+1}, fx_{n+1}),$$
$$G_p(fx_n, fx_n, fx_{n+1}), G_p(fx_{n-1}, fx_n, fx_n), G_p(fx_{n-1}, fx_n, fx_n),$$
$$G_p(fx_n, fx_{n+1}, fx_{n+1}), G_p(fx_n, fx_n, fx_n), G_p(fx_{n-1}, fx_{n+1}, fx_{n+1})\}$$
$$= \max\{G_p(fx_{n-1}, fx_n, fx_n), G_p(fx_n, fx_{n+1}, fx_{n+1}), G_p(fx_n, fx_n, fx_n),$$
$$G_p(fx_{n-1}, fx_{n+1}, fx_{n+1})\}$$
$$= \max\{G_p(fx_{n-1}, fx_n, fx_n), G_p(fx_n, fx_{n+1}, fx_{n+1}),$$

$$G_p(fx_{n-1}, fx_{n+1}, fx_{n+1})\}. \tag{2.3}$$

Now, we have to examine three cases in (2.3). For the first case, assume that $M(fx_{n-1}, fx_n, fx_n) = G_p(fx_{n-1}, fx_n, fx_n)$. Then, the expression (2.2) turns into

$$G_p(fx_n, fx_{n+1}, fx_{n+1}) \le kM(fx_{n-1}, fx_n, fx_n) = kG_p(fx_{n-1}, fx_n, fx_n), \tag{2.4}$$

where $k \in [0, \frac{1}{2})$.

For the second case, assume that $M(fx_{n-1}, fx_n, fx_n) = G_p(fx_n, fx_{n+1}, fx_{n+1})$. By the inequality (2.2), we derive that

$$G_p(fx_n, fx_{n+1}, fx_{n+1}) \le kG_p(fx_n, fx_{n+1}, fx_{n+1}) \tag{2.5}$$

which is a contradiction since $k \in [0, \frac{1}{2})$.

For the last case, assume that $M(fx_{n-1}, fx_n, fx_n) = G_p(fx_{n-1}, fx_{n+1}, fx_{n+1})$. By ($G_{p4}$) and the inequality (2.2), we have

$$
\begin{aligned}
G_p(fx_n, fx_{n+1}, fx_{n+1}) &\leq kG_p(fx_{n-1}, fx_{n+1}, fx_{n+1}) \\
&\leq k[G_p(fx_{n-1}, fx_n, fx_n) + G_p(fx_n, fx_{n+1}, fx_{n+1})],
\end{aligned}
$$

which is equivalent to

$$G_p(fx_n, fx_{n+1}, fx_{n+1}) \leq hG_p(fx_{n-1}, fx_n, fx_n) \tag{2.6}$$

where $h = \dfrac{k}{1-k} < 1$ since $k \in [0, \dfrac{1}{2})$.

As a result, from (2.4)-(2.6), we conclude that

$$G_p(fx_n, fx_{n+1}, fx_{n+1}) \leq rG_p(fx_{n-1}, fx_n, fx_n),$$

where $r \in \{h, k\}$ and hence $r < 1$.

Similarly, from (2.1), it can be shown that

$$G_p(fx_{n-1}, fx_n, fx_n) \leq rG_p(fx_{n-2}, fx_{n-1}, fx_{n-1})$$

where $r < 1$.

Therefore, we deduce that

$$G_p(fx_n, fx_{n+1}, fx_{n+1}) \leq rG_p(fx_{n-1}, fx_n, fx_n) \leq \ldots \leq r^n G_p(fx_0, fx_1, fx_1)$$

for all $n \in \mathbb{N}$ and $r < 1$. We show that the sequence $\{fx_n\}$ is a G_p-Cauchy sequence in X. By the inequality (G_{p4}), we have for $m, n \in \mathbb{N}$ with $m > n$,

$$
\begin{aligned}
G_p(fx_n, fx_m, fx_m) \leq\ & G_p(fx_n, fx_{n+1}, fx_{n+1}) + G_p(fx_{n+1}, fx_m, fx_m) \\
& - G_p(fx_{n+1}, fx_{n+1}, fx_{n+1})
\end{aligned}
$$

$$\leq G_p(fx_n, fx_{n+1}, fx_{n+1}) + G_p(fx_{n+1}, fx_{n+2}, fx_{n+2}) + \ldots$$

$$+ G_p(fx_{m-1}, fx_m, fx_m) - \sum_{i=n+1}^{m-1} G_p(fx_i, fx_i, fx_i)$$

$$\leq (r^n + r^{n+1} + \ldots + r^{m-1}) G_p(fx_0, fx_1, fx_1)$$

$$= r^n (1 + r + \ldots + r^{m-n-1}) G_p(fx_0, fx_1, fx_1) \qquad (2.7)$$

$$= r^n \frac{1 - r^{m-n}}{1 - r} G_p(fx_0, fx_1, fx_1)$$

$$\leq \frac{r^n}{1 - r} G_p(fx_0, fx_1, fx_1).$$

Letting $n, m \to \infty$ in (2.7), we get that $G_p(fx_n, fx_m, fx_m) \to 0$, that is, $\{fx_n\}$ is a G_p-Cauchy sequence. By Lemma 1.4, $\{fx_n\}$ is a Cauchy sequence in (X, d_{G_p}) metric space and the completeness of (X, G_p) G_p-metric space requires the completeness of (X, d_{G_p}) metric space. Then, there exists $z \in X$ such that

$$\lim_{n\to\infty} d_{G_p}(fx_n, z) = 0. \qquad (2.8)$$

So, from Lemma 1.4 we get

$$\lim_{n\to\infty} G_p(fx_n, z, z) = \lim_{n\to\infty} G_p(fx_n, fx_n, z)$$

$$= \lim_{n,m\to\infty} G_p(fx_n, fx_m, fx_m)$$

$$= G_p(z, z, z)$$

$$= 0.$$

As f is subsequentially convergent in (X, d_{G_p}), $\{x_n\}$ has a convergent subsequence in (X, d_{G_p}). Hence, there exist $u \in X$ and a subsequence $\{x_{n_i}\}$ such that

$$\lim_{i\to\infty} d_{G_p}(x_{n_i}, u) = 0. \tag{2.9}$$

As f is continuous, (2.9) implies that

$$\lim_{i\to\infty} d_{G_p}(fx_{n_i}, fu) = 0.$$

From (2.8) and by the uniqueness of the limit in metric space (X, d_{G_p}), we obtain that $fu = z$. Consequently,

$$\lim_{i\to\infty} G_p(fx_{n_i}, fu, fu) = \lim_{i\to\infty} G_p(fx_{n_i}, fx_{n_i}, fu) = G_p(fu, fu, fu) = 0.$$

i. If T is a continuous self map on X, by Remark 2, $Tx_{n_i} \to Tu$ and $fTx_{n_i} \to fTu$ as $i \to \infty$. Since $fx_{n_i} \to fu$ as $i \to \infty$, we obtain $fu = fTu$. As f is injective, so we have $u = Tu$.

ii. If T is not continuous then by given assumption we get $x_{n_i} \prec u$ for all $i \in \mathbb{N}$. Now, assume that $Tu \neq u$. Therefore, from (2.1) we get

$$G_p(fx_{n_i+1}, fTu, fTu) = G_p(fTx_{n_i}, fTu, fTu) \le kM(fx_{n_i}, fu, fu), \tag{2.10}$$

where

$$M(fx_{n_i}, fu, fu) = \max\{G_p(fx_{n_i}, fTx_{n_i}, fu), G_p(fu, fT^2x_{n_i}, fTu),$$
$$G_p(fTx_{n_i}, fT^2x_{n_i}, fTu), G_p(fu, fTx_{n_i}, fTu),$$
$$G_p(fx_{n_i}, fu, fu), G_p(fx_{n_i}, fTx_{n_i}, fTx_{n_i}),$$
$$G_p(fu, fTu, fTu), G_p(fu, fTx_{n_i}, fTx_{n_i}),$$

$$G_p(fx_{n_i}, fTu, fTu)\}$$

$$= \max\{G_p(fx_{n_i}, fx_{n_i+1}, fu), G_p(fu, fx_{n_i+2}, fTu),$$

$$G_p(fx_{n_i+1}, fx_{n_i+2}, fTu), G_p(fu, fx_{n_i+1}, fTu),$$

$$G_p(fx_{n_i}, fu, fu), G_p(fx_{n_i}, fx_{n_i+1}, fx_{n_i+1}),$$

$$G_p(fu, fTu, fTu), G_p(fu, fx_{n_i+1}, fx_{n_i+1}),$$

$$G_p(fx_{n_i}, fTu, fTu)\}. \qquad (2.11)$$

On taking limit as $i \to \infty$ and using Lemma 1.6 in (2.10) and (2.11), we get

$$G_p(fu, fTu, fTu) \leq kG_p(fu, fTu, fTu)$$

by the rectangular property. Since $k \in [0, \frac{1}{2})$, the inequality above causes contradiction.

Hence, we have $u = Tu$.

Hence, from (i) and (ii), u is a fixed point of T.

Now, suppose that the set of fixed points of T is well ordered. Then fixed point of T is unique. Assume on contrary that, $Tu = u$ and $Tw = w$ but $u \neq w$. As u and w are comparable, we have from (2.1)

$$G_p(fu, fw, fw) = G_p(fTu, fTw, fTw) \leq kM(fu, fw, fw) \qquad (2.12)$$

where

$$M(fu, fw, fw) = \max\{G_p(fu, fTu, fw), G_p(fw, fT^2u, fTw),$$
$$G_p(fTu, fT^2u, fTw), G_p(fw, fTu, fTw), G_p(fu, fw, fw),$$
$$G_p(fu, fTu, fTu), G_p(fw, fTw, fTw), G_p(fw, fTu, fTu),$$
$$G_p(fu, fTw, fTw)\}$$

$$= \max\{G_p(fu, fu, fw), G_p(fw, fu, fw), G_p(fu, fu, fw),$$
$$G_p(fw, fu, fw), G_p(fu, fw, fw), G_p(fu, fu, fu), G_p(fw, fw, fw),$$
$$G_p(fw, fu, fu), G_p(fu, fw, fw)\}$$
$$= G_p(fu, fu, fw).$$

Hence the inequality (2.12) is equal to

$$G_p(fu, fw, fw) \leq kG_p(fu, fu, fw) = kG_p(fu, fw, fw).$$

Since $k \in [0, \frac{1}{2})$, this is a contraction and so we get $u = w$. Thus, u is the unique fixed point of T.

Conversely, if T has only one fixed point, then the set of fixed points of T being singleton is well ordered.

Since we have assumed that (f, T) is Banach pair; $\{f, T\}$ commutes at the fixed point of T. This implies that $fTu = Tfu$ for $u \in F(T)$. So, $fu = Tfu$ which gives that fu is another fixed point of T. In that case, by the uniqueness of fixed point of T $fu = u$. Hence $fu = Tu = u$, u is unique common fixed point of f and T in X.

If we take $f = I$, the identity mapping in Theorem 2.1, we get the following result:

Corollary 2.2 *Let* $(X, G_p, \prec)$ *be a partially ordered* G_p*-complete* G_p*-metric space and* $T : X \to X$ *be a nondecreasing self mapping such that*

$$G_p(Tx, Ty, Ty) \leq kM(x, y, y)$$

for all comparable $x, y \in X$, where $k \in [0, \frac{1}{2})$ and

$$M(x,y,y)=\max\{G_p(x,Tx,y), G_p(y,T^2x,Ty), G_p(Tx,T^2x,Ty), G_p(y,Tx,Ty), G_p(x,y,y), G_p(x,Tx,Tx), G_p(y,Ty,Ty), G_p(y,Tx,Tx), G_p(x,Ty,Ty)\}.$$

If there exists $x_0 \in X$ with $x_0 \prec Tx_0$ and one of the following conditions is satisfied:

i. T is a continuous self map on X ;
ii. for any nondecreasing sequence $\{x_n\}$ in $(X, \prec)$ with $x_n \to z$ it follows $x_n \prec z$ for all $n \in \mathbb{N}$;

then, T has a fixed point in X . Furthermore, the set of fixed points of T is well ordered if and only if fixed point of T is unique.

Theorem 2.3 *Let $(X, G_p, \prec)$ be a partially ordered G_p-complete G_p-metric space and $T, S : X \to X$ be weakly increasing mappings with respect to ``$\prec$''. Let $f : X \to X$ be a continuous, injective mapping and subsequentially convergent such that*

$$G_p(fTx, fSy, fSy) \le k \max \left\{ \begin{array}{c} G_p(fy, fSy, fSy) + G_p(fx, fSy, fSy), \\ 2G_p(fy, fTx, fTx) \end{array} \right\} \tag{2.13}$$

for all comparable $x, y \in X$, where $k \in [0, \frac{1}{4})$. If one of the following conditions is satisfied:

i. T or S is a continuous self mapping on X ;
ii. for any nondecreasing sequence $\{x_n\}$ in $(X, \prec)$ with $x_n \to z$ it follows $x_n \prec z$ for all $n \in \mathbb{N}$;

then, T and S have a common fixed point in X . Furthermore, the set of common fixed points of T and S is well ordered if and only if common fixed point of T

and S is unique. Moreover, if (f,T) and (f,S) are Banach pairs, then f,T and S have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point in X and define the sequence $\{x_n\}$ inductively by

$$x_{2n+1} = Tx_{2n} \ \ and \ \ x_{2n+2} = Sx_{2n+1}$$

for $n \in \mathbb{N}$. As T and S are weakly increasing mappings with respect to "$\prec$", we obtain the following:

$$x_1 = Tx_0 \prec STx_0 = x_2$$

$$x_2 = Sx_1 \prec TSx_1 = x_3$$

$$\vdots$$

$$x_{2n+1} = Tx_{2n} \prec STx_{2n} = x_{2n+2}$$

$$\vdots$$

Suppose $G_p(fx_n, fx_{n+1}, fx_{n+1}) = 0$ for some $n \in \mathbb{N}$. Without loss of generality, we assume $n = 2N$ for some $N \in \mathbb{N}$. Thus $G_p(fx_{2N}, fx_{2N+1}, fx_{2N+1}) = 0$ and by Lemma 1.5 $fx_{2N} = fx_{2N+1}$. Now, we assume $G_p(fx_{2N+1}, fx_{2N+2}, fx_{2N+2}) > 0$. Since x_{2N} and x_{2N+1} are comparable, using the contractive condition (2.13), we have

$$\begin{aligned} G_p(fx_{2N+1}, fx_{2N+2}, fx_{2N+2}) &= G_p(fTx_{2N}, fSx_{2N+1}, fSx_{2N+1}) \\ &\leq k \max \left\{ \begin{array}{c} G_p(fx_{2N+1}, fSx_{2N+1}, fSx_{2N+1}) + G_p(fx_{2N}, fSx_{2N+1}, fSx_{2N+1}), \\ 2G_p(fx_{2N+1}, fTx_{2N}, fTx_{2N}) \end{array} \right\} \\ &= k \max \left\{ \begin{array}{c} G_p(fx_{2N+1}, fx_{2N+2}, fx_{2N+2}) + G_p(fx_{2N}, fx_{2N+2}, fx_{2N+2}), \\ 2G_p(fx_{2N+1}, fx_{2N+1}, fx_{2N+1}) \end{array} \right\}, \end{aligned}$$

thus,

$$G_p(fx_{2N+1}, fx_{2N+2}, fx_{2N+2}) \leq k[G_p(fx_{2N+1}, fx_{2N+2}, fx_{2N+2}) + G_p(fx_{2N}, fx_{2N+2}, fx_{2N+2})] = 2k\, G_p(fx_{2N+1}, fx_{2N+2}, fx_{2N+2})$$

which is a contradiction since $k \in [0, \frac{1}{4})$. Then, we conclude that

$$G_p(fx_{2N+1}, fx_{2N+2}, fx_{2N+2}) = 0.$$

Hence, we have $fx_{2N+1} = fx_{2N+2}$. As f is injective, we get $x_{2N+1} = x_{2N+2}$, that is, $x_{2N} = x_{2N+1} = x_{2N+2}$. Then, x_{2N} is a common fixed point of T and S, that is, $x_{2N} = Tx_{2N} = Sx_{2N}$.

Therefore, we can suppose that the successive terms of $\{x_n\}$ are different. Then

$G_p(fx_n, fx_{n+1}, fx_{n+1}) > 0$ for all $n \in \mathbb{N}$ and the following holds:

$$G_p(fx_{2n+1}, fx_{2n+2}, fx_{2n+2}) = G_p(fTx_{2n}, fSx_{2n+1}, fSx_{2n+1})$$

$$\leq k \max \begin{Bmatrix} G_p(fx_{2n+1}, fSx_{2n+1}, fSx_{2n+1}) + G_p(fx_{2n}, fSx_{2n+1}, fSx_{2n+1}), \\ 2G_p(fx_{2n+1}, fTx_{2n}, fTx_{2n}) \end{Bmatrix}$$

$$= k \max \begin{Bmatrix} G_p(fx_{2n+1}, fx_{2n+2}, fx_{2n+2}) + G_p(fx_{2n}, fx_{2n+2}, fx_{2n+2}), \\ 2G_p(fx_{2n+1}, fx_{2n+1}, fx_{2n+1}) \end{Bmatrix}$$

$$\leq k \max \begin{Bmatrix} 2G_p(fx_{2n+1}, fx_{2n+2}, fx_{2n+2}) + G_p(fx_{2n}, fx_{2n+1}, fx_{2n+1}), \\ 2G_p(fx_{2n+1}, fx_{2n+1}, fx_{2n+1}) \end{Bmatrix}$$

thus,

$$G_p(fx_{2n+1}, fx_{2n+2}, fx_{2n+2}) \leq 2kG_p(fx_{2n+1}, fx_{2n+2}, fx_{2n+2}) + kG_p(fx_{2n}, fx_{2n+1}, fx_{2n+1})$$

and so

$$G_p(fx_{2n+1}, fx_{2n+2}, fx_{2n+2}) \leq \frac{k}{1-2k} G_p(fx_{2n}, fx_{2n+1}, fx_{2n+1}).$$

Let $r = \dfrac{k}{1-2k}$, then $r \in [0, \frac{1}{2})$ since $k \in [0, \frac{1}{4})$ and we deduce that

$$G_p(fx_{2n+1}, fx_{2n+2}, fx_{2n+2}) \leq rG_p(fx_{2n}, fx_{2n+1}, fx_{2n+1}). \tag{2.14}$$

Similarly, by (2.13), we obtain

$$\begin{aligned}
&G_p(fx_{2n}, fx_{2n+1}, fx_{2n+1}) = G_p(fx_{2n+1}, fx_{2n}, fx_{2n}) \\
&= G_p(fTx_{2n}, fSx_{2n-1}, fSx_{2n-1}) \\
&\leq k \max\left\{\begin{array}{c} G_p(fx_{2n-1}, fSx_{2n-1}, fSx_{2n-1}) + G_p(fx_{2n}, fSx_{2n-1}, fSx_{2n-1}), \\ 2G_p(fx_{2n-1}, fTx_{2n}, fTx_{2n}) \end{array}\right\} \\
&= k \max\left\{\begin{array}{c} G_p(fx_{2n-1}, fx_{2n}, fx_{2n}) + G_p(fx_{2n}, fx_{2n}, fx_{2n}), \\ 2G_p(fx_{2n-1}, fx_{2n+1}, fx_{2n+1}) \end{array}\right\} \\
&\leq k \max\left\{\begin{array}{c} G_p(fx_{2n-1}, fx_{2n}, fx_{2n}) + G_p(fx_{2n}, fx_{2n}, fx_{2n}) \\ 2[G_p(fx_{2n-1}, fx_{2n}, fx_{2n}) + G_p(fx_{2n}, fx_{2n+1}, fx_{2n+1})] \end{array}\right\}
\end{aligned}$$

so

$$G_p(fx_{2n}, fx_{2n+1}, fx_{2n+1}) \leq 2k[G_p(fx_{2n-1}, fx_{2n}, fx_{2n}) + G_p(fx_{2n}, fx_{2n+1}, fx_{2n+1})].$$

Then, for $h=\dfrac{2k}{1-2k}$, we get $h\in[0,1)$ since $k\in[0,\dfrac{1}{4})$ and

$$G_p(fx_{2n}, fx_{2n+1}, fx_{2n+1}) \leq hG_p(fx_{2n-1}, fx_{2n}, fx_{2n}). \tag{2.15}$$

As a result, from (2.14) and (2.15), for $\lambda=\max\{r,h\}$ we conclude that

$$G_p(fx_n, fx_{n+1}, fx_{n+1}) \leq \lambda G_p(fx_{n-1}, fx_n, fx_n)$$

for all $n\in\mathbb{N}$ and $\lambda\in[0,1)$. Hence, we get

$$G_p(fx_n, fx_{n+1}, fx_{n+1}) \leq \lambda G_p(fx_{n-1}, fx_n, fx_n) \leq \ldots \leq \lambda^n G_p(fx_0, fx_1, fx_1)$$

for all $n\in\mathbb{N}$ and $\lambda\in[0,1)$.

Using the same technique as in the proof of Theorem 2.1, we can conclude that $\{fx_n\}$ is a G_p-Cauchy sequence. By Lemma 1.4, $\{fx_n\}$ is a Cauchy sequence in (X,d_{G_p}) metric space and the completeness of (X,G_p) G_p-metric space requires the completeness of (X,d_{G_p}) metric space. Then, there exists $z\in X$ such that

$$\lim_{n\to\infty} d_{G_p}(fx_n, z) = 0. \tag{2.16}$$

So, from Lemma 1.4 we get

$$\begin{aligned}\lim_{n\to\infty} G_p(fx_n, z, z) &= \lim_{n\to\infty} G_p(fx_n, fx_n, z)\\ &= \lim_{n,m\to\infty} G_p(fx_n, fx_m, fx_m)\\ &= G_p(z,z,z)\\ &= 0.\end{aligned}$$

As f is subsequentially convergent in (X, d_{G_p}), $\{x_n\}$ has a convergent subsequence in (X, d_{G_p}). Hence, there exist $u \in X$ and a subsequence $\{x_{n_i}\}$ such that

$$\lim_{i\to\infty} d_{G_p}(x_{n_i}, u) = 0. \qquad (2.17)$$

As f is continuous, (2.17) implies that

$$\lim_{i\to\infty} d_{G_p}(fx_{n_i}, fu) = 0.$$

From (2.16) and by the uniqueness of the limit in metric space (X, d_{G_p}), we obtain that $fu = z$. Consequently,

$$\lim_{i\to\infty} G_p(fx_{n_i}, fu, fu) = \lim_{i\to\infty} G_p(fx_{n_i}, fx_{n_i}, fu) = G_p(fu, fu, fu) = 0.$$

Now, let us show that u is a common fixed point of T and S.

i. If T is a continuous mapping on X, then $Tx_{2n_i} \to Tu$ and $fTx_{2n_i} \to fTu$ as $i \to \infty$. Since $fx_{n_i} \to fu$ as $i \to \infty$, we obtain $fu = fTu$. As f is injective, so we have $u = Tu$.

Assume that $u \neq Su$. Since $u \prec u$, we get from (2.13)

$$\begin{aligned} G_p(fu, fSu, fSu) &= G_p(fTu, fSu, fSu) \\ &\le k \max\left\{ \begin{array}{c} G_p(fu, fSu, fSu) + G_p(fu, fSu, fSu), \\ 2G_p(fu, fTu, fTu) \end{array} \right\} \\ &= 2kG_p(fu, fSu, fSu), \end{aligned}$$

which is a contradiction since $k \in [0, \frac{1}{4})$ and hence $u = Su$.

The proof, assuming that S is continuous, is similar to above.

ii. If T and S are not continuous then by given assumption we get $x_n \prec u$ for all $n \in \mathbb{N}$. Thus for the subsequence $\{x_{2n_i}\}$ and $\{x_{2n_i+1}\}$ of $\{x_n\}$ we have $x_{2n_i} \prec u$ and $x_{2n_i+1} \prec u$. Assume that $u \neq Tu$ and $u \neq Su$. Therefore, from (2.13) we get

$$G_p(fTu, fx_{2n_i+2}, fx_{2n_i+2}) = G_p(fTu, fSx_{2n_i+1}, fSx_{2n_i+1})$$

$$\leq k \max \left\{ \begin{array}{c} G_p(fx_{2n_i+1}, fSx_{2n_i+1}, fSx_{2n_i+1}) + G_p(fu, fSx_{2n_i+1}, fSx_{2n_i+1}), \\ 2G_p(fx_{2n_i+1}, fTu, fTu) \\ \end{array} \right\}$$

$$= k \max \left\{ \begin{array}{c} G_p(fx_{2n_i+1}, fx_{2n_i+2}, fx_{2n_i+2}) + G_p(fu, fx_{2n_i+2}, fx_{2n_i+2}), \\ 2G_p(fx_{2n_i+1}, fTu, fTu) \\ \end{array} \right\}.$$

Taking the limit as $i \to \infty$ in the last inequality, we have

$$G_p(fTu, fu, fu) \leq 2kG_p(fu, fTu, fTu) = 2kG_p(fu, fu, fTu),$$

which is a contradiction and so $u = Tu.$ Similarly, it can be seen $Su = u$. Therefore, u is a common fixed point of T and S.

The uniqueness of common fixed point of T and S can be obtained easily. Also, since (f,T) and (f,S) are Banach pairs as in the proof of Theorem 2.1, it can be shown that f,T and S have a unique common fixed point in X.

Taking $f = I$, the identity mapping in Theorem 2.3, we obtain the following result:

Corollary 2.4 *Let* $(X, G_p, \prec)$ *be a partially ordered* G_p*-complete* G_p*-metric space and* $T,S: X \to X$ *be weakly increasing mappings with respect to* ``$\prec$'' *such that*

$$G_p(Tx, Sy, Sy) \leq k \max\left\{ \begin{array}{c} G_p(y, Sy, Sy) + G_p(x, Sy, Sy), \\ 2G_p(y, Tx, Tx) \\ \end{array} \right\}$$

for all comparable $x, y \in X$, where $k \in [0, \frac{1}{4})$. If one of the following conditions is satisfied:

i. T or S is a continuous self mapping on X;
ii. for any nondecreasing sequence $\{x_n\}$ in $(X, \prec)$ with $x_n \to z$ it follows $x_n \prec z$ for all $n \in \mathbb{N}$;

then, T and S have a common fixed point in X. Furthermore, the set of common fixed points of T and S is well ordered if and only if common fixed point of T and S is unique.

Putting $T = S$ in Theorem 2.3, we have the following result:

Corollary 2.5 *Let $(X, G_p, \prec)$ be a partially ordered G_p-complete G_p-metric space and $T : X \to X$ be a nondecreasing mapping. Let $f : X \to X$ be a continuous, injective mapping and subsequentially convergent such that*

$$G_p(fTx, fTy, fTy) \leq k \max\left\{ \begin{array}{c} G_p(fy, fTy, fTy) + G_p(fx, fTy, fTy), \\ 2G_p(fy, fTx, fTx) \\ \end{array} \right\}$$

for all comparable $x, y \in X$, where $k \in [0, \frac{1}{3})$. If there exists $x_0 \in X$ with $x_0 \prec Tx_0$ and one of the following conditions is satisfied:

i. T is a continuous self map on X;
ii. for any nondecreasing sequence $\{x_n\}$ in $(X, \prec)$ with $x_n \to z$ it follows $x_n \prec z$ for all $n \in \mathbb{N}$;

then, T has a fixed point in X. Furthermore, the set of fixed points of T is well ordered if and only if fixed point of T is unique. Moreover, if (f,T) is a Banach pair, then f and T have a unique common fixed point in X.

If we take $f = I$, the identity mapping in Corollary 2.5, we obtain the following result:

Corollary 2.6 *Let* $(X, G_p, \prec)$ *be a partially ordered* G_p*-complete* G_p*-metric space and* $T : X \to X$ *be a nondecreasing mapping such that*

$$G_p(Tx,Ty,Ty) \le k \max\left\{ \begin{matrix} G_p(y,Ty,Ty) + G_p(x,Ty,Ty), \\ 2G_p(y,Tx,Tx) \\ \end{matrix} \right\}$$

for all comparable $x, y \in X$, where $k \in [0, \frac{1}{3})$. If there exists $x_0 \in X$ with $x_0 \prec Tx_0$ and one of the following conditions is satisfied:

i. T is a continuous self map on X;
ii. for any nondecreasing sequence $\{x_n\}$ in $(X, \prec)$ with $x_n \to z$ it follows $x_n \prec z$ for all $n \in \mathbb{N}$;

then, T has a fixed point in X. Furthermore, the set of fixed points of T is well ordered if and only if fixed point of T is unique.

Theorem 2.7 *Let* $(X, G_p, \prec)$ *be a partially ordered* G_p*-complete* G_p*-metric space and* $T, S : X \to X$ *be weakly increasing mappings with respect to* ``$\prec$''. *Let* $f : X \to X$ *be a continuous, injective mapping and subsequentially convergent such that*

$$G_p(fTx, fSy, fSy) \le aG_p(fx, fy, fy) + bG_p(fy, fSy, fSy)$$

$$+ k \max\left\{ \begin{matrix} G_p(fx, fSy, fSy) + G_p(fy, fTx, fTx) + G_p(fy, fTx, fSy), \\ 2G_p(fy, fSy, fSy) + G_p(fx, fSy, fSy) \\ \end{matrix} \right\}$$

for all comparable $x, y \in X$, where $0 \leq a, b, k$ and $a + b + 4k < 1$. If one of the following conditions is satisfied:

i. T or S is a continuous self mapping on X;
ii. for any nondecreasing sequence $\{x_n\}$ in $(X, \prec)$ with $x_n \to z$ it follows $x_n \prec z$ for all $n \in \mathbb{N}$;

then, T and S have a common fixed point in X. Furthermore, the set of common fixed points of T and S is well ordered if and only if common fixed point of T and S is unique. Moreover, if (f, T) and (f, S) are Banach pairs, then f, T and S have a unique common fixed point in X.

Proof. The existence and uniqueness of the common fixed point of f, T and S can be obtained applying the same method as in Theorem 2.3, so we omit it.

Taking $f = I$, the identity mapping in Theorem 2.7,

Corollary 2.8 *Let* $(X, G_p, \prec)$ *be a partially ordered* G_p*-complete* G_p*-metric space and* $T, S : X \to X$ *be weakly increasing mappings with respect to* "$\prec$" *such that*

$$G_p(Tx, Sy, Sy) \leq aG_p(x, y, y) + bG_p(y, Sy, Sy)$$

$$+ k \max \left\{ \begin{array}{c} G_p(x, Sy, Sy) + G_p(y, Tx, Tx) + G_p(y, Tx, Sy), \\ 2G_p(y, Sy, Sy) + G_p(x, Sy, Sy) \end{array} \right\}$$

for all comparable $x, y \in X$, where $0 \leq a, b, k$ and $a + b + 4k < 1$. If one of the following conditions is satisfied:

i. T or S is a continuous self mapping on X;
ii. for any nondecreasing sequence $\{x_n\}$ in $(X, \prec)$ with $x_n \to z$ it follows $x_n \prec z$ for all $n \in \mathbb{N}$;

then, T and S have a common fixed point in X . Furthermore, the set of common fixed points of T and S is well ordered if and only if common fixed point of T and S is unique.

Putting $T = S$ in Theorem 2.7, we have the following result:

Corollary 2.9 *Let $(X, G_p, \prec)$ be a partially ordered G_p-complete G_p-metric space and $T : X \to X$ be a nondecreasing mapping. Let $f : X \to X$ be a continuous, injective mapping and subsequentially convergent such that*

$$G_p(fTx, fTy, fTy) \leq aG_p(fx, fy, fy) + bG_p(fy, fTy, fTy)$$

$$+k \max\left\{\begin{array}{c} G_p(fx, fTy, fTy) + G_p(fy, fTx, fTx) + G_p(fy, fTx, fTy), \\ 2G_p(fy, fTy, fTy) + G_p(fx, fTy, fTy) \end{array}\right\}$$

for all comparable $x, y \in X$, where $0 \leq a, b, k$ and $a + b + 4k < 1$. If there exists $x_0 \in X$ with $x_0 \prec Tx_0$ and one of the following conditions is satisfied:

i. T is a continuous self map on X ;
ii. for any nondecreasing sequence $\{x_n\}$ in $(X, \prec)$ with $x_n \to z$ it follows $x_n \prec z$ for all $n \in \mathbb{N}$;

then, T has a fixed point in X . Furthermore, the set of fixed points of T is well ordered if and only if fixed point of T is unique. Moreover, if (f, T) is a Banach pair, then f and T have a unique common fixed point in X .

If we take $f = I$, the identity mapping in Corollary 2.9, we obtain the following result:

Corollary 2.10 *Let $(X, G_p, \prec)$ be a partially ordered G_p-complete G_p-metric space and $T : X \to X$ be a nondecreasing mapping such that*

$$G_p(Tx, Ty, Ty) \leq aG_p(x, y, y) + bG_p(y, Ty, Ty)$$

$$+k\max\left\{\begin{array}{c} G_p(x,Ty,Ty)+G_p(y,Tx,Tx)+G_p(y,Tx,Ty), \\ 2G_p(y,Ty,Ty)+G_p(x,Ty,Ty) \end{array}\right\}$$

for all comparable $x, y \in X$, where $0 \leq a,b,k$ and $a+b+4k<1$. If there exists $x_0 \in X$ with $x_0 \prec Tx_0$ and one of the following conditions is satisfied:

i. T is a continuous self map on X;
ii. for any nondecreasing sequence $\{x_n\}$ in $(X,\prec)$ with $x_n \to z$ it follows $x_n \prec z$ for all $n \in \mathbb{N}$;

then, T has a fixed point in X. Furthermore, the set of fixed points of T is well ordered if and only if fixed point of T is unique.

3. EXAMPLES

In this section, some examples are given to illustrate the usability of the results presented herein.

Example 1 Let $X=[0,1]$ be endowed with the following relation: $x \prec y$ if and only if $y \leq x$ where "$\leq$" is usual order on X. Then, $(X,\prec)$ is a partially ordered set. Let $G_p : X \times X \times X \to [0,\infty)$ be defined by $G_p(x,y,z)=\max\{x,y,z\}$. Therefore, for any $x, y \in X$

$$d_{G_p}(x,y)=G_p(x,y,y)+G_p(y,x,x)-G_p(x,x,x)-G_p(y,y,y)=|x-y|.$$

Then (X,G_p) is G_p-complete G_p-metric space.

Define $T, f : X \to X$ as $T(x)=\dfrac{x}{4}$ and $f(x)=\dfrac{4x}{5}$. Obviously, f is injective mapping, continuous, subsequentially convergent. Indeed, let $\{x_n\}$ be a sequence converging to x in (X,G_p), then

$$\lim_{n\to\infty}\max\{x_n, x\} = \lim_{n\to\infty} G_p(x_n, x, x) = G_p(x, x, x) = x,$$

hence by definition of f , we have

$$\lim_{n\to\infty} G_p(fx_n, fx, fx) = \lim_{n\to\infty}\max\{fx_n, fx\} = \lim_{n\to\infty}\max\left\{\frac{4x_n}{5}, \frac{4x}{5}\right\}$$

$$= \frac{4}{5}\lim_{n\to\infty}\max\{x_n, x\} = \frac{4x}{5} = G_p(fx, fx, fx), \tag{3.1}$$

that is, $\{fx_n\}$ converges to fx in (X, G_p) .

On the other hand, if $\{x_n\}$ converges to x in (X, d_{G_p}) , hence

$$\lim_{n\to\infty} d_{G_p}(x_n, x) = \lim_{n\to\infty} | x_n - x | = 0.$$

Thus, by definition of d_{G_p} and f , one can find

$$\lim_{n\to\infty} d_{G_p}(fx_n, fx) = \lim_{n\to\infty}\left|\frac{4x_n}{5} - \frac{4x}{5}\right| = \frac{4}{5}\lim_{n\to\infty} | x_n - x | = 0. \tag{3.2}$$

By convergences (3.1) and (3.2) yield that f is a continuous mapping.

Now, let we show that f is subsequentially convergent. Let $\{fy_n\}$ is convergent to y in (X, d_{G_p}) . Then, we have

$$\lim_{n\to\infty} fy_n = \lim_{n\to\infty}\frac{4y_n}{5} = y,$$

which implies that $\lim_{n\to\infty} y_n = \frac{5y}{4}$. Hence, $\{y_n\}$ a is convergent sequence in (X, d_{G_p}) and so $\{y_n\}$ has a convergent sequence in (X, d_{G_p}) .

Similarly, it can be easily shown that T is a continuous mapping. Furthermore, it is clear that T is a nondecreasing mapping with respect to ``$\prec$''. Also, for $x_0 = 0$, we have $x_0 \prec Tx_0$.

In particular, for any $x \prec y$, we get

$$M(fx, fy, fy) = \max\left\{ \begin{array}{c} \frac{4x}{5}, \max\left\{\frac{4y}{5}, \frac{x}{20}, \frac{y}{5}\right\}, \frac{x}{5}, \max\left\{\frac{4y}{5}, \frac{x}{5}, \frac{y}{5}\right\}, \frac{4y}{5}, \\ \max\left\{\frac{4y}{5}, \frac{x}{4}, \frac{x}{4}\right\} \end{array} \right\} = \frac{4x}{5}.$$

Then for all $x, y \in X$ with $x \prec y$ and $k = \frac{1}{4}$, we have

$$G_p(fTx, fTy, fTy) = \max\left\{\frac{x}{5}, \frac{y}{5}\right\} = \frac{x}{5} \leq \frac{1}{4} M(fx, fy, fy).$$

Thus, all the conditions of Theorem 2.1 are satisfied. So, 0 is a unique fixed point of T.

Finally, (f, T) is a Banach pair since $fT0 = Tf0 = 0$ for $0 \in F(T)$. Therefore, 0 is a unique common fixed point of T and f.

Example 2 *Let* $X = [0,1]$ *be endowed with the following relation:* $x \prec y$ *if and only if* $y \leq x$ *where ``*$\leq$*" is usual order on* X. *Then,* $(X, \prec)$ *is a partially ordered set. Let* $G_p : X \times X \times X \to [0, \infty)$ *be defined by* $G_p(x, y, z) = \max\{x, y, z\}$. *Therefore,* (X, G_p) *is* G_p*-complete* G_p*-metric space.*

Define $T, f : X \to X$ by $T(x) = \frac{x}{6}$ and $f(x) = \frac{3x}{4}$ for all $x \in X$. Obviously, f is injective mapping, subsequentially convergent and continuous.

Also, T is a continuous and nondecreasing mapping with respect to "$\prec$". Moreover, for $x_0 = 0$, we get $x_0 \prec Tx_0$.

On the other hand, for any $x \prec y$, we obtain

$$\max\left\{G_p(fy, fTy, fTy) + G_p(fx, fTy, fTy), 2G_p(fy, fTx, fTx)\right\} = \frac{3y}{4} + \frac{3x}{4}.$$

In that case, for every $x, y \in X$ with $x \prec y$ and $k = \frac{1}{6} \in [0, \frac{1}{3})$, we have

$$G_p(fTx, fTy, fTy) = \max\left\{\frac{x}{8}, \frac{y}{8}\right\} = \frac{x}{8} \le \frac{1}{6}\left(\frac{3y}{4} + \frac{3x}{4}\right).$$

Thus, all the conditions of Corollary 2.5 are fulfilled. Hence, T has a unique fixed point. Clearly, 0 is a unique fixed point of T. Furthermore, (f, T) is a Banach pair since $fT0 = Tf0 = 0$ for $0 \in F(T)$. So, 0 is a unique common fixed point of T and f.

Example 3 *Let* $X = [0,1]$ *be endowed with the following relation:* $x \prec y$ *if and only if* $y \le x$ *where* "$\le$" *is usual order on* X. *Then,* $(X, \prec)$ *is a partially ordered set. Let* $G_p : X \times X \times X \to [0, \infty)$ *be defined by* $G_p(x, y, z) = \max\{x, y, z\}$. *Hence* (X, G_p) *is* G_p*-complete* G_p*-metric space.*

Now, define the mappings $T, S, f : X \to X$ by $T(x) = \frac{x^2}{4}$, $S(x) = \frac{x^2}{5}$ ve $f(x) = \frac{x}{2}$. It can be shown that f is injective mapping, subsequentially convergent and continuous by similar arguments in Example 2. Also, it is clear that T and S are continuous mappings.

Now, we denote that T and S are weakly increasing mappings. Let, $x \in X$. Since

$$STx = S\left(\frac{x^2}{4}\right) = \frac{1}{80}x^4 \leq \frac{1}{4}x^2 = Tx,$$

we have $Tx \prec STx$. Similarly, we can show that $Sx \prec TSx$. Thus, T and S are weakly increasing mappings.

Without loss of generality, we assume that $x \prec y$, that is, $y \leq x$. So, we get

$$G_p(fTx, fSy, fSy) = \max\left\{\frac{x^2}{8}, \frac{y^2}{10}\right\} = \frac{x^2}{8}$$

and

$$G_p(fx, fy, fy) = \max\left\{\frac{x}{2}, \frac{y}{2}\right\} = \frac{x}{2}.$$

Then, we conclude that for $a = \dfrac{1}{4}$ and $b = k = 0$

$$G_p(fTx, fSy, fSy) = \frac{x^2}{8} \leq \frac{x}{8} = \frac{1}{4}G_p(fx, fy, fy).$$

Then, all the conditions of Theorem 2.7 holds and so, T and S have a unique common fixed point which is $x = 0$. Also, (f, T) and (f, S) are Banach pairs since $fT0 = Tf0 = 0$ for $0 \in F(T)$ and $fS0 = Sf0 = 0$ for $0 \in F(S)$. Then, f, T and S have a unique common fixed point 0 in $[0,1]$.

4. ACKNOWLEDGEMENTS

The authors would like to thank the reviewers for pointing out some mistakes and misprints in the earlier version of this paper. So, They would like to express their pleasure to the reviewers for their careful reading and making some useful comments which improved the presentation of the paper. M. Kaya has been supported by the Scientific and Technological Research Council of Turkey (TUBITAK Programme, 2211-A).

REFERENCES

Altun, I., Erduran, A., (2010), "Fixed Point Theorems for Monotone Mappings on Partial Metric Spaces'', Fixed Point Theory Appl., 2011, 1–10.

Altun, I., Sola F., Şimşek, H. (2010), "Generalized contractions on partial metric spaces'', Topology Appl., 157, 2778–2785.

Altun, I, Şimşek, H., (2010), "*S*ome Fixed Point Theorems on Ordered Metric Spaces and Application'', Fixed Point Theory and Applications, 2010, 17 pages.

Aydi, H., Karapınar E., Salimi, P., (2012), "Some fixed point results in G_p-metric spaces'', J. Appl. Math., 2012, 1–15.

Banach, S., (1922), "Sur les operations dans les ensembles abstraits et leur application aux équations integrales'', Fund. Math. J., 3**,** 133–181.

Barakat M.A., Zidan, A.M. (2015*)*, "A common fixed point theorem for weak contractive maps in G_p-metric spaces'', J. Egyptian Math. Soc., 23, 309–314.

Beiranvand, A., Moradi, S., Omid, M., Pazandeh, H., (2009), "Two Fixed Point Theorems For Special Mappings'', arxiv:0903.1504v1 math.FA.

Bilgili, N., Karapınar E., Salimi, P., (2013), "Fixed point theorems for generalized contractions on G_p-metric spaces'', Journal of Inequalities and Applications, 2013:39, 1–13.

Chen, J., Li, Z., (2007), "Common Fixed Points For Banach Operator Pairs in Best Approximation'', J. Math. Anal. Appl., 336, 1466–1475.

Ciric, Lj., Alsulami, S. M., Parvaneh, V., Roshan, R., (2013), "Some fixed point results in ordered G_p-metric spaces'', Fixed Point Theory Appl., 2013:317, 1–25.

Harjani, J., Sadarangani, K., (2009), "Fixed point theorems for weakly contractive mappings in partially ordered sets'', Nonlinear Anal., 71, 3403-3410.

Karapınar, E., (2011), "Generalizations of Caristi Kirk's Theorem on Partial Metric Spaces'', Fixed Point Theory Appl., 2011:4, 1-7.

Kaya, M., Öztürk M., Furkan, H. (2016), "Some Common Fixed Point Theorems for (F, f)-Contraction Mappings in $0\text{-}G_p$-Complete G_p-Metric Spaces'', British Journal of Mathematics & Computer Science, 16(2), 1-23.

Matthews, S.G., (1994), “Partial metric topology’’, in: Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci., 728, 183–197.

Mustafa, Z., Sims, B., (2006), “A new approach to generalized metric spaces’’, J. Nonlinear Convex Anal., 7, 289–297.

Nieto, J.J., López, R.R., (2005), “Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations’’, 22, 223–239.

Oltra S., Valero, O., (2004), “Banachs fixed point theorem for partial metric spaces’’, Rend. Istit. Math. Univ. Trieste, 36, 17–26.**Parvaneh, V., Roshan, J.R., Kadelburg, Z.,** (2013), “On generalized weakly GP-contractive mappings in ordered Gp-metric spaces’’, Gulf J. Math., 1, 78–97.

Parvaneh, V., Salimi, P., Vetro, P., Nezhad A.D., Radenović, S., (2014), “Fixed point results for $GP_{(\Lambda\Theta)}$-contractive mappings’’, J. Nonlinear Sci. Appl., **7,** 150–159.

Popa, V., Patriciu, A. M., (2015), “Two general fixed point theorems for a sequence of mappings satisfying implicit relations in Gp-metric spaces’’, Appl. Gen. Topol. 16, 225-231.

Ran, A.C.M., Reurings, M.C.B., (2003), “A fixed point theorem in partially ordered sets and some applications to matrix equations’’, Proc. Amer. Math. Soc. 132, 1435–1443.

Salimi, P., Vetro, P., (2014), “A result of Suzuki type in partial G-metric spaces’’, Acta Mathematica Scientia, 34B (2):274-284.

Schellekens, M.P., (2003), “A characterization of partial metrizability: domains are quantifiable’’, Theoret. Comp. Sci., 305,. 409–432 .

Valero, O., (2005), “On Banach fixed point theorems for partial metric spaces’’, Appl. Gen. Topol., 6, 229–240.

Zand, M.R.A., Nezhad, A.D., (2011), “A generalization of partial metric spaces’’, J. Contemp. Appl. Math., 24, 86–93.