

Green's Function and Periodic Solutions of a Spring-Mass System in which the Forces are Functionally Dependent on Piecewise Constant Argument

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Abstract: In this paper, damped spring-mass systems with generalized piecewise constant argument and with functional dependence on generalized piecewise constant argument are considered. These spring-mass systems have piecewise constant forces of the forms $Ax(\gamma(t))$ and $Ax(\gamma(t)) + h(t, x_t, x_{\gamma(t)})$, respectively. These spring-mass systems are examined without reducing them into discrete equations. While doing this examination, we make use of the results which have been obtained for differential equations with functional dependence on generalized piecewise constant argument in [1]. Sufficient conditions for the existence and uniqueness of solutions of the spring-mass system with functional dependence on generalized piecewise constant argument are given. The periodic solution of the spring-mass system which has functional force is created with the help of the Green's function, and its uniqueness is proved. The obtained theoretical results are illustrated by an example. This illustration shows that the damped spring-mass systems with functional dependence on generalized piecewise constant argument with proper parameters has a unique periodic solution which can be expressed by Green's function.

Parçalı Sabit Argümana Fonksiyonel Bağımlı Kuvvetli Bir Yay-Kütle Sisteminin Green Fonksiyonu ve Periyodik Çözümleri

Anahtar Kelimeler

Genelleştirilmiş parçalı sabit argümana fonksiyonel bağımlı diferansiyel denklemler, Periyodik çözümler, Green fonksiyonu, Yay-Kütle sistemi

Özet: Bu çalışmada, genelleştirilmiş parçalı sabit argümanlı ve genelleştirilmiş parçalı sabit argümana fonksiyonel bağımlı sönümlü yay-kütle sistemleri düşünülmüştür. Bu yay-kütle sistemleri sırasıyla $Ax(\gamma(t))$ ve $Ax(\gamma(t)) + h(t, x_t, x_{\gamma(t)})$ formlarında parçalı sabit kuvvetlere sahiptirler. Bu yay-kütle sistemleri ayrık denklemlere dönüştürülmeden incelenmiştir. Bu inceleme yapılırken, genelleştirilmiş parçalı sabit argümana fonksiyonel bağımlı diferansiyel denklemler için elde edilen sonuçlardan [1] faydalanılmıştır. Genelleştirilmiş parçalı sabit argümana fonksiyonel bağımlı yay-kütle sisteminin çözümlerinin varlık ve teklifi için yeterli koşullar verilmiştir. Green fonksiyonu yardımıyla fonksiyonel kuvvetli yay-kütle sisteminin periyodik çözümü oluşturulmuştur ve teklifi ispatlanmıştır. Elde edilen teorik sonuçlar örneklendirilmiştir. Bu örnek, belli parametreler için genelleştirilmiş parçalı sabit argümana fonksiyonel bağımlı sönümlü yay-kütle sisteminin Green fonksiyonu ile ifade edilebilen tek bir periyodik çözüme sahip olduğunu göstermiştir.

1. Introduction and Preliminaries

Delay differential equations play an important role to model real world problems, and scientists [2–5] have done a lot of work on these types of equations. Differential equations with piecewise constant argument which are in the class of delay differential equations, have deviating arguments of retarded and/or advanced type [1, 6–9]. Recently, numerous problems mainly related to the existence of periodic and almost periodic solutions [8, 10–13], oscillatory behavior of solutions [10, 14–18], global attractivity of the trivial solution [19] have been investigated for differential equations with piecewise

constant argument. This type of differential equations were initiated by Cooke and Wiener [9]. Furthermore, Cooke and Wiener [20] gave a survey paper concerning theorems for existence, uniqueness, stability, and results on existence and oscillation of periodic solutions. In addition to these developments, Akhmet [21–23] introduced differential equations with piecewise constant argument of generalized type, and these equations are studied by many authors [1, 7, 24–29]. Later, he defined a new class of differential equations: retarded differential equations with functional dependence on piecewise constant argument [1]. In [1], conditions for the existence and uniqueness

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of periodic solutions using Green's function, existence and uniqueness of almost periodic solutions, exponential stability of solutions and periodic solutions, boundedness of the solutions were obtained under certain assumptions. Besides, modeling by functional differential equations provide more detailed analysis for real life problems since they represent dependency on system's past and future situations of its current time. So, functional differential equations have a great importance, and much investigation has been carried out on the solutions and on the existence of periodic solutions for various types of them (see, e.g., [30–33] and references therein).

The general form of a damped spring-mass equation whose main expression come from Newton's second law and Hooke's law for a spring is

$$mx'' + cx' + kx = 0, \tag{1}$$

which is called a damped harmonic oscillator, as well. Here, $m > 0$ is the mass, $c > 0$ is the damping coefficient, $k > 0$ is the spring constant and $x(t)$ is the displacement of the mass. The cases $\Delta > 0$, $\Delta = 0$, $\Delta < 0$ for the discriminant

$$\Delta = \frac{c^2}{m^2} - 4\frac{k}{m}$$

of the equation

$$s^2 + \frac{c}{m}s + \frac{k}{m} = 0$$

of the spring-mass system (1) show that the system (1) exhibits motion with over damped, critical damped and under damped, respectively. The spring-mass system (1) can include any external force H , in other words, the system (1) can typically be written as nonhomogeneous differential equation

$$mx'' + cx' + kx = H, \tag{2}$$

which is referred as a forced harmonic oscillator. Spring-mass systems have been widely used by many scientists in the fields such as physics [34–37], mathematics [14, 38], biomechanics [39, 40], electrical and computer engineering [41], biology [42]. Dai and Singh [14] studied the oscillation problem for the damped spring-mass equation (2) taking the piecewise constant force $Ax([t])$ instead of the external force H . In the piecewise constant force $Ax([t])$, A specifies the magnitude of the force.

Let \mathbb{Z} and \mathbb{R} be the sets of all integers and real numbers, respectively. $\|\cdot\|$ signs to the Euclidean norm in \mathbb{R}^2 . Fix two real valued sequences $\theta_i, \zeta_i, i \in \mathbb{Z}$ such that $\theta_i < \theta_{i+1}$ and $\theta_i \leq \zeta_i \leq \theta_{i+1}$ for all $i \in \mathbb{Z}$ with $|\theta_i| \rightarrow \infty$ as $|i| \rightarrow \infty$. Describe by $\mathcal{C} = C([-\tau, 0], \mathbb{R})$ and $\mathcal{H} = C([-\tau, 0], \mathbb{R}^2)$ the sets of all continuous functions mapping the interval $[-\tau, 0]$ into \mathbb{R} and \mathbb{R}^2 with the uniform norm $\|\phi\|_0 = \max_{[-\tau, 0]} \|\phi\|$ for fixed number $0 < \tau \in \mathbb{R}$, respectively.

In this paper, we are interested in the following damped spring-mass systems with generalized piecewise constant argument

$$mx''(t) + cx'(t) + kx(t) = Ax(\gamma(t)), \tag{3}$$

and with functional dependence on generalized piecewise constant argument

$$mx''(t) + cx'(t) + kx(t) = Ax(\gamma(t)) + h(t, x_t, x_{\gamma(t)}), \tag{4}$$

where $x \in \mathbb{R}, t \in \mathbb{R}$, and $\gamma(t) = \zeta_i$ if $t \in [\theta_i, \theta_{i+1}), i \in \mathbb{Z}$. m, c, k and A remark the mass, the coefficient of damping, the spring constant and the magnitude of the force, respectively. Let \mathcal{D} be a subset of the product $\mathbb{R} \times \mathcal{C} \times \mathcal{C}$ and $h : \mathcal{D} \rightarrow \mathbb{R}$ denote a continuous functional force in system (4). Let $\mathcal{C}_s = \{\phi \in \mathcal{C} \mid \|\phi\|_0 \leq s\}$ where $0 < s \in \mathbb{R}$, and $C_0(W)$ be the set of all bounded and continuous functions on W . Here, $h \in C_0(\mathbb{R} \times \mathcal{C}_V \times \mathcal{C}_V)$ for each $0 < V \in \mathbb{R}$. In system (4), x_t and $x_{\gamma(t)}$ mean that $x_t(s) = x(t+s)$ and $x_{\gamma(t)}(s) = x(\gamma(t)+s)$ for $s \in [-\tau, 0]$.

With $z_1 = x, z_2 = x'$, the damped spring-mass systems (3) and (4) can be reduced to the first-order differential equations as follows:

$$z'(t) = Bz(t) + Cz(\gamma(t)) \tag{5}$$

and

$$z'(t) = Bz(t) + Cz(\gamma(t)) + f(t, z_t, z_{\gamma(t)}), \tag{6}$$

where the matrices

$$B = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & 0 \\ \frac{A}{m} & 0 \end{bmatrix}$$

depend on the parameters of the spring-mass systems (3) and (4). Here, continuous functional force $f : \mathcal{F} \rightarrow \mathbb{R}^2$ for a subset $\mathcal{F} = \mathbb{R} \times \mathcal{H} \times \mathcal{H}$ is given by

$$f(t, z_t, z_{\gamma(t)}) = \begin{bmatrix} 0 \\ \frac{1}{m}h(t, z_t, z_{\gamma(t)}) \end{bmatrix}. \text{ The spring-mass}$$

system (5) is a linear homogeneous system with argument-function $\gamma(t)$, and the system (6) is a quasilinear system with functional dependence on argument-function $\gamma(t)$.

The aim of the present paper is to examine damped spring-mass systems with generalized piecewise constant argument and with functional dependence on generalized piecewise constant argument without transforming them into discrete equations, assuming the systems exhibit harmonic motion with under damped. The fundamental matrix of the homogeneous spring-mass system (5) is constructed in several intervals for illustration. Sufficient conditions for the existence and uniqueness of solutions of (6) are found. Existence of periodic solutions of the system (6) is investigated by using Green's function which have been obtained for differential equations with functional dependence on piecewise constant argument of generalized type [1]. Then, we prove the uniqueness of the periodic solution.

2. Material and Method

In this section, we obtain the fundamental matrix of the linear homogeneous equation without piecewise constant argument, i.e. the system $x' = Bx$. Then, we create the matrix-function and the fundamental matrix of the linear homogeneous equation (5) with piecewise constant argument in several intervals using the construction of the fundamental matrix of the differential equations with generalized piecewise constant argument [1]. Additionally, we give the assumptions needed for our study. We state the initial conditions depending on two cases of the initial value t_0 .

2.1. The fundamental matrix of the linear homogeneous equation without piecewise constant argument

Consider the system

$$x'(t) = Bx(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} x(t). \quad (7)$$

Equation given by (7) is the linear homogenous part of (5) and (6) without piecewise constant argument. Let $X(t, s)$ denote the fundamental matrix of solutions of (7) satisfying $X(s, s) = I$, $s \in \mathbb{R}$, where I is the 2×2 identity matrix. Since it is assumed that systems (5) and (6) exhibit motion with under damped, $X(t, s)$ for (7) is in the following form

$$\begin{aligned} X(t, s) &= e^{B(t-s)} \\ &= e^{-\alpha(t-s)} \begin{bmatrix} X_{11}(t, s) & X_{12}(t, s) \\ X_{21}(t, s) & X_{22}(t, s) \end{bmatrix}, \end{aligned}$$

where its indices are given by

$$X_{11}(t, s) = \cos(\beta(t-s)) + \frac{\alpha}{\beta} \sin(\beta(t-s)),$$

$$X_{12}(t, s) = \frac{1}{\beta} \sin(\beta(t-s)),$$

$$X_{21}(t, s) = -\frac{k}{m\beta} \sin(\beta(t-s)),$$

$$X_{22}(t, s) = \cos(\beta(t-s)) - \frac{\alpha}{\beta} \sin(\beta(t-s)),$$

and

$$\alpha = \frac{c}{2m}, \quad \beta = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} = \frac{\sqrt{-\Delta}}{2}.$$

We see that the fundamental matrix $X(t, s)$ has elements depending on the model we consider.

2.2. The matrix-function and fundamental matrix of the homogeneous spring-mass system (5)

In [1], a matrix-function $M_i(t)$, $i \in \mathbb{Z}$, is introduced as follows

$$M_i(t) = X(t, \zeta_i) + \int_{\zeta_i}^t X(t, s)B(s)ds$$

for the systems

$$z'(t) = B(t)z(t) + C(t)z(\gamma(t))$$

and

$$z'(t) = B(t)z(t) + C(t)z(\gamma(t)) + f(t, z_t, z_{\gamma(t)}).$$

The matrix-function is important for the investigation of existence and uniqueness of periodic solutions. We find the matrix-function $M_i(t)$ for the linear homogeneous system (5) with piecewise constant argument as

$$M_i(t) = e^{-\alpha(t-\zeta_i)} \begin{bmatrix} K_i & L_i \\ M_i & N_i \end{bmatrix}$$

and its indices K_i, L_i, M_i, N_i are in the following form:

$$K_i = \left(1 - \frac{A}{k}\right) \left(\cos(\beta(t-\zeta_i)) + \frac{\alpha}{\beta} \sin(\beta(t-\zeta_i))\right) + \frac{Ae^{\alpha(t-\zeta_i)}}{k},$$

$$L_i = \frac{1}{\beta} \sin(\beta(t-\zeta_i)),$$

$$M_i = -\frac{k}{m\beta} \left(1 - \frac{A}{k}\right) \sin(\beta(t-\zeta_i))$$

and

$$N_i = \cos(\beta(t-\zeta_i)) - \frac{\alpha}{\beta} \sin(\beta(t-\zeta_i)).$$

Let us fix $t_0 \in \mathbb{R}$ and assume without loss of generality that $\theta_i < t_0 < \zeta_i$, $i \in \mathbb{Z}$. $Z(t) = Z(t, t_0)$ with $Z(t_0) = Z(t_0, t_0) = I$ is called a fundamental matrix of the system (5). Let $\theta_i \leq t_0 < \theta_{i+1}$ for a fixed $i \in \mathbb{Z}$. For interval $t \in [t_0, \theta_{i+1}]$, the fundamental matrix is given by

$$Z(t) = M_i(t)M_i^{-1}(t_0).$$

The fundamental matrix of (5) is defined for increasing t and decreasing t as expressed in [1, 6]. In other words, if $\theta_i \leq t_0 < \theta_{i+1}$, $t \in [\theta_l, \theta_{l+1}]$, $l > i$, then

$$Z(t) = M_l(t) \left[\prod_{k=l}^{i+1} M_k^{-1}(\theta_k) M_{k-1}(\theta_k) \right] M_i^{-1}(t_0).$$

If $\theta_i \leq t_0 \leq \theta_{i+1}$, $t \in [\theta_j, \theta_{j+1}]$, $j < i$, then

$$Z(t) = M_j(t) \left[\prod_{k=j}^{i-1} M_k^{-1}(\theta_{k+1}) M_{k+1}(\theta_{k+1}) \right] M_i^{-1}(t_0).$$

In this context, we create the fundamental matrix of (5) in three intervals $t \in [\theta_i, \theta_{i+1}]$, $t \in [\theta_{i-1}, \theta_i]$ for decreasing t and $t \in [\theta_{i+1}, \theta_{i+2}]$ for increasing t for illustration. It is possible to obtain the fundamental matrix for more intervals.

First, we obtain the fundamental matrix of (5) in the following form

$$Z(t) = M_i(t)M_i^{-1}(t_0) = \frac{e^{-\alpha(t-t_0)}}{s_{i_1}} \begin{bmatrix} K_{i_1} & L_{i_1} \\ M_{i_1} & N_{i_1} \end{bmatrix}$$

for $t \in [\theta_i, \theta_{i+1}]$ with $\theta_i \leq \zeta_i \leq \theta_{i+1}$, $i \in \mathbb{Z}$. This fundamental matrix has indices as follows

$$K_{i_1} = \left(1 - \frac{A}{k}\right) \left(\cos(\beta(t-t_0)) + \frac{\alpha}{\beta} \sin(\beta(t-t_0))\right) + \frac{Ae^{\alpha(t-\zeta_i)}}{k} \left(\cos(\beta(t_0-\zeta_i)) - \frac{\alpha}{\beta} \sin(\beta(t_0-\zeta_i))\right),$$

$$L_{i_1} = \frac{1}{\beta} \left(1 - \frac{A}{k}\right) \sin(\beta(t-t_0)) + \frac{Ae^{-\alpha\zeta_i}}{\beta k} \left(e^{\alpha t_0} \sin(\beta(t-\zeta_i)) - e^{\alpha t} \sin(\beta(t_0-\zeta_i))\right),$$

$$M_{i_1} = -\frac{k}{m\beta} \left(1 - \frac{A}{k}\right) \sin(\beta(t-t_0)),$$

$$N_{i_1} = \left(1 - \frac{A}{k}\right) \left(\cos(\beta(t-t_0)) - \frac{\alpha}{\beta} \sin(\beta(t-t_0))\right) + \frac{Ae^{\alpha(t_0-\zeta_i)}}{k} \left(\cos(\beta(t-\zeta_i)) - \frac{\alpha}{\beta} \sin(\beta(t-\zeta_i))\right),$$

$$s_{i_1} = 1 - \frac{A}{k} + \frac{Ae^{\alpha(t_0-\zeta_i)}}{k} \left(\cos(\beta(t_0-\zeta_i)) - \frac{\alpha}{\beta} \sin(\beta(t_0-\zeta_i))\right).$$

Moreover, for $t \in [\theta_{i-1}, \theta_i]$ where t is decreasing, with $\theta_{i-1} \leq \zeta_{i-1} \leq \theta_i$, $i \in \mathbb{Z}$, the fundamental matrix of (5) is in the form

$$Z(t) = M_{i-1}(t)M_{i-1}^{-1}(\theta_i)M_i(\theta_i)M_i^{-1}(t_0) = \frac{e^{-\alpha(t-t_0)}}{s_{i_1}s_{i_2}} \begin{bmatrix} K_{i_2} & L_{i_2} \\ M_{i_2} & N_{i_2} \end{bmatrix},$$

where indices are listed below;

$$K_{i_2} = \left(1 - \frac{A}{k}\right)^2 \left(\cos(\beta(t-t_0)) + \frac{\alpha}{\beta} \sin(\beta(t-t_0))\right) + \frac{A}{k} \left(1 - \frac{A}{k}\right) \left(e^{\alpha(t-\zeta_{i-1})} \cos(\beta(\zeta_{i-1}-t_0)) + e^{\alpha(t-\zeta_{i-1})} \frac{\alpha}{\beta} \sin(\beta(\zeta_{i-1}-t_0))\right) + e^{\alpha(\theta_i-\zeta_i)} \cos(\beta(t-t_0-\theta_i+\zeta_i)) + e^{\alpha(\theta_i-\zeta_i)} \frac{\alpha}{\beta} \sin(\beta(t-t_0-\theta_i+\zeta_i)) - \left(\frac{k}{m\beta^2}\right) \sin(\beta(t-\theta_i)) \sin(\beta(t_0-\zeta_i)) + \frac{A}{m\beta^2} \left(1 - \frac{A}{k}\right) e^{\alpha(\theta_i-\zeta_{i-1})} \sin(\beta(t-\zeta_{i-1})) \sin(\beta(t_0-\theta_i)) + \frac{A^2}{k^2} e^{\alpha(t-\zeta_i+\theta_i-\zeta_{i-1})} \left(\cos(\beta(t_0+\theta_i-\zeta_i-\zeta_{i-1})) - \frac{\alpha}{\beta} \sin(\beta(t_0+\theta_i-\zeta_i-\zeta_{i-1}))\right) + \frac{k}{m\beta^2} \sin(\beta(\theta_i-\zeta_{i-1})) \sin(\beta(t_0-\zeta_i)),$$

$$L_{i_2} = \frac{1}{\beta} \left(1 - \frac{A}{k}\right)^2 \sin(\beta(t-t_0)) + \frac{A}{k\beta} \left(1 - \frac{A}{k}\right) \left(e^{\alpha(t-\zeta_{i-1})} \sin(\beta(\zeta_{i-1}-t_0)) + e^{\alpha(t_0-\zeta_i)} \sin(\beta(t-\zeta_i)) - e^{\alpha(\theta_i-\zeta_i)} \sin(\beta(t_0-\zeta_i)) \cos(\beta(t-\theta_i))\right) + \frac{\alpha}{\beta} \sin(\beta(t-\theta_i))$$

$$+ e^{\alpha(\theta_i-\zeta_{i-1})} \sin(\beta(t-\zeta_{i-1})) \cos(\beta(t_0-\theta_i)) + e^{\alpha(\theta_i-\zeta_{i-1})} \sin(\beta(t-\zeta_{i-1})) \frac{\alpha}{\beta} \sin(\beta(t_0-\theta_i)) + \frac{A^2}{k^2\beta} \left(-e^{\alpha(t-\zeta_{i-1}+\theta_i-\zeta_i)} \sin(\beta(t_0-\zeta_i)) \cdot (\cos(\beta(\theta_i-\zeta_{i-1})) - \frac{\alpha}{\beta} \sin(\beta(\theta_i-\zeta_{i-1})))\right) + e^{\alpha(t-\zeta_{i-1}+t_0-\zeta_i)} \sin(\beta(\zeta_{i-1}-\zeta_i)) + e^{\alpha(t_0-\zeta_{i-1}+\theta_i-\zeta_i)} \sin(\beta(t-\zeta_{i-1})) \cos(\beta(\theta_i-\zeta_i)) - \frac{\alpha}{\beta} \sin(\beta(\theta_i-\zeta_i)),$$

$$M_{i_2} = -\frac{k}{m\beta} \left(1 - \frac{A}{k}\right)^2 \sin(\beta(t-t_0)) + \frac{A}{m\beta} \left(1 - \frac{A}{k}\right) \left(\cos(\beta(t_0-\zeta_i)) - \frac{\alpha}{\beta} \sin(\beta(t_0-\zeta_i))\right) \cdot (-1)e^{\alpha(\theta_i-\zeta_i)} \sin(\beta(t-\theta_i)) - e^{\alpha(\theta_i-\zeta_{i-1})} \sin(\beta(\theta_i-\zeta_i)) \cos(\beta(t-\zeta_{i-1}+t_0-\zeta_i)) - \frac{\alpha}{\beta} \sin(\beta(t-\zeta_{i-1}+t_0-\zeta_i)) + e^{\alpha(\theta_i-\zeta_{i-1})} \sin(\beta(t_0-\zeta_i)) \cdot (\cos(\beta(t-\zeta_{i-1}+\theta_i-\zeta_i)) - \frac{\alpha}{\beta} \sin(\beta(t-\zeta_{i-1}+\theta_i-\zeta_i))),$$

$$N_{i_2} = \left(1 - \frac{A}{k}\right)^2 \left(\cos(\beta(t-t_0)) - \frac{\alpha}{\beta} \sin(\beta(t-t_0))\right) + \frac{A}{k} \left(1 - \frac{A}{k}\right) \left(e^{\alpha(t_0-\zeta_i)} \cos(\beta(t-\zeta_i)) - \frac{\alpha}{\beta} \sin(\beta(t-\zeta_i))\right) + (e^{\alpha(\theta_i-\zeta_i)} \sin(\beta(t_0-\zeta_i)) \sin(\beta(t-\theta_i)) \frac{k}{m\beta^2}) + e^{\alpha(\theta_i-\zeta_{i-1})} \cos(\beta(t-t_0+\theta_i-\zeta_{i-1})) - \frac{\alpha}{\beta} \sin(\beta(t-t_0+\theta_i-\zeta_{i-1})) - e^{\alpha(\theta_i-\zeta_{i-1})} \sin(\beta(t-\zeta_{i-1})) \cos(\beta(t_0-\theta_i)) \frac{k}{m\beta^2}) + \frac{A^2}{k^2} e^{\alpha(\theta_i-\zeta_{i-1}+t_0-\zeta_i)} \left(\cos(\beta(t-\zeta_{i-1}+\theta_i-\zeta_i)) - \frac{\alpha}{\beta} \sin(\beta(t-\zeta_{i-1}+\theta_i-\zeta_i))\right) + \sin(\beta(t-\zeta_{i-1})) \sin(\beta(\theta_i-\zeta_i)) \frac{k}{m\beta^2},$$

$$s_{i_1} = 1 - \frac{A}{k} + \frac{Ae^{\alpha(t_0 - \zeta_i)}}{k} \left(\cos(\beta(t_0 - \zeta_i)) - \frac{\alpha}{\beta} \sin(\beta(t_0 - \zeta_i)) \right),$$

$$s_{i_2} = 1 - \frac{A}{k} + \frac{A}{k} \left(1 - e^{\alpha(\theta_i - \zeta_{i-1})} (\cos(\beta(\theta_i - \zeta_{i-1})) - \frac{\alpha}{\beta} \sin(\beta(\theta_i - \zeta_{i-1}))) \right).$$

Finally, in $t \in [\theta_{i+1}, \theta_{i+2}]$, $\theta_{i+1} \leq \zeta_{i+1} \leq \theta_{i+2}$, $i \in \mathbb{Z}$, fundamental matrix of (5) for increasing value of t is

$$Z(t) = M_{i+1}(t)M_{i+1}^{-1}(\theta_{i+1})M_i(\theta_{i+1})M_i^{-1}(t_0)$$

$$= \frac{e^{-\alpha(t-t_0)}}{s_{i_1}s_{i_3}} \begin{bmatrix} K_{i_3} & L_{i_3} \\ M_{i_3} & N_{i_3} \end{bmatrix},$$

where indices are given by

$$K_{i_3} = \left(1 - \frac{A}{k} \right)^2 \left(\cos(\beta(t-t_0)) + \frac{\alpha}{\beta} \sin(\beta(t-t_0)) \right)$$

$$+ \frac{A}{k} \left(1 - \frac{A}{k} \right) \left(e^{\alpha(t-\zeta_{i+1})} (\cos(\beta(t_0 - \zeta_{i+1})) - \frac{\alpha}{\beta} \sin(\beta(t_0 - \zeta_{i+1}))) \right)$$

$$+ e^{\alpha(\theta_{i+1} - \zeta_i)} (\cos(\beta(t-t_0 - \theta_{i+1} + \zeta_i)) - \frac{\alpha}{\beta} \sin(\beta(t-t_0 - \theta_{i+1} + \zeta_i)))$$

$$- \frac{k}{m\beta^2} \sin(\beta(t - \theta_{i+1})) \sin(\beta(t_0 - \zeta_i))$$

$$+ \frac{A}{m\beta^2} \left(1 - \frac{A}{k} \right) e^{\alpha(\theta_{i+1} - \zeta_{i+1})} \sin(\beta(t - \zeta_{i+1})) \cdot \sin(\beta(t_0 - \theta_{i+1}))$$

$$+ \frac{A^2}{k^2} e^{\alpha(t-\zeta_i + \theta_{i+1} - \zeta_{i+1})} \left(\cos(\beta(t_0 + \theta_{i+1} - \zeta_i - \zeta_{i+1})) - \frac{\alpha}{\beta} \sin(\beta(t_0 + \theta_{i+1} - \zeta_i - \zeta_{i+1})) \right)$$

$$+ \sin(\beta(\theta_{i+1} - \zeta_{i+1})) \sin(\beta(t_0 - \zeta_i)) \frac{k}{m\beta^2},$$

$$L_{i_3} = \frac{1}{\beta} \left(1 - \frac{A}{k} \right)^2 \sin(\beta(t-t_0))$$

$$+ \frac{A}{k\beta} \left(1 - \frac{A}{k} \right) \left(-e^{\alpha(t-\zeta_{i+1})} \sin(\beta(t_0 - \zeta_{i+1})) \right)$$

$$+ e^{\alpha(\theta_{i+1} - \zeta_{i+1})} \sin(\beta(t - \zeta_{i+1})) (\cos(\beta(t_0 - \theta_{i+1})) - \frac{\alpha}{\beta} \sin(\beta(t_0 - \theta_{i+1})))$$

$$- e^{\alpha(\theta_{i+1} - \zeta_i)} \sin(\beta(t_0 - \zeta_i)) (\cos(\beta(t - \theta_{i+1})) - \frac{\alpha}{\beta} \sin(\beta(t - \theta_{i+1})))$$

$$+ e^{\alpha(t_0 - \zeta_i)} \sin(\beta(t - \zeta_i))$$

$$+ \frac{A^2}{k^2\beta} \left(e^{\alpha(t_0 - \zeta_{i+1} + \theta_{i+1} - \zeta_i)} \sin(\beta(t - \zeta_{i+1})) \cdot (\cos(\beta(\theta_{i+1} - \zeta_i)) - \frac{\alpha}{\beta} \sin(\beta(\theta_{i+1} - \zeta_i))) \right)$$

$$- e^{\alpha(t - \zeta_{i+1} + t_0 - \zeta_i)} \sin(\beta(\zeta_i - \zeta_{i+1}))$$

$$- e^{\alpha(t - \zeta_{i+1} + \theta_{i+1} - \zeta_i)} \sin(\beta(t_0 - \zeta_i)) (\cos(\beta(\theta_{i+1} - \zeta_{i+1})) - \frac{\alpha}{\beta} \sin(\beta(\theta_{i+1} - \zeta_{i+1})))$$

$$- \frac{\alpha}{\beta} \sin(\beta(\theta_{i+1} - \zeta_{i+1}))),$$

$$M_{i_3} = -\frac{k}{m\beta} \left(1 - \frac{A}{k} \right) \sin(\beta(t-t_0))$$

$$+ \frac{A}{m\beta} \left(1 - \frac{A}{k} \right) \left(-e^{\alpha(\theta_{i+1} - \zeta_i)} \sin(\beta(t - \theta_{i+1})) (\cos(\beta(t_0 - \zeta_i)) - \frac{\alpha}{\beta} \sin(\beta(t_0 - \zeta_i))) \right)$$

$$+ e^{\alpha(\theta_{i+1} - \zeta_{i+1})} \sin(\beta(t_0 - \theta_{i+1})) (\cos(\beta(t - \zeta_{i+1})) - \frac{\alpha}{\beta} \sin(\beta(t - \zeta_{i+1})))$$

$$- \frac{\alpha}{\beta} \sin(\beta(t - \zeta_{i+1}))),$$

$$N_{i_3} = \left(1 - \frac{A}{k} \right)^2 \left(\cos(\beta(t-t_0)) - \frac{\alpha}{\beta} \sin(\beta(t-t_0)) \right)$$

$$+ \frac{A}{k} \left(1 - \frac{A}{k} \right) \left(e^{\alpha(t_0 - \zeta_i)} (\cos(\beta(t - \zeta_i)) - \frac{\alpha}{\beta} \sin(\beta(t - \zeta_i))) \right)$$

$$+ (e^{\alpha(\theta_{i+1} - \zeta_i)} \sin(\beta(t_0 - \zeta_i)) \sin(\beta(t - \theta_{i+1})) \frac{k}{m\beta^2})$$

$$+ e^{\alpha(\theta_{i+1} - \zeta_{i+1})} (\cos(\beta(t-t_0 + \theta_{i+1} - \zeta_{i+1})) - \frac{\alpha}{\beta} \sin(\beta(t-t_0 + \theta_{i+1} - \zeta_{i+1})))$$

$$- (\sin(\beta(t - \zeta_{i+1})) \sin(\beta(t_0 - \theta_{i+1})) \frac{k}{m\beta^2})$$

$$+ \frac{A^2}{k^2} e^{\alpha(\theta_{i+1} - \zeta_{i+1} + t_0 - \zeta_i)} \left(\cos(\beta(t - \zeta_{i+1} + \theta_{i+1} - \zeta_i)) - \frac{\alpha}{\beta} \sin(\beta(t - \zeta_{i+1} + \theta_{i+1} - \zeta_i)) \right)$$

$$+ \sin(\beta(t - \zeta_{i+1})) \sin(\beta(\theta_{i+1} - \zeta_i)) \frac{k}{m\beta^2},$$

$$s_{i_1} = 1 - \frac{A}{k} + \frac{Ae^{\alpha(t_0 - \zeta_i)}}{k} \left(\cos(\beta(t_0 - \zeta_i)) - \frac{\alpha}{\beta} \sin(\beta(t_0 - \zeta_i)) \right)$$

and

$$s_{i_3} = 1 - \frac{A}{k} + \frac{Ae^{\alpha(\theta_{i+1} - \zeta_{i+1})}}{k} \left(\cos(\beta(\theta_{i+1} - \zeta_{i+1})) - \frac{\alpha}{\beta} \sin(\beta(\theta_{i+1} - \zeta_{i+1})) \right).$$

The fundamental matrix can be obtained as shown above for any other intervals.

Besides, it can be shown that $Z(t,s) = Z(t)Z^{-1}(s)$, $t, s \in \mathbb{R}$, and a solution $z(t)$, $z(t_0) = z_0$, $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^2$, of (6) is equal to $z(t) = Z(t, t_0)z_0$, $t \in \mathbb{R}$.

2.3. Assumptions

For the damped spring-mass systems (5) and (6), we shall need the following assumptions throughout the paper:

(S1) h satisfies the Lipschitz condition for some constant $L > 0$:

$$|h(t, \mu_1, \eta_1) - h(t, \mu_2, \eta_2)| \leq L(|\mu_1 - \mu_2| + |\eta_1 - \eta_2|),$$

where (t, μ_1, η_1) and $(t, \mu_2, \eta_2) \in \mathcal{D}$;

(S2) there exist positive numbers $\bar{\theta}, \bar{\zeta} > 0$ such that $\theta_{i+1} - \theta_i \leq \bar{\theta}, \zeta_{i+1} - \zeta_i \leq \bar{\zeta}, i \in \mathbb{Z}$;

(S3) $\left(1 - e^{\alpha(t-\zeta_i)} \left(\cos(\beta(t-\zeta_i)) - \frac{\alpha}{\beta} \sin(\beta(t-\zeta_i))\right)\right) \neq \frac{k}{A}, \forall t \in [\theta_i, \theta_{i+1}], i \in \mathbb{Z}$;

(S2) and (S3) imply the existence of constants $0 < \bar{m}, 0 < \bar{M}, 0 < M$ such that $\bar{m} \leq \|Z(t, s)\| \leq M, \|X(t, s)\| \leq \bar{M}$ for $t, s \in [\theta_i, \theta_{i+1}], i \in \mathbb{Z}$ [1].

(S4) $2\bar{M}L(1+M)\bar{\theta} < 1$.

Moreover, assume that system (6) is ω -periodic with the following conditions:

(S5) there are two numbers $\omega \in \mathbb{R}$ and $p \in \mathbb{Z}$ such that $\theta_{k+p} = \theta_k + \omega$ and $\zeta_{k+p} = \zeta_k + \omega, k \in \mathbb{Z}$;

(S6) $h(t + \omega, \mu, \eta) = h(t, \mu, \eta), t \in \mathbb{R}, \mu, \eta \in \mathcal{C}$.

We can infer from (S1) and (S6) that the functional force

$f(t, z_t, z_{\gamma(t)}) = \begin{bmatrix} 0 \\ \frac{1}{m}h(t, z_t, z_{\gamma(t)}) \end{bmatrix}$ also satisfies the Lipschitz condition and periodicity condition.

In addition to the above conditions, we can define the initial conditions for the damped spring-mass system (6) with the functional force for the cases corresponding to $t_0 \leq \zeta_i$ or $\zeta_i < t_0$ for $t \geq t_0$. It is clear that $\gamma(t_0) \geq t_0$ and $\gamma(t_0) < t_0$ if $\theta_i \leq t_0 \leq \zeta_i < \theta_{i+1}$ and $\theta_i \leq \zeta_i < t_0 < \theta_{i+1}$, respectively. For a fixed number $t_0 \in \mathbb{R}$, the functions $\mu, \eta \in \mathcal{K}$ and some $i \in \mathbb{Z}$, if $\theta_i \leq t_0 < \theta_{i+1}$:

(K1) there exists a solution $z(t) = z(t, t_0, \mu)$ satisfying the initial condition $z_{t_0}(s) = \mu(s), \mu \in \mathcal{K}, s \in [-\tau, 0]$ if $\gamma(t_0) \geq t_0$;

(K2) there exists a solution $z(t) = z(t, t_0, \mu, \eta)$ satisfying the initial conditions $z_{t_0}(s) = \mu(s)$ and $z_{\gamma(t_0)}(s) = \eta(s)$ with $\mu, \eta \in \mathcal{K}, s \in [-\tau, 0]$ if $\gamma(t_0) < t_0$.

Definition 2.1. [1] A function $z(t)$ is a solution of (6) with (K1) or (K2) on an interval $[t_0, t_0 + a), a > 0$, if:

- (i) it satisfies the initial condition,
- (ii) $z(t)$ is continuous on $[t_0, t_0 + a)$,
- (iii) the derivative $z'(t)$ exists for $t \geq t_0$ with the possible exception of the points θ_i , where one-sided derivatives exists,
- (iv) equation (6) is satisfied by $z(t)$ for all $t > t_0$, except, possibly, the points of θ and it holds for the right derivative of $z(t)$ at points θ_i .

Definition 2.2. [1] A function $z(t)$ is a solution of (6)((5)) on \mathbb{R} if:

- (i) $z(t)$ is continuous,
- (ii) the derivative $z'(t)$ exists for all $t \in \mathbb{R}$ with the possible exception of the points $\theta_i, i \in \mathbb{Z}$, where one-sided derivatives exists,
- (iii) equation (6)((5)) is satisfied by $z(t)$ for all $t \in \mathbb{R}$, except, points of θ and it holds for the right derivative of $z(t)$ at points θ_i .

3. Results

In this section, we give the sufficient conditions for the existence and uniqueness of solutions and periodic solutions of the damped spring-mass system (6). We create the periodic solution using Green's function with the initial condition corresponding to the Poincare criterion for differential equations with generalized piecewise constant argument.

3.1. Existence and uniqueness of solutions

The following lemmas give necessary conditions for existence and uniqueness of solutions of the damped spring-mass system (6).

Lemma 3.1. Assume that the conditions (S1)-(S4) hold. Then for fixed $i \in \mathbb{Z}$ and for every $(t_0, \mu, \eta) \in [\theta_i, \theta_{i+1}] \times \mathcal{K} \times \mathcal{K}$ there exists a unique solution $z(t) = z(t, t_0, \mu, \eta)$ of (6) on $[t_0, \theta_{i+1}]$.

Proof. Consider the initial condition (K1) and so a solution of the form $z(t) = (z_1(t), z_2(t))^T = z(t, t_0, \mu)$ with $\theta_i \leq t_0 \leq \zeta_i < \theta_{i+1}$ for fixed $i \in \mathbb{Z}$. The proof for the initial condition (K2) can be performed as in the case of functional differential equations [43].

Existence. With $z^0(t) = Z(t, t_0) \mu(t_0)$, state a sequence $\{z^k(t)\}, k \geq 0$ by

$$z^{k+1}(t) = \mu(t - t_0), t \in [t_0 - \tau, t_0],$$

$$z^{k+1}(t) = Z(t, t_0) \left[\mu(t_0) + \int_{t_0}^{\zeta_i} e^{B(\zeta_i - s)} f(s, z_s^k, z_{\zeta_i}^k) ds \right] + \int_{\zeta_i}^t e^{B(t-s)} f(s, z_s^k, z_{\zeta_i}^k) ds, t \in [t_0, \theta_{i+1}].$$

A value $M_V \in (0, \infty)$ can be found such that $\|f(t, z_t, z_{\gamma(t)})\| \leq M_V$ since $f \in C_0(\mathbb{R} \times \mathcal{K}_V \times \mathcal{K}_V), 0 < V \in \mathbb{R}$. Hence,

$$\max_{[t_0, \theta_{i+1}]} \|z^{k+1}(t) - z^k(t)\| \leq [2\bar{M}L(1+M)\bar{\theta}]^k \Omega,$$

where $\Omega = \bar{M}(1+M)\bar{\theta}M_V$. So, (S4) shows that there exists a solution $z(t) = z(t, t_0, \mu)$ of the equation (6) on

$[t_0, \theta_{i+1}]$ given by

$$z(t) = Z(t, t_0) \left[\mu(t_0) + \int_{t_0}^{\zeta_i} e^{B(\zeta_i-s)} f(s, z_s, z_{\zeta_i}) ds \right] + \int_{\zeta_i}^t e^{B(t-s)} f(s, z_s, z_{\zeta_i}) ds.$$

Uniqueness. Denote the solutions of (6) by $z^j(t) = z^j(t, t_0, \mu)$, $j = 1, 2$. With $\|\mu\|_\infty = \sup_{\mathbb{R}} \|\mu(t)\|$, it is found that

$$\begin{aligned} \|z^1(t) - z^2(t)\| &\leq \|Z(t, t_0)\| \int_{t_0}^{\zeta_i} \|e^{B(\zeta_i-s)}\| \cdot \|f(s, z_s^1, z_{\zeta_i}^1) - f(s, z_s^2, z_{\zeta_i}^2)\| ds \\ &+ \int_{\zeta_i}^t \|e^{B(t-s)}\| \cdot \|f(s, z_s^1, z_{\zeta_i}^1) - f(s, z_s^2, z_{\zeta_i}^2)\| ds \\ &\leq M \int_{t_0}^{\zeta_i} \overline{ML} \left[\|z_s^1 - z_s^2\|_0 + \|z_{\zeta_i}^1 - z_{\zeta_i}^2\|_0 \right] ds \\ &+ \int_{\zeta_i}^t \overline{ML} \left[\|z_s^1 - z_s^2\|_0 + \|z_{\zeta_i}^1 - z_{\zeta_i}^2\|_0 \right] ds \\ &= M \int_{t_0}^{\zeta_i} \overline{ML} \max_{\sigma \in [-\tau, 0]} \|z_s^1(\sigma) - z_s^2(\sigma)\| ds \\ &+ M \int_{t_0}^{\zeta_i} \overline{ML} \max_{\sigma \in [-\tau, 0]} \|z_{\zeta_i}^1(\sigma) - z_{\zeta_i}^2(\sigma)\| ds \\ &+ \int_{\zeta_i}^t \overline{ML} \max_{\sigma \in [-\tau, 0]} \|z_s^1(\sigma) - z_s^2(\sigma)\| ds \\ &+ \int_{\zeta_i}^t \overline{ML} \max_{\sigma \in [-\tau, 0]} \|z_{\zeta_i}^1(\sigma) - z_{\zeta_i}^2(\sigma)\| ds \\ &\leq 2M \int_{t_0}^{\zeta_i} \overline{ML} \sup_{\mathbb{R}} \|z^1(t) - z^2(t)\| ds \\ &+ 2 \int_{\zeta_i}^t \overline{ML} \sup_{\mathbb{R}} \|z^1(t) - z^2(t)\| ds \\ &\leq 2M \int_{t_0}^{\zeta_i} \overline{ML} \|z^1 - z^2\|_\infty ds \\ &+ 2 \int_{\zeta_i}^t \overline{ML} \|z^1 - z^2\|_\infty ds \\ &\leq 2M \int_{t_0}^{\zeta_i} \overline{ML} \|z^1 - z^2\| ds \\ &+ 2 \int_{\zeta_i}^t \overline{ML} \|z^1 - z^2\| ds \end{aligned}$$

$$\begin{aligned} &\leq 2\overline{ML}(1+M)\overline{\theta} \|z^1 - z^2\| \\ &\leq 2\overline{ML}(1+M)\overline{\theta} \max_{[t_0, \theta_{i+1}]} \|z^1 - z^2\|. \end{aligned}$$

It is seen that $z^1(t) = z^2(t)$, in other words, uniqueness of the solution is proved using the condition (S4). \square

Lemma 3.2. Assume that conditions (S1)-(S4) hold and fix $i \in \mathbb{Z}$. Then for every $(t_0, \mu, \eta) \in [\theta_i, \theta_{i+1}] \times \mathcal{K} \times \mathcal{K}$ there exists a unique solution $z(t) = z(t, t_0, \mu, \eta)$, $t \geq t_0$, of (6) and it satisfies the following equation

$$z(t) = Z(t, t_0) \left[\mu(t_0) + \int_{t_0}^{\zeta_i} e^{B(t_0-s)} f(s, z_s, z_{\gamma(s)}) ds \right] + \sum_{k=i}^{j-1} Z(t, \theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} e^{B(\theta_{k+1}-s)} f(s, z_s, z_{\gamma(s)}) ds + \int_{\zeta_j}^t e^{B(t-s)} f(s, z_s, z_{\gamma(s)}) ds,$$

where $\theta_i \leq t_0 \leq \theta_{i+1}$, $\theta_j \leq t \leq \theta_{j+1}$, $i < j$.

3.2. Existence and uniqueness of periodic solutions

In addition to the assumptions, let $\zeta_0 = 0$ without loss of generality, and $\zeta_0 = t_0$. $Z(\omega) = Z(\omega, 0)$ is called as the monodromy matrix. Assume that $\det[I - Z(\omega)]^{-1} \neq 0$. Besides, the matrix Q is the monodromy matrix defined by

$$Q := \prod_{k=1}^p G_k = \prod_{k=1}^p M_k^{-1}(\theta_k) M_{k-1}(\theta_k), k \in \mathbb{Z}, \quad (8)$$

where $p \in \mathbb{Z}$ such that $\theta_{k+p} = \theta_k + \omega$ and $\zeta_{k+p} = \zeta_k + \omega$ for $\omega \in \mathbb{R}$, $k \in \mathbb{Z}$. Eigenvalues of the matrix Q or $Z(\omega)$, ρ_j , $j = 1, 2$, are called multipliers [1]. For the spring-mass system (6), using the formula (8) we find the matrix G_k as follows

$$G_k = \frac{e^{\alpha(\zeta_{k-1} - \zeta_k)}}{g_{k_5}} \begin{bmatrix} g_{k_1} & g_{k_2} \\ g_{k_3} & g_{k_4} \end{bmatrix},$$

where

$$\begin{aligned} g_{k_1} &= \left(1 - \frac{A}{m(\alpha^2 + \beta^2)} \right) (\cos(\beta(\zeta_{k-1} - \zeta_k))) \\ &- \frac{\alpha}{\beta} \sin(\beta(\zeta_{k-1} - \zeta_k)) + \frac{Ae^{\alpha(\theta_k - \zeta_{k-1})}}{m(\alpha^2 + \beta^2)} (\cos(\beta(\theta_k - \zeta_k))) \\ &- \frac{\alpha}{\beta} \sin(\beta(\theta_k - \zeta_k)), \end{aligned}$$

$$g_{k_2} = -\frac{\sin(\beta(\zeta_{k-1} - \zeta_k))}{\beta},$$

$$g_{k_3} = \left(1 - \frac{A}{m(\alpha^2 + \beta^2)} \right)^2 \left(\frac{\alpha^2 + \beta^2}{\beta} \right) \sin(\beta(\zeta_{k-1} - \zeta_k))$$

$$\begin{aligned}
 & + \left(1 - \frac{A}{m(\alpha^2 + \beta^2)}\right) \frac{Ae^{\alpha\theta_k}}{m\beta} (e^{-\alpha\zeta_{k-1}} \sin(\beta(\theta_k - \zeta_k)) \\
 & - e^{-\alpha\zeta_k} \sin(\beta(\theta_k - \zeta_{k-1}))), \\
 g_{k4} & = \left(1 - \frac{A}{m(\alpha^2 + \beta^2)}\right) (\cos(\beta(\zeta_{k-1} - \zeta_k)) \\
 & + \frac{\alpha}{\beta} \sin(\beta(\zeta_{k-1} - \zeta_k))) \\
 & + \frac{Ae^{\alpha(\theta_k - \zeta_k)}}{m(\alpha^2 + \beta^2)} (\cos(\beta(\theta_k - \zeta_{k-1})) - \frac{\alpha}{\beta} \sin(\beta(\theta_k - \zeta_{k-1}))) \\
 & \text{and}
 \end{aligned}$$

$$\begin{aligned}
 g_{k5} & = 1 - \frac{A}{m(\alpha^2 + \beta^2)} + \frac{Ae^{\alpha(\theta_k - \zeta_k)}}{m(\alpha^2 + \beta^2)} (\cos(\beta(\theta_k - \zeta_k)) \\
 & - \frac{\alpha}{\beta} \sin(\beta(\theta_k - \zeta_k))).
 \end{aligned}$$

The matrix Q can be obtained in terms of the matrices G_k with the value $p \in \mathbb{Z}$ corresponding the sequences $\theta = (\theta_i)$ and $\zeta = (\zeta_i)$, $i \in \mathbb{Z}$. So, the multipliers can be found, and periodicities of the solutions of the spring-mass system (6) can be researched. As a result, the existence of periodic solutions is certain if there exists a unit multiplier. However, periodic solutions can also be found in the absence of unit multipliers i.e. in the non-critical case. In this study, a periodic solution of the damped spring-mass system (6) is created for the non-critical case with the help of Green's function. In the interval $t \in [\theta_j, \theta_{j+1}]$, with $Z(t) = Z(t, 0)$, $t \in \mathbb{R}$, the solution $z(t) = z(t, 0, z_0)$ satisfies the following integral equation

$$\begin{aligned}
 z(t) & = Z(t)z_0 \\
 & + \sum_{k=0}^{j-1} Z(t, \theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} e^{B(\theta_{k+1}-s)} f(s, z_s, z_{\gamma(s)}) ds \quad (9) \\
 & + \int_{\zeta_j}^t e^{B(t-s)} f(s, z_s, z_{\gamma(s)}) ds.
 \end{aligned}$$

The solution (9) is a periodic solution of the system (6) if the initial condition is taken according to Poincare criterion [24], [25], in the following form

$$\begin{aligned}
 z_0 & = [I - Z(\omega)]^{-1} \sum_{k=0}^{p-1} Z(\omega, \theta_{k+1}) \\
 & \int_{\zeta_k}^{\zeta_{k+1}} e^{B(\theta_{k+1}-s)} f(s, z_s, z_{\gamma(s)}) ds, \quad (10)
 \end{aligned}$$

where $\det[I - Z(\omega)]^{-1} \neq 0$. Substiting the initial condition (10) in the equation (9), we obtain the integral equation

$$\begin{aligned}
 z(t) & = \sum_{k=0}^{j-1} Z(t) [I - Z(\omega)]^{-1} Z^{-1}(\theta_{k+1}) \\
 & \int_{\zeta_k}^{\zeta_{k+1}} e^{B(\theta_{k+1}-s)} f(s, z_s, z_{\gamma(s)}) ds
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{k=j}^{p-1} Z(t) [I - Z(\omega)]^{-1} Z(\omega) Z^{-1}(\theta_{k+1}) \\
 & \int_{\zeta_k}^{\zeta_{k+1}} e^{B(\theta_{k+1}-s)} f(s, z_s, z_{\gamma(s)}) ds \\
 & + \int_{\zeta_j}^t e^{B(t-s)} f(s, z_s, z_{\gamma(s)}) ds.
 \end{aligned}$$

This solution $z(t)$ is a continuous function. Thus, Green's function $G_p(t, s)$, $t, s \in [0, \omega]$ for the periodic solution can be constructed in $t \in [\theta_j, \theta_{j+1}]$, $j = 0, 1, \dots, p-1$ as follows

$$G_p(t, s) = \begin{cases} Z(t)[I - Z(\omega)]^{-1} Z^{-1}(\theta_{k+1}) e^{B(\theta_{k+1}-s)}, & s \in [\zeta_k, \zeta_{k+1}], k < j, \\ Z(t)[I - Z(\omega)]^{-1} Z(\omega) Z^{-1}(\theta_{k+1}) e^{B(\theta_{k+1}-s)}, & s \in [\zeta_k, \zeta_{k+1}] \setminus [\zeta_j, t], k \geq j, \\ Z(t)[I - Z(\omega)]^{-1} Z(\omega) Z^{-1}(\theta_{k+1}) e^{B(\theta_{k+1}-s)} + e^{B(t-s)}, & s \in [\zeta_j, t]. \end{cases}$$

So, the periodic solution of the system (6) is expressed in the form

$$z(t) = \int_0^\omega G_p(t, s) f(s, z_s, z_{\gamma(s)}) ds. \quad (11)$$

In the next theorem, the sufficient conditions for the spring-mass system (6) to have a unique ω -periodic solution are given.

Theorem 3.3. *Let $\tilde{R} = \max_{t,s \in [0, \omega]} \|G_p(t, s)\| < \infty$ and $2\tilde{R}L\omega < 1$. Suppose that conditions (S1) – (S6) are valid and $\det[I - Z(\omega)] \neq 0$. Then, the spring-mass system (6) admits a unique ω -periodic solution.*

Proof. Let the complete metric space $C_\lambda(\mathbb{R})$ denote the sets of all continuous and ω -periodic functions on \mathbb{R} . Define on $C_\lambda(\mathbb{R})$ an operator such that

$$\Pi \tilde{S}(t) = \int_0^\omega G_p(t, s) f(s, \tilde{S}_s, \tilde{S}_{\gamma(s)}) ds,$$

where $t \in [\theta_j, \theta_{j+1}]$, $j = 0, 1, 2, \dots, p-1$ and

$$G_p(t, s) = \begin{cases} Z(t)[I - Z(\omega)]^{-1} Z^{-1}(\theta_{k+1}) e^{B(\theta_{k+1}-s)}, & s \in [\zeta_k, \zeta_{k+1}], k < j, \\ Z(t)[I - Z(\omega)]^{-1} Z(\omega) Z^{-1}(\theta_{k+1}) e^{B(\theta_{k+1}-s)}, & s \in [\zeta_k, \zeta_{k+1}] \setminus [\zeta_j, t], k \geq j, \\ Z(t)[I - Z(\omega)]^{-1} Z(\omega) Z^{-1}(\theta_{k+1}) e^{B(\theta_{k+1}-s)} + e^{B(t-s)}, & s \in [\zeta_j, t]. \end{cases}$$

It can be seen that $\Pi : C_\lambda(\mathbb{R}) \rightarrow C_\lambda(\mathbb{R})$. Let $u, v \in C_\lambda(\mathbb{R})$,

then we have

$$\begin{aligned} \|\prod u(t) - \prod v(t)\| &\leq \int_0^\omega \|G_p(t,s)\| \|f(s, u_s, u_{\gamma(s)}) - \\ &\quad - f(s, v_s, v_{\gamma(s)})\| ds \\ &\leq \int_0^\omega \tilde{R}L \left(\|u_s - v_s\|_0 + \right. \\ &\quad \left. + \|u_{\gamma(s)} - v_{\gamma(s)}\|_0 \right) ds \\ &\leq 2 \int_0^\omega \tilde{R}L \|u - v\| ds \\ &\leq 2\tilde{R}L\omega \|u - v\|. \end{aligned}$$

So, the condition $2\tilde{R}L\omega < 1$ shows the uniqueness of the periodic solution (11). The proof is completed. \square

4. An Example

Example 4.1. Taking the parameters $k = 10$, $m = 1$, $c = 0.1$, $A = 0.001$ and the sequences $\theta_i = \zeta_i = \frac{1}{10}i$, consider the linear nonhomogeneous spring-mass system with piecewise constant argument of generalized type and delayed argument

$$\begin{aligned} x''(t) + 0.1x'(t) + 10x(t) &= 10^{-3}x(\gamma(t)) + 10^{-1} \\ &\quad + 10^{-1}x(t - \tau) \quad (12) \\ &\quad + 10^{-1}x(\gamma(t) - \tau). \end{aligned}$$

Let $\tau = 1$ and the initial condition $z_{t_0}(s) = \mu(s) = (0.02029980811, -0.01018243128)^T$, $s \in [-1, 0]$. Taking $z_1 = x$ and $z_2 = x'$, spring-mass system (12) can be reduced to the following nonhomogeneous differential equation

$$\begin{aligned} z'(t) &= \begin{bmatrix} 0 & 1 \\ -10 & -0.1 \end{bmatrix} z(t) + \begin{bmatrix} 0 & 0 \\ 10^{-3} & 0 \end{bmatrix} z(\gamma(t)) \\ &\quad + \begin{bmatrix} 0 \\ 10^{-1}(1 + z_1(t-1) + z_1(\gamma(t)-1)) \end{bmatrix}. \end{aligned} \quad (13)$$

The conditions (S1) – (S4) are satisfied with $L = 10^{-1}$, $\bar{\theta} = \bar{\zeta} = 0.1$, $M = 1.552565377$, $\bar{M} = 1.528461349$, and the system (13) is $\omega = 0.1$ -periodic since it satisfies (S5) – (S6). $t = \omega = 0.1 \in [\theta_0, \theta_1] = [0, 0.1]$ for $i = 0$. Then, for $t \in [0, 0.1]$, the monodromy matrix is in the form

$$Z(0.1) = \begin{bmatrix} 0.9505848166 & 0.0978515763 \\ -0.9784179112 & 0.9950124792 \end{bmatrix}.$$

In the interval $t \in [0, 0.1]$, $Z(t) = Z(t, 0)$, $t \in \mathbb{R}$ is found as follows

$$Z(t) = \begin{bmatrix} Z_{11}(t) & Z_{12}(t) \\ Z_{21}(t) & Z_{22}(t) \end{bmatrix},$$

and its indices are

$$\begin{aligned} Z_{11}(t) &= e^{-t/20} \frac{9999}{10000} \cos(3.161882351t) \\ &+ e^{-t/20} 0.01581178375 \sin(3.161882351t) + \frac{1}{10000}, \end{aligned}$$

$$Z_{12}(t) = 0.3162673019e^{-t/20} \sin(3.161882351t),$$

$$Z_{21}(t) = -3.162356751e^{-t/20} \sin(3.161882351t),$$

$$\begin{aligned} Z_{22}(t) &= e^{-t/20} (\cos(3.161882351t) \\ &- 0.01581336509 \sin(3.161882351t)). \end{aligned}$$

So, in the interval $t \in [0, 0.1]$, the solution $z(t) = z(t, 0, z_0)$ satisfies the following integral equation

$$\begin{aligned} z(t) &= Z(t)z_0 \\ &+ Z(t) \int_{-0.1}^0 e^{-Bs} \begin{bmatrix} 0 \\ 10^{-1}(1 + z_1(s-1) + z_1(-1.1)) \end{bmatrix} ds \\ &+ Z(t, 0.1) \int_0^{0.1} e^{-B(s-0.1)} \begin{bmatrix} 0 \\ 10^{-1}(1 + z_1(s-1) + z_1(-1)) \end{bmatrix} ds \\ &+ \int_0^t e^{B(t-s)} \begin{bmatrix} 0 \\ 10^{-1}(1 + z_1(s-1) + z_1(-1)) \end{bmatrix} ds. \end{aligned}$$

From here,

$$\begin{aligned} z(t) &= Z(t)z_0 \\ &+ Z(t) \int_{-0.1}^0 e^{-Bs} \begin{bmatrix} 0 \\ 0.1040599616 \end{bmatrix} ds \\ &+ Z(t, 0.1) \int_0^{0.1} e^{B(0.1-s)} \begin{bmatrix} 0 \\ 0.1040599616 \end{bmatrix} ds \\ &+ \int_0^t e^{B(t-s)} \begin{bmatrix} 0 \\ 0.1040599616 \end{bmatrix} ds. \end{aligned} \quad (14)$$

Thus, it can be concluded that the solution (14) is a periodic solution of the system (13) if the initial condition is taken in the following form

$$z_0 = \begin{bmatrix} 0.02029980811 \\ -0.01018243128 \end{bmatrix},$$

where $\det[I - Z(0.1)]^{-1} = 10.41816491 \neq 0$. Thus, in $t \in [\theta_0, \theta_1] = [0, 0.1]$, Green's function $G_1(t, s)$, $t, s \in [0, 0.1]$ for the periodic solution is as follows

$$G_1(t, s) = \begin{cases} G_{11}, & s \in [-0.1, 0), \\ G_{12}, & s \in [0, 0.1] \setminus [0, \hat{t}], \\ G_{13}, & s \in [0, \hat{t}]. \end{cases}$$

Here, we have

$$G_{11} = \begin{bmatrix} g_1^{*1} & g_1^{*2} \\ g_1^{*3} & g_1^{*4} \end{bmatrix}$$

and its indices are

$$\begin{aligned} g_1^{*1} &= \left[0.05196081420e^{-t/20} \right. \\ &\quad \left(\frac{9999}{10000} \cos(3.161882351t) \right. \\ &\quad \left. + 0.01581178375 \sin(3.161882351t) + \frac{e^{t/20}}{10000} \right) \\ &\quad \left. - 3.223813545e^{-t/20} \sin(3.161882351t) \right] e^{s/20} \\ &\quad (\cos(3.161882351s)) \end{aligned}$$

$$\begin{aligned}
 & -0.01581336509 \sin(3.161882351s)) \\
 & +3.162673019 \left[1.019433859 e^{-t/20} \right. \\
 & \left. \left(\frac{9999}{10000} \cos(3.161882351t) \right. \right. \\
 & \left. \left. +0.01581178375 \sin(3.161882351t) + \frac{e^{t/20}}{10000} \right) \right. \\
 & \left. +0.1628193186 e^{-t/20} \sin(3.161882351t) \right] \\
 & e^{s/20} \sin(3.161882351s),
 \end{aligned}$$

$$\begin{aligned}
 g_1^{*2} = & -0.3162673019 \left[0.05196081420 e^{-t/20} \right. \\
 & \left(\frac{9999}{10000} \cos(3.161882351t) \right. \\
 & \left. +0.01581178375 \sin(3.161882351t) \right. \\
 & \left. + \frac{e^{t/20}}{10000} \right) \\
 & \left. -3.223813545 e^{-t/20} \sin(3.161882351t) \right] \\
 & e^{s/20} \sin(3.161882351s) + \left[1.019433859 e^{-t/20} \right. \\
 & \left(\frac{9999}{10000} \cos(3.161882351t) \right. \\
 & \left. +0.01581178375 \sin(3.161882351t) \right. \\
 & \left. + \frac{e^{t/20}}{10000} \right) \\
 & \left. +0.1628193186 e^{-t/20} \sin(3.161882351t) \right] \\
 & e^{s/20} (\cos(3.161882351s) \\
 & +0.01581336509 \sin(3.161882351s)),
 \end{aligned}$$

$$\begin{aligned}
 g_1^{*3} = & \left(-0.1643186316 e^{-t/20} \sin(3.161882351t) \right. \\
 & \left. -10.19331915 e^{-t/20} (\cos(3.161882351t) \right. \\
 & \left. -0.01581336509 \sin(3.161882351t)) \right) \\
 & e^{s/20} (\cos(3.161882351s) \\
 & -0.01581336509 \sin(3.161882351s)) \\
 & +3.162673019 \\
 & \left(-3.223813546 e^{-t/20} \sin(3.161882351t) \right. \\
 & \left. +0.5148155299 e^{-t/20} (\cos(3.161882351t) \right. \\
 & \left. -0.01581336509 \sin(3.161882351t)) \right) \\
 & e^{s/20} \sin(3.161882351s),
 \end{aligned}$$

$$\begin{aligned}
 g_1^{*4} = & -0.3162673019 \\
 & \left(-0.1643186316 e^{-t/20} \sin(3.161882351t) \right. \\
 & \left. -10.19331915 e^{-t/20} (\cos(3.161882351t) \right. \\
 & \left. -0.01581336509 \sin(3.161882351t)) \right) \\
 & e^{s/20} \sin(3.161882351s) \\
 & + \left(-3.223813546 e^{-t/20} \sin(3.161882351t) \right. \\
 & \left. +0.5148155299 e^{-t/20} (\cos(3.161882351t) \right.
 \end{aligned}$$

$$\begin{aligned}
 & -0.01581336509 \sin(3.161882351t)) e^{s/20} \\
 & (\cos(3.161882351s) \\
 & +0.01581336509 \sin(3.161882351s)).
 \end{aligned}$$

Similarly, we find

$$G_{1_2} = \begin{bmatrix} g_2^{*1} & g_2^{*2} \\ g_2^{*3} & g_2^{*4} \end{bmatrix},$$

where

$$\begin{aligned}
 g_2^{*1} = & \left[0.0519608143 e^{-t/20} \left(\frac{9999}{10000} \cos(3.161882351t) \right. \right. \\
 & \left. \left. +0.01581178375 \sin(3.161882351t) + \frac{e^{t/20}}{10000} \right) \right. \\
 & \left. -3.223813545 e^{-t/20} \sin(3.161882351t) \right] \\
 & e^{-0.005 + \frac{s}{20}} \\
 & (\cos(-0.3161882351 + 3.161882351s) \\
 & -0.01581336509 \\
 & \sin(-0.3161882351 + 3.161882351s)) \\
 & +3.162673019 \left[1.019433859 e^{-t/20} \right. \\
 & \left(\frac{9999}{10000} \cos(3.161882351t) \right. \\
 & \left. +0.01581178375 \sin(3.161882351t) + \frac{e^{t/20}}{10000} \right) \\
 & \left. +0.1628193187 e^{-t/20} \sin(3.161882351t) \right] \\
 & e^{-0.005 + \frac{s}{20}} \sin(-0.3161882351 + 3.161882351s),
 \end{aligned}$$

$$\begin{aligned}
 g_2^{*2} = & -0.3162673019 \left[0.0519608143 e^{-t/20} \right. \\
 & \left(\frac{9999}{10000} \cos(3.161882351t) \right. \\
 & \left. +0.01581178375 \sin(3.161882351t) \right. \\
 & \left. + \frac{e^{t/20}}{10000} \right) \\
 & \left. -3.223813545 e^{-t/20} \sin(3.161882351t) \right] \\
 & e^{-0.005 + \frac{s}{20}} \\
 & \sin(-0.3161882351 + 3.161882351s) \\
 & + \left[1.019433859 e^{-t/20} \right. \\
 & \left(\frac{9999}{10000} \cos(3.161882351t) \right. \\
 & \left. +0.01581178375 \sin(3.161882351t) \right. \\
 & \left. + \frac{e^{t/20}}{10000} \right) \\
 & \left. +0.1628193187 e^{-t/20} \sin(3.161882351t) \right] \\
 & e^{-0.005 + \frac{s}{20}} \\
 & (\cos(-0.3161882351 + 3.161882351s) \\
 & +0.01581336509 \\
 & \sin(-0.3161882351 + 3.161882351s)),
 \end{aligned}$$

$$g_2^{*3} = \left[-0.164318632e^{-t/20} \sin(3.161882351t) - 10.19331915e^{-t/20} (\cos(3.161882351t)) - 0.01581336509 \sin(3.161882351t) \right] e^{-0.005 + \frac{s}{20}} (\cos(-0.3161882351 + 3.161882351s) - 0.01581336509 \sin(-0.3161882351 + 3.161882351s)) + 3.162673019 \left[-3.223813546e^{-t/20} \sin(3.161882351t) + 0.5148155299e^{-t/20} (\cos(3.161882351t)) - 0.01581336509 \sin(3.161882351t) \right] e^{-0.005 + \frac{s}{20}} \sin(-0.3161882351 + 3.161882351s),$$

$$g_2^{*4} = -0.3162673019 \left[-0.164318632e^{-t/20} \sin(3.161882351t) - 10.19331915e^{-t/20} (\cos(3.161882351t)) - 0.01581336509 \sin(3.161882351t) \right] e^{-0.005 + \frac{s}{20}} \sin(-0.3161882351 + 3.161882351s) + \left[-3.223813546e^{-t/20} \sin(3.161882351t) + 0.5148155299e^{-t/20} (\cos(3.161882351t)) - 0.01581336509 \sin(3.161882351t) \right] e^{-0.005 + \frac{s}{20}} (\cos(-0.3161882351 + 3.161882351s) + 0.01581336509 \sin(-0.3161882351 + 3.161882351s)).$$

Finally,

$$G_{13} = G_{12} + \begin{bmatrix} g_3^{*1} & g_3^{*2} \\ g_3^{*3} & g_3^{*4} \end{bmatrix},$$

where

$$g_3^{*1} = e^{-(t-s)/20} (\cos(-3.161882351t + 3.161882351s) - 0.01581336509 \sin(-3.161882351t + 3.161882351s)),$$

$$g_3^{*2} = -0.3162673019e^{-(t-s)/20} \sin(-3.161882351t + 3.161882351s),$$

$$g_3^{*3} = 3.162673019e^{-(t-s)/20} \sin(-3.161882351t + 3.161882351s),$$

$$g_3^{*4} = e^{-(t-s)/20} (\cos(-3.161882351t + 3.161882351s) + 0.01581336509 \sin(-3.161882351t + 3.161882351s)).$$

So, the periodic solution of the system (13) can be expressed in the form

$$z(t) = \int_0^{0.1} G_1(t, s) \left[10^{-1} (1 + z_1(s-1) + z_1(\gamma(s)-1)) \right] ds. \tag{15}$$

As a result of Theorem 3.1, the spring-mass system (13) admits a unique 0.1-periodic solution since $\tilde{R} = \max_{t,s \in [0,0.1]} \|G_1(t, s)\| = 10.24608221 < \infty$ and $2\tilde{R}L\omega = 0.2049216442 < 1$, the conditions (S1) – (S6) are valid and $\det[I - Z(0.1)] \neq 0$.

5. Discussion and Conclusion

Differential equations are important to model real world problems in many areas. Nevertheless, the modeling the problems with differential equations may not reflect reality if we ignore the effects of delays and discontinuities. For this reason, differential equations with deviating argument that produce more realistic models have great importance. The differential equations with deviating argument include delay differential equations, functional differential equations, differential equations with piecewise constant argument and differential equations with generalized piecewise constant argument. Many scientists have worked on the theory and applications of these equations. Moreover, Akhmet contributed to these studies by introducing differential equations with functional dependence on generalized piecewise constant argument. This contribution increases the realism of the models. In applications, the spring-mass system has an importance in many areas such as physic, mathematics, biomechanics, biology. In our study, we modeled the spring-mass systems using differential equations with generalized piecewise constant argument and with functional dependence on generalized piecewise constant argument. So, we obtain more realistic and detailed analysis. Later, we analyze the qualitative behaviors of these spring-mass systems, and give the sufficient conditions for the existence and uniqueness of periodic solutions. Periodicity provides information about behavior of solution for the other intervals, with knowledge of the qualitative behavior of the system in a particular interval. Therefore, periodicity in a system is a desired feature, and a lot of study about existence of periodic solutions are available in the literature. Our examination is considerable since it is obvious that periodicity is significant in both theory and practice. In the literature, differential equations with deviating argument are generally studied by reducing into discrete equations. We examine our models without reducing them into discrete equations. It shows our work's novelty from the other studies in the literature.

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