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Complex Eigenvalue Analysis of a Linear Pencil: An Experimental Study

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Abstract. This paper is concerned with a finite-dimensional example of a linear pencil which leads to a class of non-self-adjoint matrices. We consider the linear pencil $H_c - \lambda L$, where H_c is a tri-diagonal matrix with a constant parameter c on the main diagonal and off-diagonal entries equal to one, and L is a diagonal matrix whose elements decrease linearly from one to minus one. In general, the spectra of operator polynomials may contain non-real eigenvalues as well as real eigenvalues. Nevertheless, they exhibit certain patterns. Our aim in this research is to carry out a variety of numerical investigation on the eigenvalues so as to understand the eigenvalue behaviour of such pencils from different points of view. In accordance with our numerical findings, a series of conjectures are offered and various heuristics has been discussed.

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Keywords: Linear operator pencils, non-self-adjoint matrices, eigenvalues, spectral theory.

1. Introduction

Operator pencils are family of operators depending on a spectral parameter λ, that is, operator polynomials of the form

$$\mathcal{P}(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 + \cdots + \lambda^m A_m,$$

where $\lambda \in \mathbb{C}$ and A_k, $k \in \{0, 1, \ldots, m\}$, are linear operators acting in a Hilbert space. A family whose operator coefficients are self-adjoint, i.e. $A_k = (A_k)^*$, is called a self-adjoint operator pencil. The eigenvalue problem is to find $\lambda \in \mathbb{C}$ for which the equation $\mathcal{P}(\lambda)\mathbf{u} = \mathbf{0}$ admits a non-trivial solution. There are various problems in physics and engineering which can be described by using polynomial operator pencils. They arise for instance in transport theory [7], quantum physics [1], magnetohydrodynamics [4], vibrating structures [13], and control theory [3], to name a few. Further applications can be found in [2], and also in the monographs [11, 12]. Several researchers contributed to the development of the theory over the past few decades, and the reader is referred to [11, 12, 14] for a historical survey.

This research focuses on linear operator pencils acting in finite-dimensional Hilbert spaces, which sometimes called as matrix polynomials, and are of the form $\mathcal{P}(\lambda) = H - \lambda L$. We say that the complex number λ_0 is an eigenvalue of the pencil $\mathcal{P}$ if

$$H\mathbf{u} = \lambda_0 L\mathbf{u}$$

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for $\mathbf{u} \in \mathbb{C}^N$, $\mathbf{u} \neq \mathbf{0}$, or in other words if $\mathcal{P}(\lambda_0)$ is not invertible. The set of all eigenvalues of the pencil $\mathcal{P}$ is called the spectrum which we denote by $\mathrm{Spec}(\mathcal{P})$. The spectrum equals to the set of all roots of the polynomial $\det(\mathcal{P}(\lambda))$, i.e.

$$\mathrm{Spec}(\mathcal{P}) = \{\lambda \in \mathbb{C} \ : \ \det(H - \lambda L) = 0\}.$$

The spectral problem for a linear pencil can be converted into a non-self-adjoint eigenvalue problem. If L is invertible, then one can reduce the problem to the eigenvalue problem for an operator $L^{-1}H$, which is non-self-adjoint in general, and $\mathrm{Spec}(\mathcal{P}) \equiv \mathrm{Spec}(L^{-1}H)$. If one takes L to be the identity operator I, then the eigenvalue problem for the pencil $\mathcal{P}(\lambda) = H - \lambda I$ is equivalent to the standard spectral problem for the operator H. It is well-known that the eigenvalues of a non-self-adjoint operator can lie in anywhere in the complex plane, and locating the spectrum is not an easy task.

Our main interest will be on the non-real eigenvalues which appear when both H and L are indefinite. If all eigenvalues of a matrix have the same sign, either positive or negative, then the matrix is called sign-definite. For a linear pencil $\mathcal{P}(\lambda)$, if either H or L is sign-definite, then one can convert the problem to an eigenvalue problem for a self-adjoint operator so that the spectrum of the pencil $\mathcal{P}$ is real. For instance, suppose that H is positive-definite, i.e. $H > 0$. Then $H^{1/2} > 0$ and using the change of variables $\mu = 1/\lambda$ and $\mathbf{v} = H^{1/2}\mathbf{u}$, we see that the eigenvalue problem for the pencil $\mathcal{P}$ reduces to a self-adjoint spectral problem:

$$H\mathbf{u} = \lambda L\mathbf{u} \quad \Longleftrightarrow \quad H^{1/2}\mathbf{v} = \lambda L H^{-1/2}\mathbf{v} \quad \Longleftrightarrow \quad \mu\mathbf{v} = H^{-1/2} L H^{-1/2}\mathbf{v}.$$

It can be seen that $H^{-1/2}LH^{-1/2}$ is a product of three self-adjoint operator, hence it is self-adjoint and the spectrum is real. The only time when we cannot proceed with a similar reasoning is only if both H and L are indefinite (or sign-indefinite). Then there may be some complex eigenvalues in the spectrum. In this case, the pencil $\mathcal{P}$ in which either H or L has both positive and negative eigenvalues is called indefinite linear pencil.

In this article, we deal with a particular type of an indefinite self-adjoint linear pencil written as

$$\mathcal{P}_c = \mathcal{P}_{c;N}(\lambda) := H_c^{(N)} - \lambda L^{(N)}, \tag{1.1}$$

where $H_c^{(N)}$ is tri-diagonal and $L^{(N)}$ is diagonal with entries

$$\left(H_c^{(N)}\right)_{i,j} = \begin{cases} c & \text{if } i = j, \\ 1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad \left(L^{(N)}\right)_{i,j} = \begin{cases} V_j & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

where the size of both matrices is $N \times N$, c is a real parameter and $V_j = 1 - \frac{2(j-1)}{N-1}$, $i, j \in \{1, 2, \ldots, N\}$. For simplicity, the superscript (N) in $H_c^{(N)}$ and $L^{(N)}$ will be omitted whenever it is clear from the context, i.e. $H_c := H_c^{(N)}$ and $L := L^{(N)}$. The eigenvalue problem we consider is $\mathcal{P}_c\mathbf{x} = \mathbf{0}$, where $\mathcal{P}_c$ is given as in (1.1), and the number $\lambda \in \mathbb{C}$ and the non-zero vector $\mathbf{x} \in \mathbb{C}^N$ are the sought eigenvalue and eigenvector, respectively. The spectrum of the pencil $\mathcal{P}_c$ is the set of all its eigenvalues, denoted by $\mathrm{Spec}(\mathcal{P}_c)$.

When N is odd, the matrix L is singular due to its construction. We therefore need to analyse the roots of the polynomial $\det(H_c - \lambda L) = 0$. However, as N increases, finding the roots of high degree polynomials takes a significant amount of time and we loose the precision dramatically. On the other hand, when N is even, the matrix L is invertible so that we can compute the spectrum of the non-self-adjoint matrix $L^{-1}H_c$ instead, which gives more accurate results. For this reason, we set $N = 2n$ and choose to work with the $(2n) \times (2n)$ matrices. We shall consider the odd dimensions only when the double eigenvalues of $\mathcal{P}_c$ are investigated.

In general, the spectra of operator polynomials may contain non-real eigenvalues as well as real eigenvalues. Nevertheless, they exhibit certain patterns. Recently, there has been increased interest in studying such pencils and the literature is extensive. One can find similar problems for instance in [5, 6, 8, 9] where the problems that seem trivial require highly non-trivial analysis. Our aim in this research is to consider a finite dimensional example of a linear pencil and investigate, at least numerically, the eigenvalues. Several computational experiments are conducted so as to understand the behaviour of the eigenvalues of such pencils. Supported by our experiments, numerous questions and conjectures have been posed.

2. Numerical Experiments and Heuristics

We start with some preliminaries. It can be observed that the matrix H_c is a tri-diagonal Toeplitz matrix, and the eigenvalues of H_0 are given by

$$\mu_j = \mu_j^{(N)} = \cos\left(\frac{\pi j}{N+1}\right), \qquad j \in \{1, \dots, N\}.$$

Since $\mathrm{Spec}(H_c) = \mathrm{Spec}(H_0 + cI)$, the matrix H_c is indefinite only if $c \in (\mu_N, \mu_1) \subset (-2, 2)$. It is also straightforward to see that the diagonal matrix L is indefinite. Since we want to focus on the non-real eigenvalues of $\mathcal{P}_c$, it will be sufficient to consider $c \in (-2, 2)$.

It is well-known that the spectra of a linear pencil with self-adjoint coefficients is symmetric with respect to the real axis. This is because if $\lambda \in \mathrm{Spec}(\mathcal{P}_c)$, then $H_c - \lambda L$ is not invertible and so $(H_c - \lambda L)^* = H_c - \overline{\lambda} L$ is not invertible. Hence, $\overline{\lambda}$ is also an eigenvalue of $\mathcal{P}_c$.

One approach to deal with the eigenvalue problem for a linear pencil is to re-write it in a block-matrix structure. We shall take an example to explain this approach. Let $n = 3$ and $\mathbf{x} = (\mathbf{u}, \mathbf{v})^T$, then

$$\lambda \mathbf{x} = L^{-1} H_c \mathbf{x} = \begin{pmatrix} c & 1 & & & & \\ \frac{5}{3} & \frac{5c}{3} & \frac{5}{3} & & & \\ & 5 & 5c & 5 & & \\ & & -5 & -5c & -5 & \\ & & & -\frac{5}{3} & -\frac{5c}{3} & -\frac{5}{3} \\ & & & & -1 & -c \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix},$$

which is a skew-centrosymmetric, non-self-adjoint matrix. One can treat this problem as a system of linear equations. If we change the enumeration of eigenvectors and write them as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_1)^T$ instead of $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)^T$, one can obtain the following

$$\lambda \tilde{\mathbf{x}} = \begin{pmatrix} c & 1 & & & & \\ \frac{5}{3} & \frac{5c}{3} & \frac{5}{3} & & & \\ & 5 & 5c & & & 5 \\ & & & -c & -1 & \\ & & & -\frac{5}{3} & -\frac{5c}{3} & -\frac{5}{3} \\ & & -5 & & -5 & -5c \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{v}_3 \\ \mathbf{v}_2 \\ \mathbf{v}_1 \end{pmatrix}.$$

In other words, if we act by $\mathcal{P}_c$ on vectors written as

$$\mathbf{x} = (\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{v}_n, \dots, \mathbf{v}_1)^T,$$

then the eigenvalue problem for the linear pencil $\mathcal{P}_c$ can be converted into the eigenvalue problem for the following class of non-self-adjoint block operator matrices

$$\lambda \mathbf{x} = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix},$$

where $A \neq A^*$, and $B = B^*$ with $(B)_{n,n} = N - 1$ and all other entries of B are zeros. We remark that $N = 2n$ and $V_n = 1/(N-1)$, and since the matrix L is diagonal, we have $(L^{-1}H_c)_{n,n+1} = N - 1$. One of the difficulties in our particular example is that several eigenvalue inclusion sets, for instance Gershgorin-type sets, does not provide a tight region for the spectra. This is due to the fact that the norm $\|B\|$ gets large as N increases and the matrix A is non-self-adjoint.

We shall now proceed with the numerical investigation of the non-real eigenvalues of $\mathcal{P}_c$. In our first illustration, the proportion of non-real eigenvalues of $\mathcal{P}_c$ is shown as c changes, see Figure 1. The first thing we notice is that all eigenvalues seem to be non-real when $c = 0$. In addition, as n increases the curve is getting smoother and it is not difficult to predict the number of non-real eigenvalues of $\mathcal{P}_c$ for a given c. Therefore, the figure indicates that there should be a relation between the proportion of non-real eigenvalues and number-theoretical properties of some quantities including c and n.

Conjecture 2.1. *If* $c = 0$, *then* $\mathrm{Spec}(\mathcal{P}_0) \cap \mathbb{R} = \varnothing$. *In addition, if* $\lambda \in \mathrm{Spec}(\mathcal{P}_0)$ *then* $\mathrm{Re}(\lambda) \neq 0$.

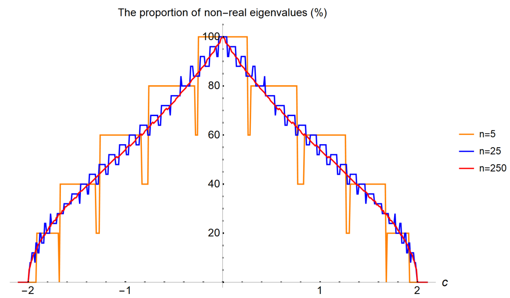

FIGURE 1. The proportion of non-real eigenvalues of $\mathcal{P}_c$ is shown as c changes from -2.1 to 2.1 with the step-size of 0.01, for $n = 5$ (orange line), $n = 25$ (blue line) and $n = 250$ (red line).

Another important feature we observe from Figure 1 is that the spectra $\mathrm{Spec}(\mathcal{P}_c)$ seems to invariant under the symmetry $c \to -c$. However, our next experiment reveals that this is not true for the non-real spectra. Nevertheless, we see that the symmetry $c \to -c$ corresponds to the symmetry $\lambda \to -\lambda$ for the non-real eigenvalues of $\mathcal{P}_c$. In Figure 2, we show the non-real eigenvalues only as the numerics indicate that the real eigenvalues are symmetric. $\mathrm{Spec}(\mathcal{P}_c)$ is shown by red dots and blue dots represent $\mathrm{Spec}(\mathcal{P}_{-c})$.

Conjecture 2.2. *For all values of $N \in \mathbb{N}$, the real spectrum $\mathrm{Spec}(\mathcal{P}_c) \cap \mathbb{R}$ is invariant under the symmetry $\lambda \to -\lambda$, and also the symmetry $c \to -c$.*

Conjecture 2.3. *For all values of $N \in \mathbb{N}$, if $\lambda \in \mathrm{Spec}(\mathcal{P}_c)$, then $-\lambda \in \mathrm{Spec}(\mathcal{P}_{-c})$.*

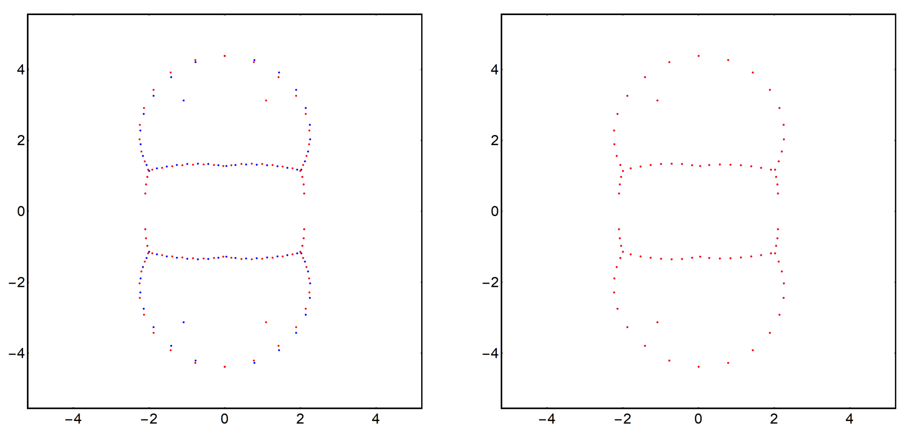

FIGURE 2. For $n = 50$, $c = 0.3$, the non-real eigenvalues are shown in coordinates $(\mathrm{Re}(\lambda), \mathrm{Im}(\lambda))$ and the real eigenvalues are omitted for simplicity. Left: $\mathrm{Spec}(\mathcal{P}_c)$ (red dots) and $\mathrm{Spec}(\mathcal{P}_{-c})$ (blue dots). Right: $-\mathrm{Spec}(\mathcal{P}_c)$ (red dots) and $\mathrm{Spec}(\mathcal{P}_{-c})$ (blue dots).

As can be seen from Figure 2 that $\mathrm{Spec}(\mathcal{P}_c)$ and $\mathrm{Spec}(\mathcal{P}_{-c})$ are very similar. We therefore illustrate several typical pictures of the spectra by taking some values of c in $[0, 2)$. The top of Figure 3 demonstrates the eigenvalues of $\mathcal{P}_c$ for $n = 1000$. The non-real eigenvalues which are close to the real line behave differently than the others. Therefore, we also zoom in near the real line in the bottom of Figure 3. The non-real eigenvalues whose imaginary parts are not close to zero seem to approximately lie on two circles. It can also be seen that there are some non-real eigenvalues which lie inside of these circles and they do not form a pattern.

As mentioned earlier, the eigenvalues of a sign-indefinite linear pencil exhibit some common features. Computational results indicate that the eigenvalues lie on or under a set of curves. Davies and Levitin [5] showed that this is true in some cases. Namely, they studied the eigenvalues of the pencil $\mathcal{A}_c = H_c - \lambda D$ as the dimension N diverges to

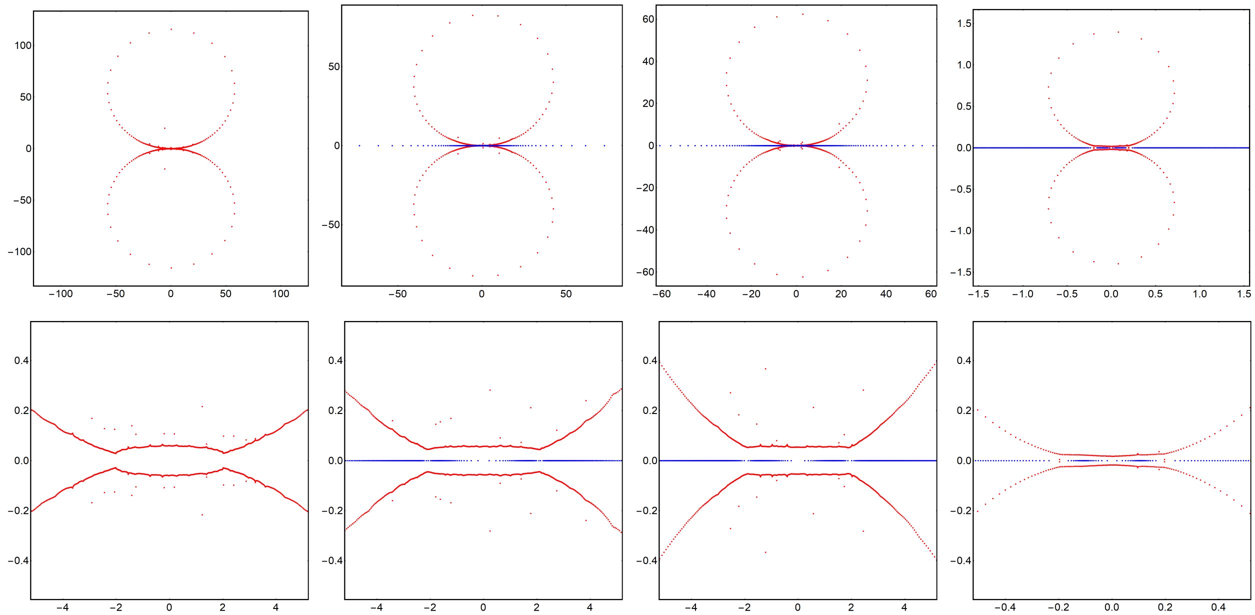

Figure 3. For $n = 1000$, the real eigenvalues (blue dots) and the non-real eigenvalues (red dots) of the pencil $\mathcal{P}_c$ is illustrated in the $(\mathrm{Re}(\lambda), \mathrm{Im}(\lambda))$-plane. Some large real eigenvalues are not shown for simplicity. Top: for $c = 0$, $c = 0.4$, $c = 0.7$ and $c = 1.9$ from left to right. Bottom: Zooming in near the origin in the same order.

infinity, where H_c is given as in (1.2) and

$$D = \begin{pmatrix} I & \\ & -I \end{pmatrix}.$$

Davies and Levitin [5] proved that when $c = 0$ the spectrum $\mathrm{Spec}(\mathcal{A}_0)$ approximately lies on a particular curve. On the other hand, when $c \neq 0$ the spectrum of $\mathcal{A}_c$ is localised in a certain region and Davies and Levitin [5] are able to derive the curve explicitly which bounds all eigenvalues. However, there are still several open questions regarding the eigenvalues of the pencil $\mathcal{A}_c$, some of which has been also discussed in [10].

For a given n and c, it is not easy to extract some information about $\mathrm{Spec}(\mathcal{P}_c)$ directly from Figure 3. However, if we fix n and take different values of c and then superimpose the spectra of $\mathcal{P}_c$, one can see some common patterns for all values of c, see Figure 4. When n is small, the non-real eigenvalue curves can be seen clearly. On the other hand, after around $n = 23$, the eigenvalues start to be more scattered as n increases, but still preserves a certain shape. This scattering starts from the ones with the largest imaginary part.

One approach to see the relation between c and the non-real eigenvalues is to repeat the same experiment using a different coordinate system. In the top of Figure 5, we present the non-real eigenvalues in the $(c, \mathrm{Im}(\lambda))$-plane, whereas the bottom of Figure 5 illustrates the non-real eigenvalues in the $(c, \mathrm{Re}(\lambda))$-plane. As n increases, a similar scattering can be observed.

In Figure 6, the superimposition of the non-real eigenvalues of $\mathcal{P}_c$ for a given c by taking n from 2 to 250 is shown in coordinates $(\mathrm{Re}(\lambda), \mathrm{Im}(\lambda))$. The spectral picture when $c = 0$ is slightly different than the others, i.e. the non-real eigenvalues are more lined up when $c = 0$. In addition, there exists an additional clustering near the origin. Moreover, the figure indicates that there seems to be a region near the real axis which does not contain any non-real eigenvalue.

In order to get a better understanding of the spectra for a particular value of n, we superimpose the real eigenvalue curves (blue) and the real part of non-real eigenvalue curves (red) in coordinates $(\mathrm{Re}(\lambda), c)$ in Figure 7. The left of the figure indicates that when $|c|$ is near 2, there are two large real eigenvalues of $\mathrm{Spec}(\mathcal{P}_c)$. In addition, in each real eigenvalue curves there are some local extremum points. It can be observed from the right of Figure 7 that the non-real eigenvalue curves do not always cross at these extremum points. The eigenvalues at the point where the non-real eigenvalue curves cross the extremum points of the real eigenvalue curves are indeed the double eigenvalues of $\mathcal{P}_c$.

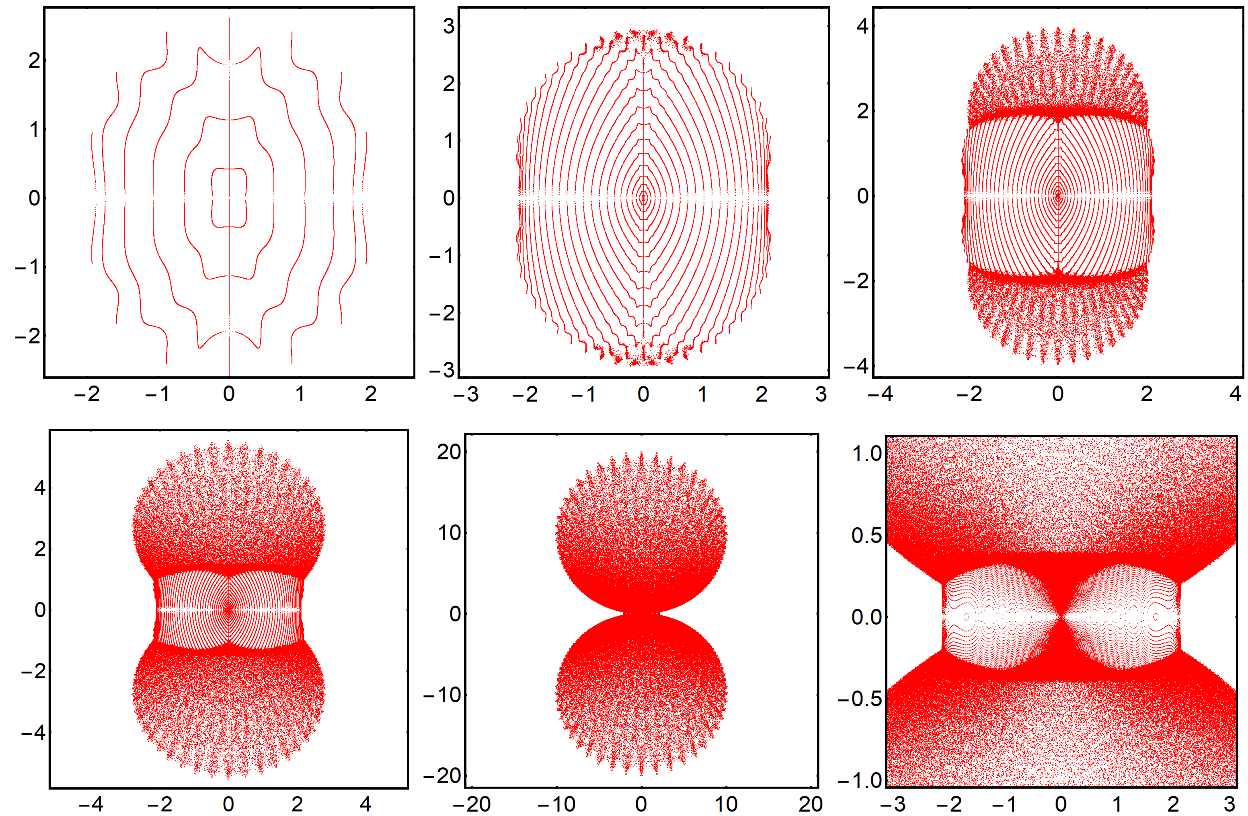

FIGURE 4. By taking c from -2 to 2 with the step-size of 0.001, the superimposition of the non-real eigenvalues of $\mathcal{P}_c$ is illustrated in the $(\mathrm{Re}(\lambda), \mathrm{Im}(\lambda))$-plane. Top: for $n = 7$, $n = 26$ and $n = 35$ from left to right. Bottom: for $n = 48$, $n = 170$ and zooming in near the origin when $n = 170$.

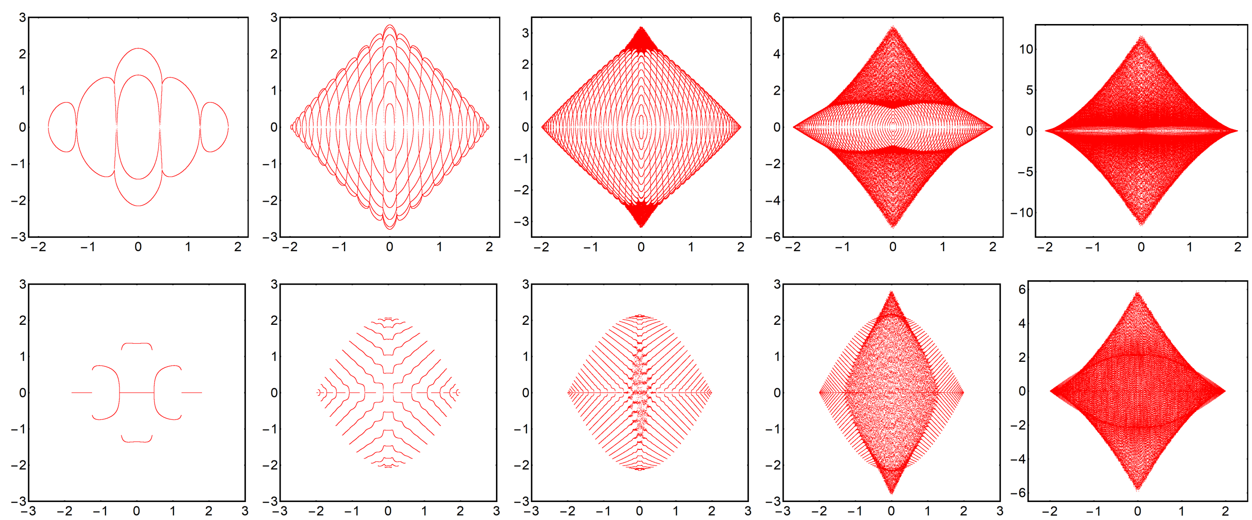

FIGURE 5. $\bigcup_{c=-2}^{2} \mathrm{Spec}(\mathcal{P}_c) \setminus \mathbb{R}$ is shown for $n = 3$, $n = 13$, $n = 28$, $n = 48$ and $n = 100$ from left to right. Top: in coordinates $(c, \mathrm{Im}(\lambda))$. Bottom: in coordinates $(c, \mathrm{Re}(\lambda))$.

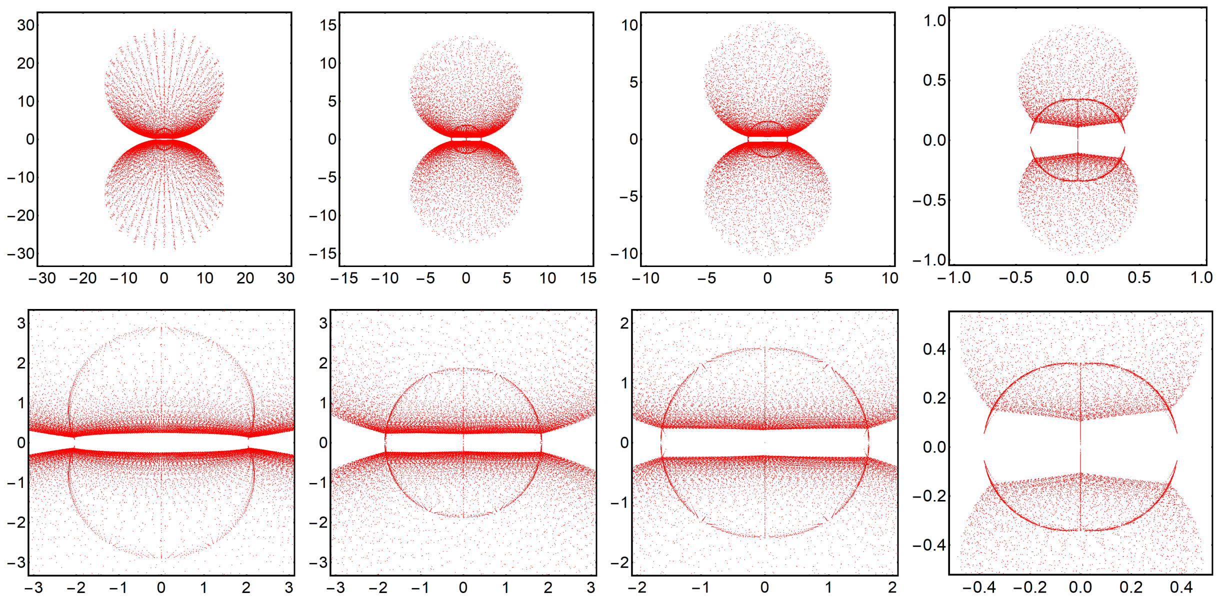

FIGURE 6. $\bigcup_{n=2}^{250} \mathrm{Spec}(\mathcal{P}_c) \setminus \mathbb{R}$ is shown in the $(\mathrm{Re}(\lambda), \mathrm{Im}(\lambda))$-plane. Top: for $c = 0$, $c = 0.8$, $c = 1$ and $c = 1.8$ from left to right. Bottom: zooming in near the origin in the same order.

We remark also that some non-real eigenvalue curves meet before they cross the real eigenvalue curves. The points where two red curves meet and produce another red curves are the non-real double eigenvalues. As can be seen from the figure that the non-real eigenvalue collisions occur only when $\mathrm{Re}(\lambda) = 0$. We shall now turn to a more detailed discussion of the double eigenvalues of $\mathcal{P}_c$.

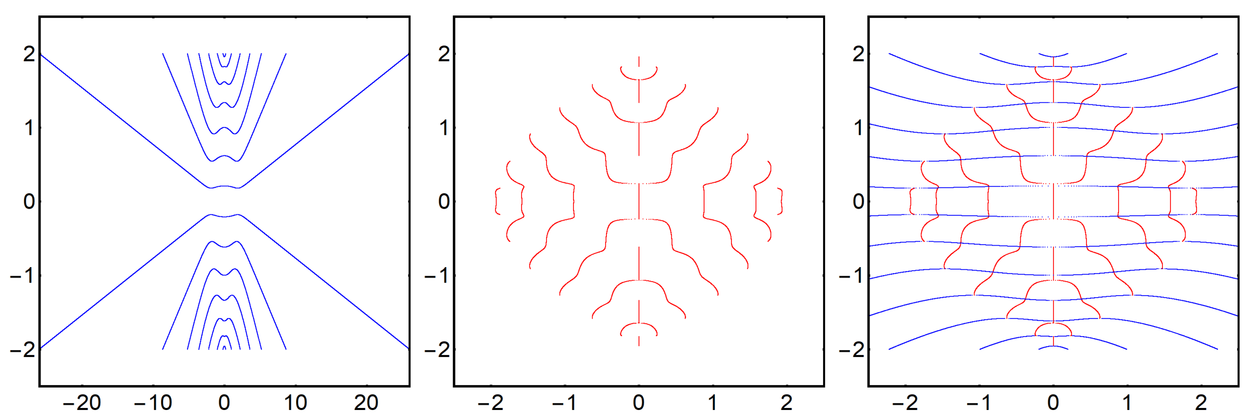

FIGURE 7. For $n = 7$, $\bigcup_{c=-2}^{2} \mathrm{Spec}(\mathcal{P}_c)$ is illustrated in the $(\mathrm{Re}(\lambda), c)$-plane. Left: the real eigenvalues (blue dots). Middle: the real part of the non-real eigenvalues (red dots). Right: the superimposition of the real and non-real eigenvalues.

Recall that $\mathcal{P}_c$ has dimension N. As mentioned earlier, the method of finding the roots of $\det(H - \lambda L) = 0$ causes enormous precision loss as N gets bigger. Nevertheless, the method gives accurate results up until around $N = 60$. Therefore, we shall use this method to find the double eigenvalue of $\mathcal{P}_c$ and consider the odd dimensions as well. If λ^*

is a double eigenvalue of the pencil $\mathcal{P}_c$, then λ^* should satisfy the system

$$\begin{cases} \det(H_c - \lambda L) & = 0, \\ \dfrac{d}{d\lambda}[\det(H_c - \lambda L)] & = 0. \end{cases}$$

However, if N is odd, then $\det(H - \lambda L)$ becomes c multiplied by a polynomial which is in λ of degree $N - 1$ with non-zero leading coefficient. This can be seen if one expands the determinant along the n^{th} row. Let A_n be a tri-diagonal matrix with diagonal elements $a_1, \ldots, a_n$ and off-diagonal elements equal to one, and B_n be a tri-diagonal matrix with diagonal elements $b_n, \ldots, b_1$ and off-diagonal elements equal to one. We then have when N is odd that

$$\begin{aligned} \det(H_c - \lambda L) &= \begin{vmatrix} a_1 & 1 & & & & & \\ 1 & \ddots & \ddots & & & & \\ & \ddots & a_n & 1 & & & \\ & & 1 & c & 1 & & \\ & & & 1 & b_n & \ddots & \\ & & & & \ddots & \ddots & 1 \\ & & & & & 1 & b_1 \end{vmatrix} \\ &= c\det(A_n)\det(B_n) - \det(A_{n-1})\det(B_n) - \det(A_n)\det(B_{n-1}), \end{aligned}$$

and using the recurrence relation for the determinant

$$\begin{aligned} \det(A_n) &= a_n \det(A_{n-1}) - \det(A_{n-2}), \\ \det(B_n) &= b_n \det(B_{n-1}) - \det(B_{n-2}), \end{aligned}$$

we obtain

$$\begin{aligned} \det(H_c - \lambda L) =& c\det(A_n)\det(B_n) - (a_n + b_n)\det(A_{n-1})\det(B_{n-1}) \\ & - \det(A_{n-1})\det(B_{n-2}) - \det(A_{n-2})\det(B_{n-1}). \end{aligned}$$

Since $a_n = c - \frac{\lambda}{N+1}$ and $b_n = c + \frac{\lambda}{N+1}$, we have $a_n + b_n = 2c$. If we continue to expand the determinant in the same way, it can be seen that c will be the common factor in the summation. For instance, if $N = 3$, then

$$\begin{cases} \det(H - \lambda L) & = c(-\lambda^2 + c^2 - 2) = 0, \\ \dfrac{d}{d\lambda}[\det(H - \lambda L)] & = -2\lambda c = 0. \end{cases}$$

It can be seen that if $c = 0$, then both equations are satisfied trivially for any $\lambda \in \mathbb{C}$. Since the system does not give any particular solution, we divide the characteristic equation by c and then look for solutions so that we obtain the double eigenvalues as $(\lambda, c) = (0, \pm\sqrt{2})$. Therefore, we shall solve the following system instead:

$$\begin{cases} \det(H - \lambda L)/c & = 0, \\ \dfrac{d}{d\lambda}[\det(H - \lambda L)/c] & = 0. \end{cases}$$

In addition, numerics indicate the following.

Conjecture 2.4. *If $c = 0$, then there are no double eigenvalues of the pencil $\mathcal{P}_c$ for any $N \in \mathbb{N}$.*

We note that finding the double eigenvalues takes a long time for large N (e.g. it takes approximately 57 hours to compute the double eigenvalues for $N = 50$). Therefore, the location of double eigenvalues, obtained by solving numerically the system of these two equations, will be computed for all $N \in \{3, 4, \ldots, 50\}$.

In the first place, we observe that the spectra $\bigcup_{c\in(-2,2)} \operatorname{Spec}(\mathcal{P}_c)$ contains the non-real double eigenvalues as well as the real ones, and our numerics indicate that the location of the non-real double eigenvalues always occur on the imaginary axis.

Conjecture 2.5. *Let $N \geq 3$ and $c \in (-2, 2)$. If λ^* is a non-real double eigenvalue of the pencil $\mathcal{P}_c$, then $\operatorname{Re}(\lambda^*) = 0$.*

As explained earlier, this claim can also be observed at the right of Figure 7. We now give additional examples to support our claim, see Figure 8. Blue dots have been used to represent the real double eigenvalues of $\mathcal{P}_c$ and red dots depict the non-real double eigenvalues. In the left of Figure 8, the superimposition of all double eigenvalues for all $N \in \{3, \ldots, 50\}$ are shown in the $(c, \mathrm{Re}(\lambda))$-plane. As claimed, red dots which depict the real part of the non-real double eigenvalues lie on the line $\mathrm{Re}(\lambda) = 0$. On the other hand, the right of Figure 8 shows the superimposition of all double eigenvalues in the $(c, \mathrm{Im}(\lambda))$-plane. An important feature in both coordinates that the double eigenvalues seem to be localised in a certain region and they both form interesting patterns.

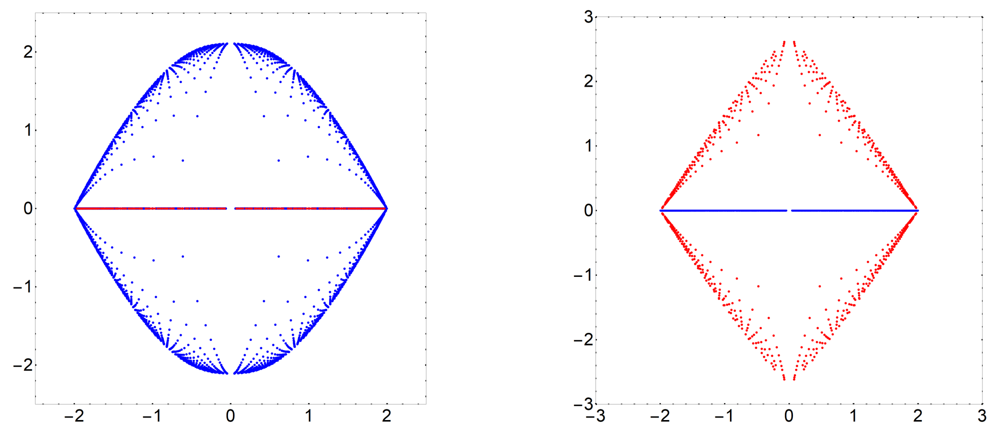

Figure 8. The superimposition of double eigenvalues of $\mathcal{P}_c$ is illustrated by taking N from 3 to 50. The real double eigenvalues are shown as blue dots whereas red dots represent the non-real double eigenvalues. Left: in coordinates $(c, \mathrm{Re}(\lambda))$. Right: in coordinates $(c, \mathrm{Im}(\lambda))$.

Actually, it is straightforward to find the location of double eigenvalues of $\mathcal{P}_c$ by solving the eigenvalue problem directly.

Lemma 2.6. *Subject to Conjecture 2.1 and Conjecture 2.5, the number of double eigenvalues which take place at the origin, i.e.* $\lambda^* = 0$, *is given by*

$$\# \bigcup_{c \in (-2,2)} \{\lambda^* \in \mathrm{Spec}(\mathcal{P}_c) : \lambda^* = 0\} = \begin{cases} N, & \text{if } N \text{ is even}, \\ N-1, & \text{if } N \text{ is odd}. \end{cases}$$

Proof. If $\lambda^* = 0$, then

$$\mathcal{P}_c \mathbf{x} = (H_0 + cI - \lambda L)\mathbf{x} = 0 \qquad \Longleftrightarrow \qquad c = \mu_j = 2\cos\left(\frac{\pi j}{N+1}\right) \in \mathrm{Spec}(H_0),$$

where $j \in \{1, \ldots, N\}$. This also means that there are N double eigenvalues which take place at the origin. However, if N is odd, then $c = 0 \in \mathrm{Spec}(H_0)$. As we claimed in Conjecture 2.1, all eigenvalues of $\mathcal{P}_c$ are non-real when $c = 0$ and they do not lie on the imaginary axis. Since all non-real double eigenvalues occur at the imaginary axis by Conjecture 2.5, we need to exclude $c = 0$ so that there are $N - 1$ double eigenvalues at $\lambda^* = 0$ if N is odd. □

We note that $\mathrm{Spec}(\mathcal{P}_c)$ for a given c may not contain any double eigenvalue. As can be seen from the figure that the double eigenvalues are rare and they form a discrete set. It can be said that the double eigenvalues λ^* do not depend continuously on c. We therefore look at the union of all c's in Figure 8. If one wants to look at the double eigenvalues for a particular N, then one observes the following.

Conjecture 2.7. *For each given value of* $N \in \mathbb{N}$,

(a) The real double eigenvalues $\lambda^* \in \bigcup_{c \in (-2,2)} \mathrm{Spec}(\mathcal{P}_c) \cap (\mathbb{R} \setminus \{0\})$ *lie on a particular curve in the* $(c, \mathrm{Re}(\lambda))$*-plane.*

(b) If there exists a non-real double eigenvalue $\lambda^* \in \bigcup_{c \in (-2,2)} \mathrm{Spec}(\mathcal{P}_c) \setminus \mathbb{R}$, *then it lies on a particular line in the* $(c, \mathrm{Im}(\lambda))$*-plane.*

To support our claim, we illustrate the double eigenvalues for a specific value of N in Figure 9. In the left of the figure, the first thing we see is that there is no non-real double eigenvalues of $\mathcal{P}_c$ when $N = 36$. Second, the real double eigenvalues, which do not lie at the origin $\lambda^* = 0$, seem to lie on a particular curve in the $(c, \mathrm{Re}(\lambda))$-plane. On the other hand, the non-real double eigenvalues seem to approximately form a line in the $(c, \mathrm{Im}(\lambda))$-plane, see the right of the figure for $N = 23$.

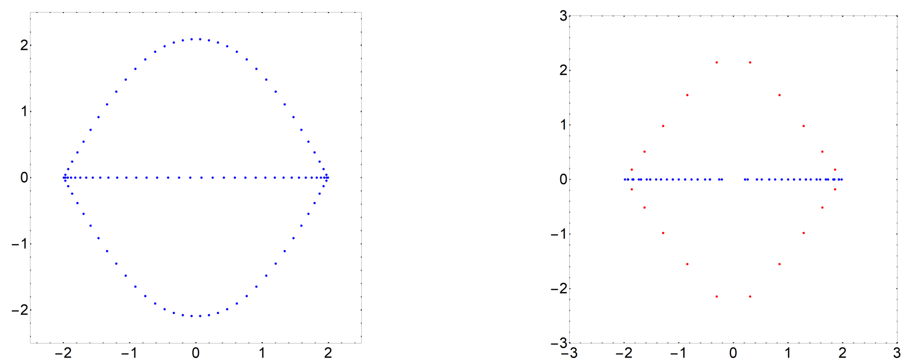

Figure 9. The double eigenvalues in the union $\bigcup_{c\in(-2,2)} \mathrm{Spec}(\mathcal{P}_c)$ are shown. Left: for $N = 36$ in coordinates $(c, \mathrm{Re}(\lambda))$. Right: for $N = 23$ in coordinates $(c, \mathrm{Im}(\lambda))$.

Using the numerical results, one can count the number of double eigenvalues of $\mathcal{P}_c$ for a given N as in Table 1. It can be observed from the table that the number of real double eigenvalues with $\lambda^* \neq 0$ have a certain pattern. We therefore pose the following.

Conjecture 2.8. *The number of real double eigenvalues with $\lambda^* \neq 0$ for a given $N \geq 3$ is*

$$\# \bigcup_{c\in(-2,2)} \{\lambda^* \in \mathrm{Spec}(\mathcal{P}_c) : \lambda^* \neq 0\} = 4\left(\left\lfloor \frac{N}{2} \right\rfloor - 1\right).$$

double eigenvalues	N=3	N=4	N=5	N=6	N=7	N=8	N=9	N=10	N=11	N=12	N=13	N=14	N=15	N=16	N=17	N=18	N=19	N=20	N=21	N=22	N=23	N=24	N=25	N=26
# at λ=0	2	4	4	6	6	8	8	10	10	12	12	14	14	16	16	18	18	20	20	22	22	24	24	26
# real with $\lambda \neq 0$	0	4	4	8	8	12	12	16	16	20	20	24	24	28	28	32	32	36	36	40	40	44	44	48
# non–real	0	0	0	4	4	4	4	8	8	8	8	12	12	12	12	16	16	16	16	20	20	20	20	24
# total	2	8	8	18	18	24	24	34	34	40	40	50	50	56	56	66	66	72	72	82	82	88	88	98

double eigenvalues	N=27	N=28	N=29	N=30	N=31	N=32	N=33	N=34	N=35	N=36	N=37	N=38	N=39	N=40	N=41	N=42	N=43	N=44	N=45	N=46	N=47	N=48	N=49	N=50
# at λ=0	26	28	28	30	30	32	32	34	34	36	36	38	38	40	40	42	42	44	44	46	46	48	48	50
# real with $\lambda \neq 0$	48	52	52	56	56	60	60	64	64	68	68	72	72	76	76	80	80	84	84	88	88	92	92	96
# non–real	24	24	24	28	28	28	28	8	6	0	0	0	0	36	0	40	0	40	0	44	0	44	0	48
# total	98	104	104	114	114	120	120	106	104	104	104	110	110	152	116	162	122	168	128	178	134	184	140	194

Table 1. For a given value of $N \in \{3, \ldots, 50\}$, the number of double eigenvalues at $\lambda = 0$, the real double eigenvalues with $\lambda \neq 0$, the non-real double eigenvalues, and the double eigenvalues in total are shown respectively.

Another important feature we observe from the table is that the number of non-real double eigenvalues obey a certain rule up until $N = 33$. However, after then it falls to zero, and then a different pattern continue to occur when $N \geq 40$. Another experiment that one may repeat is to take a large N, for instance $N = 70$, and observe the dynamics of the eigenvalues of $\mathcal{P}_{c,70}$, as c diminishes from 2.1 to 0. Then, one would realize that the behaviour of the eigenvalues are completely different after around $c = 0.85$.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed the published version of the article.

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