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ON A CLASS OF BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER RELATED TO FABER POLYNOMIALS AND $q-$SĂLĂGEAN OPERATOR

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Abstract. In this paper, we define a new class of bi-univalent functions of complex order $\sum_q^n(\tau, \zeta; \phi)$ which is defined by subordination in the open unit disc $\mathbb{D}$. By using $\mathcal{D}_q^n F(\varsigma)$ operator. Furthermore, using the Faber polynomial expansions, we get upper bounds for the coefficients of function belonging to this class.

1. Introduction

Let $\mathcal{A}$ be the class of functions

$$F(\varsigma) = \varsigma + \sum_{\rho=2}^{\infty} a_\rho \varsigma^\rho, \tag{1}$$

defined in $\mathbb{D} = \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$ normalized by the conditions $F(0) = F'(0) - 1 = 0$ for every $\varsigma \in \mathbb{D}$ and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{D}$. For every $F \in \mathcal{S}$ there exists an inverse function F^{-1} which is defined in some neighborhood of the origin, and satisfying the conditions

$$F^{-1}(F(\varsigma)) = \varsigma, \quad (\varsigma \in \mathbb{D}),$$

and

$$F(F^{-1}(\omega)) = \omega, \quad (|\omega| < r_0(F);\ r_0(F) \geq \frac{1}{4}),$$

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where

$$\begin{aligned} g(\omega) &= F^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 + -(5a_2^3 - 5a_2a_3 + a_4)\omega^4 + ... \\ &= \omega + \sum_{\rho=2}^{\infty} A_\rho \omega^\rho. \end{aligned} \tag{2}$$

If both F and F^{-1} are univalent in $\mathbb{D}$, then $F \in \mathcal{A}$ is called bi-univalent in $\mathbb{D}$ and the class of these functions is denoted by σ. For more study this class (see [5, 7, 13, 24, 26, 27]).

In [17] Faber introduced a polynomial which bears his name and is very important role in geometric function theory.

By using the expansion of this polynomial for $F \in \mathcal{S}$, the coefficients of its inverse $g = F^{-1}$ may be expressed, (see [3] and [4]) as

$$g(\omega) = F^{-1}(\omega) = \omega + \sum_{\rho=2}^{\infty} \frac{1}{\rho} \chi_{\rho-1}^{-\rho}(a_2, a_3, ..., a_\rho)\omega^\rho, \tag{3}$$

where

$$\begin{aligned} \chi_{\rho-1}^{-\rho} = {} & \frac{(-\rho)!}{(-2\rho+1)!(\rho-1)!} a_2^{\rho-1} + \frac{(-\rho)!}{(2(-\rho+1))!(\rho-3)!} a_2^{\rho-3} a_3 \\ & + \frac{(-\rho)!}{(-2\rho+3)!(\rho-4)!} a_2^{\rho-4} a_4 + \frac{(-\rho)!}{(2(-\rho+2))!(\rho-5)!} a_2^{\rho-5} \\ & \times \left[a_5 + (-\rho+2)a_3^2\right] + \frac{(-\rho)!}{(-2\rho+5)!(\rho-6)!} a_2^{\rho-6} \times \left[a_6 + (-2\rho+5)a_3a_4\right] \\ & + \sum_{j\geq 7}^{\infty} a_2^{\rho-j} V_j, \end{aligned}$$

such that V_j with $7 \leq j \leq \rho$ is a homogeneous polynomial in the variables $a_2, a_3, ..., a_\rho$, (see [4]). The first three terms of $\chi_{\rho-1}^{-\rho}$ are

$$\chi_1^{-2} = -2a_2, \ \chi_2^{-3} = 3(2a_2^2 - a_3), \ \chi_3^{-4} = -4(5a_2^3 - 5a_2a_3 + a_4).$$

In general, for any $p \in \mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$, an expansion of χ_ρ^p is (see for details [3,33] or [4])

$$\chi_\rho^p = pa_{\rho+1} + \frac{p(p-1)}{2} D_\rho^2 + \frac{p!}{(p-3)!3!} D_\rho^3 + ... + \frac{p!}{(p-\rho)!\rho!} D_\rho^\rho,$$

where $D_\rho^p = D_\rho^p(a_2, a_3, ...)$ and by [22] (see for details [2, 14, 16, 20, 23, 31–33, 35])

$$D_\rho^m(a_2, a_3, ..., a_{\rho+1}) = \sum_{\rho=0}^{\infty} \frac{m!(a_2)^{\mu_1} ... (a_{\rho+1})^{\mu_\rho}}{\mu_1! ... \mu_\rho!}, \tag{4}$$

where the sum is taken $\forall \ \mu_1, ..., \mu_\rho \in \mathbb{N} = \{1, 2, ...\}$ satisfying

$$\begin{cases} \mu_1 + \mu_2 + ... + \mu_\rho = m, \\ \mu_1 + 2\mu_2 + ... + k\mu_\rho = \rho. \end{cases}$$

Note that $D_{\rho}^{\rho}(a_2, a_3, ..., a_{\rho+1}) = a_2^{\rho}$.

In the rest of this paper, assume that ϕ is an analytic function with positive real part in $\mathbb{D}$, satisfying $\phi(0) = 1,\ \phi^{'}(0) > 0$ and $\phi(\mathbb{D})$ is symmetric w. r. to the real axis and has the expansion

$$\phi(\varsigma) = 1 + \psi_1\varsigma + \psi_2\varsigma^2 + \psi_3\varsigma^3 + ... \qquad (\psi_1 > 0).$$

Let $u(\varsigma)$ and $v(\omega)$ are analytic in $\mathbb{D}$ with $u(0) = v(0) = 0,\ |u(\varsigma)| < 1,\ |v(\omega)| < 1,$ and

$$u(\varsigma) = \varsigma(p_1 + \sum_{\rho=2}^{\infty} p_\rho \varsigma^{\rho-1}) \ and\ v(\omega) = \omega(q_1 + \sum_{\rho=2}^{\infty} q_\rho \omega^{\rho-1}) \quad (\varsigma, \omega \in \mathbb{D}). \tag{5}$$

Then

$$|p_1| \leq 1,\ |p_\rho| \leq 1 - |p_1|^2,\ |q_1| \leq 1,\ |q_\rho| \leq 1 - |q_1|^2,\ (\rho \geq 2), \tag{6}$$

see ([28]).

The Jackson [21] $q-$derivative, $0 < q < 1$, was defined by (see also [6], [8,9,11], [18], [30]):

$$\nabla_q F(\varsigma) = \begin{cases} \frac{F(\varsigma) - F(q\varsigma)}{(1-q)\varsigma} & ,\varsigma \neq 0 \\ F^{'}(0) & ,\varsigma = 0 \end{cases},$$

that is

$$\nabla_q F(\varsigma) = 1 + \sum_{\rho=2}^{\infty} [\rho]_q a_\rho \varsigma^{\rho-1}, \tag{7}$$

where

$$[j]_q = \frac{1 - q^j}{1 - q},\ [0]_q = 0. \tag{8}$$

As $q \to 1^-$, $[j]_q = j$ and $\nabla_q F(\varsigma) = F^{'}(\varsigma)$.

Now [19,34] defined $q-$ Sălăgean operator by

$$\mathcal{D}_q^0 F(\varsigma) = F(\varsigma)$$

$$\mathcal{D}_q^1 F(\varsigma) = \varsigma\nabla_q(F(\varsigma)) = \varsigma + \sum_{\rho=2}^{\infty} [\rho]_q a_\rho \varsigma^{\rho},$$

$$\mathcal{D}_q^2 F(\varsigma) = \varsigma\nabla_q(\mathcal{D}_q F(\varsigma)),$$

and

$$\begin{aligned} \mathcal{D}_q^n F(\varsigma) &= \varsigma\nabla_q(\mathcal{D}_q^{n-1} F(\varsigma)) \\ &= \varsigma + \sum_{\rho=2}^{\infty} [\rho]_q^n a_\rho \varsigma^{\rho}, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \end{aligned} \tag{9}$$

Note that: Putting $q \to 1^-$in (9) we have the Sălăgean operator $\mathcal{D}^n$ ([29]).

Definition 1. *([12, 25]) For F and g, analytic in $\mathbb{D}$, F is subordinate to g in $\mathbb{D}$ written $F \prec g$, if $\exists\ \Omega(\varsigma)$, analytic in $\mathbb{D}$, with $\Omega(0) = 0$ and $|\Omega(\varsigma)| < 1 \quad (\varsigma \in \mathbb{D})$, such that $F(\varsigma) = g(\Omega(\varsigma)) \quad (\varsigma \in \mathbb{D})$.*

Definition 2. *For* $\tau \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $0 \leq \zeta \leq 1$, $0 < q < 1$, $n \in \mathbb{N}_0$ *and* $F \in \sigma$, $F \in \sum_q^n(\tau, \zeta; \phi)$ *if for all* $\varsigma, \omega \in \mathbb{D}$:

$$1 + \frac{1}{\tau}\left[\nabla_q(\mathcal{D}_q^n F(\varsigma)) + \zeta\varsigma\nabla_q(\nabla_q\mathcal{D}_q^n F(\varsigma)) - 1\right] \prec \phi(\varsigma), \tag{10}$$

and

$$1 + \frac{1}{\tau}\left[\nabla_q(\mathcal{D}_q^n g(\omega)) + \zeta\omega\nabla_q(\nabla_q\mathcal{D}_q^n g(\omega)) - 1\right] \prec \phi(\omega), \tag{11}$$

where $g(\omega) = F^{-1}(\omega)$.

Note that:
(i) $\sum_q^0(\tau, \zeta; \phi) = \sum_q(\tau, \zeta; \phi)$;
(ii) $\sum_q^0(1, \zeta; \phi) = \sum_q(\zeta; \phi)$;
(iii) $\sum_q^n((1-\alpha)e^{-i\theta}\cos\theta, \zeta; \phi) = \sum_q^n(\zeta, \alpha, \theta; \phi)$ $(0 \leq \alpha < 1,\ |\theta| < \frac{\pi}{2})$, where

$$= \begin{cases} F \in \sigma : \frac{e^{i\theta}\left[\nabla_q(\mathcal{D}_q^n F(\varsigma)) + \zeta\varsigma\nabla_q(\nabla_q\mathcal{D}_q^n F(\varsigma))\right] - (\alpha\cos\theta + i\sin\theta)}{(1-\alpha)\cos\theta} \prec \phi(\varsigma) \\ g \in \sigma : \frac{e^{i\theta}\left[\nabla_q(\mathcal{D}_q^n g(\omega)) + \zeta\omega\nabla_q(\nabla_q\mathcal{D}_q^n g(\omega))\right] - (\alpha\cos\theta + i\sin\theta)}{(1-\alpha)\cos\theta} \prec \phi(\omega) \end{cases};$$

(iv) $\lim_{q\to1^-}\sum_q^0(\tau, \zeta; \phi) = \sum(\tau, \zeta; \phi)$ (see [15]);
(v) $\lim_{q\to1^-}\sum_q^0(1, \zeta; \phi) = \sum(\zeta; \phi)$ (see [1]);
(vi) $\lim_{q\to1^-}\sum_q^0((1-\alpha)e^{-i\theta}\cos\theta, \zeta; \frac{1+\varsigma}{1-\varsigma}) = \sum(\zeta, \alpha, \theta; \frac{1+\varsigma}{1-\varsigma})$ $(0 \leq \alpha < 1, |\theta| < \frac{\pi}{2})$, where

$$= \begin{cases} F \in \sigma : \frac{e^{i\theta}\left[F'(\varsigma) + \zeta\varsigma F''(\varsigma)\right] - (\alpha\cos\theta + i\sin\theta)}{(1-\alpha)\cos\theta} \prec \frac{1+\varsigma}{1-\varsigma} \\ g \in \sigma : \frac{e^{i\theta}\left[g'(\omega) + \zeta\omega g''(\omega)\right] - (\alpha\cos\theta + i\sin\theta)}{(1-\alpha)\cos\theta} \prec \frac{1+\omega}{1-\omega} \end{cases}.$$

2. Main Results

We assume that $\tau \in \mathbb{C}^*$, $0 < q < 1$, $0 \leq \zeta \leq 1$, $n \in \mathbb{N}_0$ and $F(\varsigma) \in \sigma$.
In this section we obtain some inequalities for the function class $\sum_q^n(\tau, \zeta; \phi)$.

Theorem 1. *Let* $F \in \sum_q^n(\tau, \zeta; \phi)$. *If* $a_\varepsilon = 0$ *for* $2 \leq \varepsilon \leq \rho - 1$, *then*

$$|a_\rho| \leq \frac{\psi_1 |\tau|}{(1 + \zeta[\rho-1]_q)\,[\rho]_q^{n+1}} \qquad (\rho \geq 3), \tag{12}$$

Proof. For functions $\mathcal{D}_q^n F(\varsigma)$ given by (9) and $g = F^{-1}$, we have

$$\begin{aligned} &1 + \frac{1}{\tau}\left[\nabla_q(\mathcal{D}_q^n F(\varsigma)) + \zeta\varsigma\nabla_q(\nabla_q\mathcal{D}_q^n F(\varsigma)) - 1\right] \\ = \;& 1 + \frac{1}{\tau}\sum_{\rho=2}^{\infty}(1 + \zeta[\rho-1]_q)\,[\rho]_q^{n+1} a_\rho \varsigma^{\rho-1}, \end{aligned} \tag{13}$$

$$1 + \frac{1}{\tau}\left[\nabla_q(\mathcal{D}_q^n g(\omega)) + \zeta\omega\nabla_q(\nabla_q\mathcal{D}_q^n g(\omega)) - 1\right]$$

$$= \quad 1+\frac{1}{\tau}\sum_{\rho=2}^{\infty}\left(1+\zeta[\rho-1]_q\right)[\rho]_q^{n+1}A_\rho\omega^{\rho-1}. \tag{14}$$

Using (3), we have

$$\begin{aligned} & 1+\frac{1}{\tau}\left[\nabla_q(\mathcal{D}_q^n g(\omega))+\zeta\omega\nabla_q(\nabla_q\mathcal{D}_q^n g(\omega))-1\right] \\ = \quad & 1+\frac{1}{\tau}\sum_{\rho=2}^{\infty}\left(1+\zeta[\rho-1]_q\right)[\rho]_q^{n+1}\frac{1}{\rho}\chi_{\rho-1}^{-\rho}(a_2,a_3,...,a_\rho)\omega^{\rho-1}. \end{aligned} \tag{15}$$

Considering (10) and (11), there are two Schwarz functions $u,\ v:\mathbb{D}\to\mathbb{D}$ with $u(0)=v(0)=0$, which are given by (5), so that

$$1+\frac{1}{\tau}\left[\nabla_q(\mathcal{D}_q^n F(\varsigma))+\zeta\varsigma\nabla_q(\nabla_q\mathcal{D}_q^n F(\varsigma))-1\right]=\phi(u(\varsigma)), \tag{16}$$

$$1+\frac{1}{\tau}\left[\nabla_q(\mathcal{D}_q^n g(\omega))+\zeta\omega\nabla_q(\nabla_q\mathcal{D}_q^n g(\omega))-1\right]=\phi(v(\omega)). \tag{17}$$

Also, by (4) we get

$$\begin{aligned} \phi(u(\varsigma)) \quad &= \quad 1+\psi_1p_1\varsigma+(\psi_1p_2+\psi_2p_1^2)\varsigma^2+... \\ &= \quad 1+\sum_{\rho=1}^{\infty}\sum_{\varepsilon=1}^{\rho}\psi_\varepsilon D_\rho^\varepsilon(p_1,p_2,...,p_\rho)\varsigma^\rho \quad (\varsigma\in\mathbb{D}), \end{aligned} \tag{18}$$

and

$$\begin{aligned} \phi(v(\omega)) \quad &= \quad 1+\psi_1q_1\omega+(\psi_1q_2+\psi_2q_1^2)\omega^2+... \\ &= \quad 1+\sum_{\rho=1}^{\infty}\sum_{\varepsilon=1}^{\rho}\psi_\varepsilon D_\rho^\varepsilon(q_1,q_2,...,q_\rho)\omega^\rho \quad (\omega\in\mathbb{D}). \end{aligned} \tag{19}$$

Comparing the coefficients of (13) and (16) with (18), we get

$$\frac{1}{\tau}\left(1+\zeta[\rho-1]_q\right)[\rho]_q^{n+1}a_\rho=\sum_{\varepsilon=1}^{\rho-1}\psi_\varepsilon D_{\rho-1}^\varepsilon(p_1,p_2,...,p_{\rho-1}) \qquad (\rho\geq 2). \tag{20}$$

Similarly, from (15) and (17) with (19), we get

$$\frac{1}{\tau}\left(1+\zeta[\rho-1]_q\right)[\rho]_q^{n+1}\frac{1}{\rho}\chi_{\rho-1}^{-\rho}(a_2,a_3,...,a_\rho)=\sum_{\varepsilon=1}^{\rho-1}\psi_\varepsilon D_{\rho-1}^\varepsilon(q_1,q_2,...,q_{\rho-1}) \quad (\rho\geq 2). \tag{21}$$

Now, from $a_\varepsilon=0$ for $2\leq\varepsilon\leq\rho-1$, we have $A_\rho=-a_\rho$ and the equalities (20) and (21) yield

$$\begin{aligned} \left(1+\zeta[\rho-1]_q\right)[\rho]_q^{n+1}a_\rho \quad &= \quad \tau\psi_1p_{\rho-1}, \\ -\left(1+\zeta[\rho-1]_q\right)[\rho]_q^{n+1}a_\rho \quad &= \quad \tau\psi_1q_{\rho-1}. \end{aligned} \tag{22}$$

Taking the modulus of each of the two equations in (22) and using (6), we obtain (12). $\square$

Corollary 1. *For* $\phi(\varsigma) = (\frac{1+\varsigma}{1-\varsigma})^{\alpha}$ $(0 < \alpha \leq 1)$, *let* $F \in \sum_q^n(\tau, \zeta; \phi)$, *then*

$$|a_\rho| \leq \frac{2\alpha\,|\tau|}{(1+\zeta[\rho-1]_q)\,[\rho]_q^{n+1}} \qquad (\rho \geq 3). \tag{23}$$

Corollary 2. *For* $\phi(\varsigma) = \frac{1+(1-2\beta)\varsigma}{1-\varsigma}$ $(0 \leq \beta < 1)$, *let* $F \in \sum_q^n(\tau, \zeta; \phi)$, *then*

$$|a_\rho| \leq \frac{2\,|\tau|\,(1-\beta)}{(1+\zeta[\rho-1]_q)\,[\rho]_q^{n+1}} \qquad (\rho \geq 3). \tag{24}$$

Remark 1. *For* $\tau = 1$, $n = 0$, $q \to 1^-$ *Corollary 2, reduces to results for [31, Theorem 1], for all* $0 \leq \zeta \leq 1$.

Theorem 2. *Let* $F \in \sum_q^n(\tau, \zeta; \phi)$. *Then*

$$|a_2| \leq \frac{\psi_1\sqrt{\psi_1}\,|\tau|}{\sqrt{\psi_1[2]_q^{2n+2}\,(1+\zeta)^2 + \left|\tau[3]_q^{n+1}\,(1+\zeta[2]_q)\,\psi_1^2 - [2]_q^{2n+2}\,(1+\zeta)^2\,\psi_2\right|}}, \tag{25}$$

$$|a_3| \leq \min\left\{\mathcal{K}(\zeta), \mathcal{L}(\zeta)\right\}, \tag{26}$$

where

$$\mathcal{L}(\zeta) = \begin{cases} \dfrac{\psi_1|\tau|}{[3]_q^{n+1}(1+\zeta[2]_q)} \times \dfrac{[3]_q^{n+1}(1+\zeta[2]_q)|\tau|\psi_1^2 + \left|[3]_q^{n+1}(1+\zeta[2]_q)\tau\psi_1^2 - [2]_q^{2n+2}(1+\zeta)^2\psi_2\right|}{[2]_q^{2n+2}(1+\zeta)^2\psi_1 + \left|[3]_q^{n+1}(1+\zeta[2]_q)\tau\psi_1^2 - [2]_q^{2n+2}(1+\zeta)^2\psi_2\right|} & , \psi_1 \geq \dfrac{[2]_q^{2n+2}(1+\zeta)^2}{[3]_q^{n+1}(1+\zeta[2]_q)|\tau|} \\ \dfrac{\psi_1|\tau|}{[3]_q^{n+1}(1+\zeta[2]_q)}, & 0 \leq \psi_1 \leq \dfrac{[2]_q^{2n+2}(1+\zeta)^2}{[3]_q^{n+1}(1+\zeta[2]_q)|\tau|} \end{cases} \tag{27}$$

and

$$\mathcal{K}(\zeta) = \begin{cases} \dfrac{|\psi_2||\tau|}{[3]_q^{n+1}(1+\zeta[2]_q)} & , |\psi_2| > \psi_1 \\ \dfrac{\psi_1|\tau|}{[3]_q^{n+1}(1+\zeta[2]_q)} & , |\psi_2| \leq \psi_1 \end{cases}. \tag{28}$$

Proof. If we set $\rho = 2$ and $\rho = 3$ in (20) and (21), respectively, we have

$$\frac{1}{\tau}[2]_q^{n+1}\,(1+\zeta)\,a_2 = \psi_1 p_1, \tag{29}$$

$$\frac{1}{\tau}[3]_q^{n+1}\,(1+\zeta[2]_q)\,a_3 = \psi_1 p_2 + \psi_2 p_1^2, \tag{30}$$

$$-\frac{1}{\tau}[2]_q^{n+1}\,(1+\zeta)\,a_2 = \psi_1 q_1, \tag{31}$$

and

$$\frac{1}{\tau}[3]_q^{n+1}\,(1+\zeta[2]_q)\,(2a_2^2 - a_3) = \psi_1 q_2 + \psi_2 q_1^2. \tag{32}$$

From (29) and (31), we obtain

$$p_1 = -q_1. \tag{33}$$

Adding (30) and (32), and using (33), we have

$$\frac{2}{\tau}[3]_q^{n+1}\left(1+\zeta[2]_q\right)a_2^2 - 2p_1^2\psi_2 = \psi_1(p_2+q_2). \tag{34}$$

From (29), we get

$$\left[2\tau[3]_q^{n+1}\psi_1^2\left(1+\zeta[2]_q\right) - 2[2]_q^{2n+2}\left(1+\zeta\right)^2\psi_2\right]a_2^2 = \tau^2\psi_1^3(p_2+q_2). \tag{35}$$

By (6), (29) and (33), we obtain

$$\begin{aligned} &\left|2\tau[3]_q^{n+1}\psi_1^2\left(1+\zeta[2]_q\right) - 2[2]_q^{2n+2}\left(1+\zeta\right)^2\psi_2\right|\left|a_2\right|^2 \\ \leq\ & \left|\tau\right|^2\psi_1^3(\left|p_2\right|+\left|q_2\right|) \\ \leq\ & 2\left|\tau\right|^2\psi_1^3(1-\left|p_1\right|^2) \\ =\ & 2\left|\tau\right|^2\psi_1^3 - 2[2]_q^{2n+2}\left(1+\zeta\right)^2\psi_1\left|a_2\right|^2. \end{aligned} \tag{36}$$

Consequently

$$\left|a_2\right|^2 \leq \frac{\left|\tau\right|^2\psi_1^3}{[2]_q^{2n+2}\left(1+\zeta\right)^2\psi_1 + \left|\tau[3]_q^{n+1}\psi_1^2\left(1+\zeta[2]_q\right) - [2]_q^{2n+2}\left(1+\zeta\right)^2\psi_2\right|}.$$

So we obtain the bound on $|a_2|$ in (25).

Next, in order to find the bound on the coefficient $|a_3|$, by subtracting (32) from (30), and using (33), we get

$$\frac{-2}{\tau}[3]_q^{n+1}\left(1+\zeta[2]_q\right)a_2^2 + \frac{2}{\tau}[3]_q^{n+1}\left(1+\zeta[2]_q\right)a_3 = \psi_1(p_2-q_2). \tag{37}$$

Using (6), we have

$$\begin{aligned} 2[3]_q^{n+1}\left(1+\zeta[2]_q\right)\left|a_3\right| &\leq 2[3]_q^{n+1}\left(1+\zeta[2]_q\right)\left|a_2\right|^2 + \left|\tau\right|\psi_1(\left|p_2\right|+\left|q_2\right|) \\ &\leq 2[3]_q^{n+1}\left(1+\zeta[2]_q\right)\left|a_2\right|^2 + 2\left|\tau\right|\psi_1(1-\left|p_1\right|^2). \end{aligned} \tag{38}$$

From (29), we get

$$\begin{aligned} [3]_q^{n+1}\left(1+\zeta[2]_q\right)\left|\tau\right|\psi_1\left|a_3\right| &\leq \left|\tau\right|^2\psi_1^2 \\ &+ \left[\left|\tau\right|[3]_q^{n+1}\psi_1\left(1+\zeta[2]_q\right) - [2]_q^{2n+2}\left(1+\zeta\right)^2\right]\left|a_2\right|^2. \end{aligned} \tag{39}$$

On the other hand from (30), we have

$$[3]_q^{n+1}\left(1+\zeta[2]_q\right)\left|a_3\right| \leq \left|\tau\right|\left[\psi_1(1-\left|p_1\right|^2) + \left|\psi_2\right|\left|p_1\right|^2\right].$$

Consequently,

$$\left|a_3\right| \leq \begin{cases} \frac{\left|\psi_2\right|\left|\tau\right|}{[3]_q^{n+1}(1+\zeta[2]_q)} & , \left|\psi_2\right| > \psi_1 \\ \frac{\psi_1\left|\tau\right|}{[3]_q^{n+1}(1+\zeta[2]_q)} & , \left|\psi_2\right| \leq \psi_1 \end{cases}. \tag{40}$$

Hence, from (39) and (40), we obtain (26). □

By letting $\tau = 1,\ n = 0$, we have:

Corollary 3. *Let* $F \in \sum_q^0(1, \zeta; \phi)$. *Then*

$$|a_3| \le \min\{\mathcal{K}(\zeta), \mathcal{L}(\zeta)\}, \tag{41}$$

where

$$\mathcal{L}(\zeta) = \begin{cases} \frac{\psi_1}{[3]_q(1+\zeta[2]_q)} \times \frac{[3]_q(1+\zeta[2]_q)\psi_1^2 + \left|[3]_q(1+\zeta[2]_q)\psi_1^2 - [2]_q^2(1+\zeta)^2\psi_2\right|}{[2]_q^2(1+\zeta)^2\psi_1 + \left|[3]_q(1+\zeta[2]_q)\psi_1^2 - [2]_q^2(1+\zeta)^2\psi_2\right|} & , \psi_1 \ge \frac{[2]_q^2(1+\zeta)^2}{[3]_q(1+\zeta[2]_q)} \\ \frac{\psi_1}{[3]_q(1+\zeta[2]_q)} & , 0 \le \psi_1 \le \frac{[2]_q^2(1+\zeta)^2}{[3]_q(1+\zeta[2]_q)} \end{cases}, \tag{42}$$

and

$$\mathcal{K}(\zeta) = \begin{cases} \frac{|\psi_2|}{[3]_q(1+\zeta[2]_q)} & , |\psi_2| > \psi_1 \\ \frac{\psi_1}{[3]_q(1+\zeta[2]_q)} & , |\psi_2| \le \psi_1 \end{cases}. \tag{43}$$

3. Future Work

The authors suggest to find upper bounds for the coefficients of function class $\sum_{\lambda,q}^m(\tau, \zeta; \phi)$ for all $\vartheta, \omega \in \mathbb{D}$:

$$1 + \frac{1}{\tau}\left[\nabla_q(\mathcal{D}_{\lambda,q}^m F(\varsigma)) + \zeta\varsigma\nabla_q(\nabla_q \mathcal{D}_{\lambda,q}^m F(\varsigma)) - 1\right] \prec \phi(\varsigma), \tag{44}$$

and

$$1 + \frac{1}{\tau}\left[\nabla_q(\mathcal{D}_{\lambda,q}^m g(\omega)) + \zeta\omega\nabla_q(\nabla_q \mathcal{D}_{\lambda,q}^m g(\omega)) - 1\right] \prec \phi(\omega), \tag{45}$$

where

$$\nabla_{\lambda,q}^m(F(\varsigma)) = \varsigma + \sum_{k=2}^{\infty}[1 + \lambda([k]_q - 1)]^m a_k \varsigma^k,\ \lambda \ge 0,\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{46}$$

is the $q-$ Al-Oboudi operator is defined by Aouf et al. [10].

4. Conclusions

Throughout the paper, we defined a new subclass of bi-univalent functions of complex order $\sum_q^n(\tau, \zeta; \phi)$ by using $\mathcal{D}_q^n F(\varsigma)$ operator. Furthermore, using the Faber polynomial expansions, we find the initial coefficient bounds for this function class.

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