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Optimal Pension Fund Management with Stochastic Additional Contribution Rate in a Defined Contribution Pension Plan

Chiedozie B. Ibe[1*] and Charles. I. Nkeki[1]

[1]*Department of Mathematics, Faculty of Physical Sciences, University of Benin, Benin City, Edo State, Nigeria*
[*]*Corresponding author*

Abstract

This paper examines an optimal pension fund management in which a pension plan member (PPM) make a flow of contributions that comprises of two parts: mandatory contribution with fixed contribution rate, and additional voluntary contribution (AVC) with stochastic contribution rate. The mandatory part is a fixed fraction of a PPM's stochastic salary while the latter part is assume to be stochastic with contribution growth rate (CGR) and volatility depending on the contribution rate, the salary and the wealth processes of a PPM over time. The sum of the two contribution rates make up the total stochastic contribution rate of a PPM. Two asset classes are consider: the risky and the riskless assets. The latter pays a stochastic interest rate that follows uncontrolled Vasisek model. Also, it is assume that the PPM consume some portion of his or her wealth continuously over time. This paper aims at determining the optimal investment, optimal consumption rate, the optimal contribution rate and effect of AVC rate on the investment portfolio. The optimal investment portfolio, optimal contribution rate, optimal consumption rate and explicit expression of the CGR of a PPM's contribution rate are obtained. We found that the investment portfolio depend inversely on the contribution rate and the interest rate, and directly on the contribution of a PPM. The inter-temporary hedging portfolio strategies with respect to contribution rate, contribution process and interest rate are obtained. Furthermore, we found that the CGR depend on the optimal wealth, optimal contribution rate, interest rate, salary, investment risk, salary risk, contribution risk, interest rate risk, and coefficient of constant relative risk averse of a PPM. We also found that the CGR depend directly on the salary and the contribution processes and inversely on the contribution rate and interest rate of a PPM over time. Also, we found that the CGR react to changes in the salary process and its risk, wealth process, contribution rate and contribution risk, interest rate and interest rate risk, and coefficient of risk averse.

Keywords: *optimal pension fund, stochastic salary, stochastic interest rate, defined contribution, additional voluntary contribution rate, contribution growth rate, pension plan member, Vasisek model*

2010 Mathematics Subject Classification: *39A06; 39A21; 39A30*

1. Introduction

Effective management of pension fund to ensure maximum return at retirement is of fundamental importance for any pension plan member (PPM). The focus of any PPM is to choose a suitable strategy that will maximize his or her expected returns at retirement. Again, a PPM may want to increase his or her wealth at retirement by making additional voluntary contributions (AVCs) into the plan. This paper considers an optimal investment of a pension fund and a stochastic contribution rate of a plan member's stochastic salary in a defined contribution (DC) pension plan. The contribution of a PPM is made up of two parts: mandatory with fixed contribution rate and AVC with stochastic contribution rate. The mandatory part is a fixed fraction of a plan member's stochastic salary, and the latter part is a stochastic contribution rate of a PPM's stochastic salary with time, salary and wealth processes dependent drift and volatility. Since the investment in pension fund is expected to better the life of a PPM at retirement, hence, a PPM will indeed strategically invest his or her contributions to maximize his or her wealth at retirement. Some of the fundamental parameters that may influence the contribution rate of a PPM are considered in this paper. In other to determine when to continue or discontinue AVC into the pension plan, we classify the control strategy into two: the class of strategy that captured the maximum portfolio value loss in a risky asset, and the class of strategy that captured the maximum portfolio gain in a risky asset. Since no reasonable investor will want to invest when there is portfolio value loss, we assume that little or no AVCs will be made if the investment portfolio in the risky asset is in class of portfolio value loss, and vice versa.

There are two classes of assets consider in this paper. They are riskless and risky assets. The riskless asset is assumed to pay a stochastic interest rate that follows uncontrolled Vasisek model. It is assume that a PPM will consume continuously part of his or her wealth over time. In the optimal control problem, we have two control variables and three state variables. The two control variables are investment and consumption strategies. And the three state variables are the wealth process, the stochastic interest rate and the stochastic contribution rate of

Email addresses: **chiedozie.ibe@uniben.edu (Chiedozie B. Ibe), iwebuke.nkeki@uniben.edu; nkekicharles2003@yahoo.com (Charles. I. Nkeki)**

a PPM. The control strategy is divided into two classes, referred to as 'class of u_1' and 'class of u_2' (In Brachetta and Ceci (2021), they did similar thing but in different context). The class of u_1 represents the control strategy that gives maximum portfolio value loss in a risky asset while class of u_2 represents the control strategy that gives maximum portfolio value gain in a risky asset. The essence of this classification, is to guide the PPM on how to make more or less contributions into the plan. When the control strategy is of class of u_1, PPM may make little or no contributions into the pension plan and when is of class of u_2, PPM may make more contributions into the pension plan.
There are so many studies of pension fund with constant contribution rate in the literature, see for instance Gerrard *el.al* (2004), Battocchio and Menoncin (2004), Zhang *el.al* (2007), Hϕjgaard and E. Vigna (2007), Gao (2008), Vigna (2010), Nkeki and Nwozo (2012), Nkeki (2012), Nkeki (2013), Nkeki (2015). Nkeki (2017) studied pension fund management with deterministic contribution rate in a DC pension plan. They assumed that the contribution rate is a linear function of time. Nkeki (2018) considered a theoretical and an empirical study of an optimal pension fund in a jump-diffusion environment. A plan member was expected to contribute continuously a time-consistent proportion of his or her income into the scheme. But, stochastic contribution rate is yet to be studied in the literature to the best of my knowledge. In this paper, we consider the optimal investment and optimal consumption with stochastic contribution rate. The stochastic contribution rate is made up of mandatory and AVC rates. In this paper, the AVC rate is assumed to be stochastic, and its drift term and the volatility are assumed to depend on the stochastic contribution rate, the stochastic salary and the wealth process of a PPM over time. Nkeki and Modugu (2020) studied an investor's optimal investment, optimal net debt ratio with collateral security and optimal consumption plan in an economy that faces both diffusion and jump risks. The underlying assets considered in their paper are tangible and intangible assets.
The contributions of this paper are as follows. The AVC rate of a PPM is consider, and assume to be stochastic. We derive the PPM's wealth dynamics under two classes of control strategies, which includes maximum portfolio loss strategy and maximum portfolio gain strategy. The minimum and maximum expected wealth of a PPM over time are derived using the two classes of our control strategies. The derived wealth process of a PPM is solved using dynamic programming approach for a stochastic differential equation. We show, under the general assumptions on functional consumption process, that our value function is locally bounded and is a continuous viscosity solution to our resulting Hamilton-Jacobi-Bellman (HJB) equation. We provides a Verification Theorem based on the classical solutions of the HJB equation for our problem. From the resulting HJB equation, the optimal investment and optimal consumption under a stochastic contribution rate are obtained. Also obtain, is the explicit expression of the CGR of a PPM stochastic contribution rate over time. We rigorously derive the explicit form of our HJB equation. We also consider the sensitivity analysis of the CGR of a PPM stochastic contribution rate.
The remainder of the paper is organized as follows. In Section 2, we presents our financial models formulation. The dynamics of salary process and the dynamics of the contribution rate processes are presented in Section 3. Section 4 deals with the wealth dynamics of a PPM. The admissible strategies, optimal controls and the characteristics of our value functions are presented in Section 5. In Section 6, we present the optimal investment, optimal consumption rate of a PPM over time and the explicit form of our HJB equation. Section 7 presents the sensitivity analyses of the CGR of a PPM's contribution rate. Section 8 concludes the paper.

2. Financial Models Formulation

In this section, we consider an investor (PPM) who selects an amount to be invested in a risky asset with price process $S(t)$ and a riskless asset with price process $B(t)$ which pays a stochastic interest rate $r(t)$ at time $t \in [0,T]$, where T is the retirement period. Let $(\Omega, \mathscr{F}, \mathbf{F}(\mathscr{F}), \mathbf{P})$ be a complete filtated probability space which is endowed with $\mathbf{F}(\mathscr{F}) = \{\mathscr{F}_t : t \in [0,T]\}$ be a complete and right continuous fitration, where $\mathscr{F}_t = \sigma(W(s) : s \leq t)$, $W(s) = (W_S(s), W_Y(s), W_r(s), W_c(s))'$ and $W_S(t)$, $W_Y(t)$, $W_r(t)$ and $W_c(t)$ are defined on a given filtered probability space $(\Omega, \mathscr{F}, \mathbf{F}(\mathscr{F}), \mathbf{P})$, where $W_S(t)$ is a 1-dimensional Brownian motions with respect to stock market risks, $W_Y(t)$ is a 1-dimensional Brownian motions with respect to salary risk $W_r(t)$ is a 1-dimensional Brownian motions with respect to interest rate and $W_c(t)$ is a 1-dimensional Brownian motions with respect to contribution rate risk at time t. $\mathbf{P}$ denotes the real world probability measure. In this paper, (') denotes transpose.
The dynamics of the underlying assets considered in this paper, that is, a riskless asset with price process $B(t)$ and a risky asset (stock) with price process $S(t)$ at time t are as follows:
The dynamics of the riskless asset is given by:

$$dB(t) = r(t)B(t)dt, B(0) = 1, \tag{2.1}$$

where $r(t)$ is the short-term interest rate at time t, and its satisfies the following Vasicek model:

$$dr(t) = \kappa(\theta - r(t))dt + \sigma_r(t)dW(t), r(0) = r_0, \tag{2.2}$$

where κ is the speed of mean reversion, θ is the long-run mean, $\sigma_r(t) = (\sigma_{r_1}(t), \sigma_{r_2}(t), \sigma_{r_3}(t), \sigma_{r_4}(t))$, $\sigma_{r_1}(t)$ is the volatility of the short-term interest rate with respect to stock at time t, $\sigma_{r_2}(t)$ is the volatility of the short-term interest rate with respect to salary process at time t and $\sigma_{r_3}(t)$ is the volatility of the short-term interest rate at time t and $\sigma_{r_4}(t)$ is the volatility of the short-term interest rate with respect to contribution rate at time t.

Assumption 2.1. *The following assumption is needed in the sequel:* $r : [0,T] \times \mathscr{R} \to \mathscr{R}$ *is bounded above with upper bound being* R *such that* $0 < r(t) \leq R \; \forall \, t \in [0,T]$.

Next, we considered the dynamics of the risky asset at time t. It is given by the following dynamics:

$$dS(t) = S(t)(\mu(t)dt + \sigma_S(t)dW(t)), S(0) = s_0, \tag{2.3}$$

where $\mu(t)$ is the expected growth rate of stocks at time t, $\sigma_S(t) = (\sigma_1(t), \sigma_2(t),$
$\sigma_3(t), \sigma_4(t))$, $\sigma_1(t)$ is the volatility of stock at time t, $\sigma_2(t)$ is the volatility of stock with respect to salary process of a PPM at time t, $\sigma_3(t)$ is the volatility of stock with respect to the short-term interest rate at time t and $\sigma_4(t)$ is the volatility of stock with respect to contribution rate process of a PPM at time t.

Assumption 2.2. *The following assumptions are also needed in the sequel:*

- $\mu : [0,T] \times \mathscr{R} \to \mathscr{R}$ *such that* $\mu(t) \leq \mu_S \; \forall \, t \in [0,T]$ *for a given constant* $\mu_S \in \mathscr{R}$.
- $\sigma_S : [0,T] \times \mathscr{R}^4 \to \mathscr{R}^4$ *such that* $\sigma_S(t) \leq \sigma_1 \; \forall \, t \in [0,T]$ *for a given constant vector* $\sigma_1 \in \mathscr{R}^4$.

We now give the salary process of a plan member at time t, in the next section.

3. The Salary and Contribution rate Processes of a PPM

We now give the salary process of a plan member at time t as follows:

$$dY(t) = Y(t)(\beta(t)dt + \sigma_Y(t)dW(t)), Y(0) = y_0 > 0, \tag{3.1}$$

where $\beta(t)$ is the expected growth rate of a PPM's salary at time t, $\sigma_Y(t) = (\sigma_{Y_1}(t), \sigma_{Y_2}(t), \sigma_{Y_3}(t), \sigma_{Y_4}(t))$, $\sigma_{Y_1}(t)$ is the volatility of of PPM's salary with respect to stock at time t, $\sigma_{Y_2}(t)$ is the volatility of of PPM's salary process at time t and $\sigma_{Y_3}(t)$ is the volatility of PPM's salary with respect to short-term interest rate at time t and $\sigma_{Y_4}(t)$ is the volatility of of PPM's salary with respect to the contribution rate at time t.

Assumption 3.1. *The following assumptions are also needed in the sequel:*

- $\beta : [0,T] \times \mathscr{R} \to \mathscr{R}$ *bounded above with upper bound being* β_Y *such that* $\beta(t) \le \beta_Y \ \forall \ t \in [0,T]$ *for a given constant* $\beta_Y \in \mathscr{R}$.
- $\sigma_Y : [0,T] \times \mathscr{R}^4 \to \mathscr{R}^4$ *such that* $\sigma_Y(t) \le \sigma_2 \ \forall \ t \in [0,T]$ *for a given constant vector* $\sigma_2 \in \mathscr{R}^4$.

Since the fund available to a PPM at retirement is determined by the conbination of a PPM's contributions and the investment returns generated on these contributions minus the flow of consumption process, then a PPM may want to improve on his or her retirement benefits by making AVCs in addition to the normal (mandatory) contributions into the pension scheme. This AVCs will indeed go a long way in enhancing a PPM retirement benefits, if it is properly managed. We now give the following definition.

Definition 3.2. *[Nkeki, 2017] Let* $\{c(t) = c_0 + c_A(t) : 0 < c_A(t) < 1\}$ *be the contribution rate of a PPM at time t, where* $c_0 \in \mathscr{R}_+$ *is the mandatory contribution rate and* $c_A(t) \in ([0,T] \times \mathscr{R})$ *is the AVC rate at time t, then we define the dynamics of the contribution process of the PPM at time t as* $d\Phi(t) := c(t)dY(t)$*, where* $Y(t)$ *satisfies (3.1).*

Remark 3.3. *Note that in Nkeki (2017), the contribution rate is taken to be a linear function of time t, that is,* $c(t) = c_0 + c_A t$. *In this paper, we take the contribution rate to be a stochastic differential equation.*

Next, we consider the contribution rate of a plan member, and is given by the following stochastic differential equation:

$$dc(t) = \psi(t,c,X,Y)dt + \sigma_c(t,c,X,Y)dW(t), c(0) = c_0 > 0, \tag{3.2}$$

where $\psi(t,c,X,Y)$ is the drift term of contribution rate of a PPM at time t, $\sigma_c(t,c,X,Y) = (\sigma_{c_1}(t,c,X,Y), \sigma_{c_2}(t,c,X,Y), \sigma_{c_3}(t,c,X,Y), \sigma_{c_4}(t,c,X,Y))$, $\sigma_{c_1}(t,c,X,Y)$ is the volatility of contribution rate of a PPM with respect to stock at time t, $\sigma_{c_2}(t,c,X,Y)$ the volatility of contribution rate of a PPM with respect to salary at time t, $\sigma_{c_3}(t,c,X,Y)$ the volatility of contribution rate of a PPM with respect to the short-term interest rate at time t, $\sigma_{c_4}(t,c,X,Y)$ the volatility of contribution rate of a PPM at time t, and $X(t)$ is the wealth process of a PPM at time t, and it satisfies (4.7) below.

Assumption 3.4. *The following assumption is also needed in the sequel:* $c : [0,T] \times \mathscr{R} \to \mathscr{R}$ *such that* $0 < c(t) \le K \ \forall \ t \in [0,T]$ *for a given constant* $K > 0$.

Remark 3.5. *The drift term,* $\psi(t,c,X,Y)$ *and the volatility* $\sigma_c(t,c,X,Y)$ *of a PPM's contribution rate are assumed to depend on contribution rate, wealth process of a PPM, the salary process of a PPM and others market parameters at time t.*

Remark 3.6. *It is assumed in this paper that the short-term interest rate, the stock market, salary process and contribution rate of a PPM at time t, are all corrolated.*

Solving (3.2), we have

$$c(t) = c_0 + \int_0^t [\psi(s,c,X,Y)ds + \sigma_c(s,c,X,Y)dW(s)]. \tag{3.3}$$

From (3.3), we have that the quantity $\int_0^t [\psi(s,c,X,Y)ds + \sigma_c(s,c,X,Y)dW(s)]$ is equal to the AVC rate, $c_A(t)$ of a PPM at time t.

4. The Wealth Dynamics of a PPM

The fund invested in stock, $S(t)$ at time t is given by $\Delta(t)$ and the remainder $\Delta_0(t) = X(t) - \Delta(t)$ is invested in riskless asset at time t. Nigeria Pension Reform Act 2014, stipulates that PPM may make withdrawer of part of his or her wealth over time. Let $C(t)$ be the amount of consumption withdrawn from the wealth process by a PPM at time t such that

$$dC(t) := \alpha(t)X(t)dt, \tag{4.1}$$

where $\alpha(t)$ is the withdrawer rate or fraction of wealth withdrawn by a PPM at any time $t > 0$. Let $Q(t)$ be the fund invested in the financial market at time t such that

$$dQ(t) = \Delta_0(t)\frac{dB(t)}{B(t)} + \Delta(t)\frac{dS(t)}{S(t)}, Q(0) = q_0 \in \mathscr{R}_+. \tag{4.2}$$

Hence, the wealth process that will accrued to a PPM at time t is define as

$$X(t) := Q(t) + \Phi(t) - C(t). \tag{4.3}$$

It therefore follows that the wealth dynamics of a PPM is given as follows:

$$dX(t) = dQ(t) + d\Phi(t) - dC(t), X(0) = x_0 \in \mathscr{R}_+. \tag{4.4}$$

Using (2.1), (2.3), (3.1), (4.1), (4.2) and (4.3) on (4.4), we obtain the following:

$$dX(t) = \Delta_0(t)r(t)dt + \Delta(t)(\mu(t)dt + \sigma_S(t)dW(t)) + c(t)Y(t)(\beta(t)dt + \sigma_Y(t)dW(t)) - \alpha(t)X(t)dt, X(0) = x_0 \in \mathscr{R}_+. \tag{4.5}$$

Using the fact that $\Delta_0(t) = X(t) - \Delta(t)$, it then follows that

$$dX(t) = (r(t)X(t) + \Delta(t)(\mu(t) - r(t)) + \beta(t)c(t)Y(t) - \alpha(t)X(t))dt + (\Delta(t)\sigma_S(t) + c(t)Y(t)\sigma_Y(t))dW(t), X(0) = x_0 \in \mathscr{R}_+. \tag{4.6}$$

Hence,

$$\frac{dX(t)}{X(t)} = (r(t) + \pi(t)(\mu(t) - r(t)) + \beta(t)c(t)y(t) - \alpha(t))dt + (\pi(t)\sigma_S(t) + c(t)y(t)\sigma_Y(t))dW(t), X(0) = x_0 \in \mathscr{R}_+, \tag{4.7}$$

where $\pi(t) = \frac{\Delta(t)}{X(t)}$ is the fraction wealth given by the investment in stock market at time t and $y(t) = \frac{Y(t)}{X(t)}$ is the fraction wealth given by the salary process of a PPM at time t. Next, we consider the optimization of our problem, in subsequent section.

Assumption 4.1. *We now give the following assumption which will be needed subsequently:*

- *The admissible portfolio strategy $\pi(t)\ \forall\ t \in [0,T]$ takes values from a compact set $[-\pi_1(t), \pi_2(t)]$, where $\pi_1(t) > 0$ denotes maximum portfolio value loss in a risky asset at time t and $\pi_2(t) > 0$ denotes the maximum portfolio value gain in a risky asset at time t;*
- *$y : [0,T] \times \mathscr{R} \to \mathscr{R}$ such that $y(t) \leq y_a\ \forall\ t \in [0,T]$ for a given constant y_a;*
- *$\alpha : [0,T] \times \mathscr{R} \to \mathscr{R}$ is an admissible consumption rate strategy and is a nonnegative, and bounded above with upper bound being H such that $0 \leq \alpha(t) \leq H < \infty$;*
- *For $\pi(t) \in [-\pi_1(t), \pi_2(t)]$ and $0 \leq \alpha(t) \leq H < \infty$, there exists an admissible control strategy $u(t)$ such that $u(t) = \{(\pi(t), \alpha(t)) : t \geq 0\}$ which is progressively measurable with respect to $\{W(s) : 0 \leq s \leq t\}$.*

We now classify our control strategy into different classes under the following definitions.

Definition 4.2. *A PPM control strategy is of class*

- *u_1, if $u_1 = (-\pi_1, \alpha)$;*
- *u_2, if $u_2 = (\pi_2, \alpha)$;*
- *u, if $u = (u_1, u_2) = (-\pi_1, \pi_2; \alpha)$.*

Assumption 4.3. *We assume that a PPM make*

- *more contributions if the risky asset generates more positive portfolio value over time;*
- *both mandatory contribution and AVC into the pension plan if the investment portfolio is of class u_2;*
- *only the mandatory contribution into the pension plan if the investment portfolio is of class u_1.*

Remark 4.4. *We remark that*

- *if the control strategy is of class u_1, it means that more of the pension fund is invested in riskless asset, and the risky asset is generating negative portfolio value.*
- *if the control strategy is of class u_2, it means that more of the pension fund is invested in risky asset, and the risky asset is generating positive portfolio value.*

Remark 4.5. *We remark that the wealth process of a PPM admits an explicit solution of the form: for $X(t) = X^u(t)$*

$$\begin{aligned} X^u(t) &= x_0 e^{\int_0^t (r(s)+\pi(s)(\mu(s)-r(s))+\beta(s)c(s)y(s)-\alpha(s))ds} \times e^{-\frac{1}{2}\int_0^t (\pi(s)\sigma_S(s)+c(s)y(s)\sigma_Y(s))(\pi(s)\sigma_S(s)+c(s)y(s)\sigma_Y(s))' ds} \\ &\times e^{\int_0^t (\pi(s)\sigma_S(s)+c(s)y(s)\sigma_Y(s))dW(s)}\ \forall t \in [0,T], \mathbb{P}-a.s.. \end{aligned} \tag{4.8}$$

Whence, by Assumption 1 to Assumption 5, for any $n > 0$, we have that

$$x_0^n e^{\delta_1 t + \pi_1\sigma_1 W(t)} \leq (X^u(t))^n \leq x_0^n e^{\delta_2 t + (\pi_2\sigma_1 + Ky_a\sigma_2)W(t)}, \tag{4.9}$$

where $\delta_1 = n(R - \pi_1(\mu_S - R) + \beta_Y K y_a - H) + \frac{n(n-1)}{2}(Ky_a\sigma_2 - \pi_1\sigma_1)(Ky_a\sigma_2 - \pi_1\sigma_1)'$, $\delta_2 = n(R + \pi_2(\mu_S - R) + \beta_Y K y_a - H) + \frac{n(n-1)}{2}(\pi_2\sigma_1 + Ky_a\sigma_2)(\pi_2\sigma_1 + Ky_a\sigma_2)'$ and $\delta_1 < \delta_2$.
Now, taking the mathematical expectation of bothsides of (4.9), we have

$$x_0^n e^{\delta_1 t} \leq E[X^u(t)]^n \leq x_0^n e^{\delta_2 t}, \forall t \in [0,T], \mathbb{P}-a.s.. \tag{4.10}$$

Then, setting $t = T$, (4.10) becomes

$$x_0^n e^{\delta_1 T} \leq E[X^u(T)]^n \leq x_0^n e^{\delta_2 T}. \tag{4.11}$$

Remark 4.6. *We observe from (4.11) that the*

- $\min_{u_1} E[X^{u_1}(T)]^n = x_0^n e^{\delta_1 T}$ *is the minimum expected wealth of a PPM at retirement, and it is obtained from class u_1.*
- $\max_{u_2} E[X^{u_2}(T)]^n = x_0^n e^{\delta_2 T}$ *is the maximum expected wealth of a PPM at retirement, and it is obtained from class u_2.*

Definition 4.7. *The control strategy u is said to be admissible if*

$$\lim_{T\to\infty} e^{-\int_0^T r(t)dt} E[X^u(T)] = 0, \mathbb{P}-a.s.. \tag{4.12}$$

5. The Admissible Strategies, Optimal Controls and Value Function

We now consider the admissible strategies, optimal controls and characterization of the value function of our problem.

5.1. The admissible strategies

For admissible portfolio strategy $\pi(t)$, $t \in [0,T]$ is given by

$$E\int_0^T \pi(t)^2 dt < \infty. \tag{5.1}$$

For admissible consumption rate $\alpha(t)$, we assume that it is a non-negative and bounded above with upper bound being H such that $0 \leq \alpha(t) \leq H < \infty$.
A strategy $u(\cdot) = \{(\pi(t), \alpha(t)) : t \geq 0\}$ which is progressively measurable with respect to $\{W(t) : 0 \leq s \leq t\}$ is referred to as admissible strategy. We denote the collection of all admissible strategies by $\mathscr{A}$. It then follows that the set of all admissible strategies $\mathscr{A}$ can be defined as follows:

$$\mathscr{A} = \{u(t) = (\pi(t), \alpha(t)) \in [0,T] \times \mathscr{R} : E\int_0^T \pi(t)^2 dt < \infty; 0 \leq \alpha(t) \leq H < \infty\}. \tag{5.2}$$

5.2. The optimal control problems

The goal of the decision maker is to choose the optimal investment and optimal consumption rate under a stochastic contribution rate and a stochastic interest rate, to maximize the expected discounted utility of consumption over a time interval $[0,T]$. For an arbitrary admissible pair of strategy $u = (\pi, \alpha)$, the objective function $G(x,r,c;u)$ follows:

$$G(x,r,c;u) = E\int_0^T e^{-\delta t} V(c(t), \alpha(t) X^u(t)) dt, \tag{5.3}$$

where the function $V(c(t), \alpha(t)X^u(t))$ is utility of consumption, $\delta > 0$ is the discount rate and $X^u(t) = x, r(t) = r, c(t) = c$.
The scalar variables X^u, r and c represent the three state variables, while π and α represent the two control variables.

Assumption 5.1. *We make the following assumptions*

- *V is nonnegative function of c, α, x and u;*
- *there exists a constant $L > 0$ and $n > 0$ such that $\forall (c, \alpha x; u) \in [0,T] \times \mathscr{R}$, we have*

$$V(c, \alpha X; u) \leq L(1 + \alpha^n x^n);$$

- $\delta > \delta_2 > \delta_1$.

Define the value function as

$$U(x,r,c) = \max_{u \in \mathscr{A}} E_t[G(x,r,c;u)], \tag{5.4}$$

subject to (2.2), (3.2) and (4.7), where $E_t[\cdot]$ stands for $E(\cdot | X^u(t) = x, r(t) = r, c(t) = c)$ and E is the mathematical expectation, at time t.

Proposition 5.2. *The admissible control strategy $u = (\pi, \alpha)$ is such that*

$$G(x,r,c;u) \leq \frac{L}{\delta}(1 - e^{-\delta T}) + \frac{LH^n x^n}{\delta_2 - \delta}(e^{(\delta_2 - \delta)T} - 1) < +\infty, \forall (x,r,c) \in [0,T] \times \mathscr{R}.$$

Proof. see Appendix 1. □

We now deduce the following Corollaries.

Corollary 5.3. *If $L\alpha^n x^n \leq L(1 + \alpha^n x^n)$, then $LH^n x^n \leq L(1 + H^n x^n)$.*

Corollary 5.4. *If $T \to +\infty$ then $\lim_{T \to +\infty} G(x,r,c;u) \leq \dfrac{L}{\delta} + \dfrac{LH^n x^n}{\delta - \delta_2} < +\infty$.*

The following proposition gives some properties of our value function.

Proposition 5.5. *Our value function given in (5.4) satisfies the following properties:*

- *$U(x,r,c) \geq 0 \; \forall (x,r,c) \in [0,T] \times \mathscr{R}$;*
- *there exists a constant $M > 0$ and $n > 0$ such that $\forall (x,r,c) \in [0,T] \times \mathscr{R}$, we have*

$$U(x,r,c) \leq M(1 + H^n x^n).$$

Proof. see Appendix 2. □

We again deduce the following Corollaries.

Corollary 5.6. *If $MH^n x^n \leq M(1 + H^n x^n)$, then $Mx^n \leq M(1 + x^n)$.*

Corollary 5.7. *If $T \to +\infty$, then $\lim_{T \to +\infty} G(x,r,c;u) \leq M_1(1 + H^n x^n)$ with $M_1 = \dfrac{L}{\delta - \delta_2} > 0$.*

Proposition 5.8. *Suppose that V is increasing in $x \in [0,T] \times \mathscr{R}$, then U is increasing in $x > 0$. It therefore follows that for $0 < x < x^*$, it implies that $U(x,r,c) \leq U(x^*,r,c)$ $\forall (x^*,r,c) \in [0,T] \times \mathscr{R}$.*

Proof. see Appendix 3. □

Proposition 5.9. *For all $u \in \mathscr{A}$ and $(x,r,c) \in [0,T] \times \mathscr{R}$, the following holds*

$$\max_u \left\{ \lim_{T \to +\infty} e^{-\delta T} E[U(X^u(T), r(T), c(T))] \leq 0 \right\} = 0.$$

Proof. see Appendix 4. □

Proposition 5.10. *The following hypotheses hold:*

- *σ_r is continuous function on $t \in [0,T]$;*
- *ψ and σ_c are Lipschitz continuous functions on $(c,y,x) \in [0,T] \times \mathscr{R}$, uniformly in $u \in \mathscr{A}$;*
- *V is continuous in $(x,r,c) \in [0,T] \times \mathscr{R}$, uniformly in $u \in \mathscr{A}$;*
- *r is continuous in $t \in [0,T]$;*

Then, our value function U is continuous in $(x,r,c) \in [0,T] \times \mathscr{R}$.

Proof. see Appendix 5. □

5.3. Characterization of our value function

In solving a stochastic control problem, the techniques of dynamic programming is adopted. It requires the consideration of the operator of the controlled process and use it to derive a partial differential equation referred to as HJB equation that is satisfied by the value function. For simplicity, we shall drop t in our functionals from now on. The solution of the HJB equation then yields the optimal controls and the optimal value, by assuming the existence of optimal control, for an arbitrary value function $U(\cdot,\cdot) \in \mathscr{C}^{2,2}(\mathscr{R} \times \mathscr{R})$, define an operator $\mathscr{L}^u$ by

$$\begin{aligned} \mathscr{L}^u U(x,r,c) &= \kappa(\theta - r)U_r + \frac{1}{2}\sigma_r\sigma_r' U_{rr} + \psi(c,X,Y)U_c + \frac{1}{2}\sigma_c(c,X,Y)\sigma_c(c,X,Y)'U_{cc} \\ &+ x(r + \pi(\mu - r) + \beta cy - \alpha)U_x + \frac{1}{2}x^2(\pi\sigma_S + cy\sigma_Y)(\pi\sigma_S + cy\sigma_Y)'U_{xx} \\ &+ \sigma_r\sigma_c(c,X,Y)'U_{rc} + x(\pi\sigma_S + cy\sigma_Y)\sigma_r' U_{rx} + x(\pi\sigma_S + cy\sigma_Y)\sigma_c(c,X,Y)'U_{cx}, \end{aligned} \tag{5.5}$$

where U_r, U_{rx}, U_{rc}, U_{rr}, U_x, U_{xx}, U_{cx}, U_c, U_{cc} denote the first-and second-order derivatives with respect to r, c and x. Formally, the value function (5.4) satisfies the HJB equation

$$\max_u \{\mathscr{L}^u U(x,r,c) + V(c,C)\} = 0, \tag{5.6}$$

where $V(c,C)$ is the utility function with respect to consumption of a PPM. We assume that the consumption process of a PPM depends on the contribution rate. In proving the proposition for the viscosity solution, we follow the framework of Brachetta and Ceci (2021).
We now give the definition of viscosity solution to (5.6).

Definition 5.11. *Let $\nu : [0,T] \times \mathscr{R} \to [0,T]$ be locally bounded and continuous function in $(\bar{x},\bar{r},\bar{c})[0,T] \times \mathscr{R}$. We define*

- *ν as a viscosity subsolution of (5.6), if*

$$\inf_{u \in \mathscr{A}} \{\mathscr{L}^u \bar{U}(\bar{x},\bar{r},\bar{c}) + V(\bar{c},\bar{C})\} \geq 0, \tag{5.7}$$

  for all $(\bar{x},\bar{r},\bar{c}) \in [0,T] \times \mathscr{R}$ and for all $\bar{U} \in \mathscr{C}^{2,2}([0,T] \times \mathscr{R})$ such that $(\bar{x},\bar{r},\bar{c})$ gives the maximum point of $\nu - \bar{U}$;
- *ν as a viscosity supersolution of (5.6), if*

$$\inf_{u \in \mathscr{A}} \{\mathscr{L}^u \bar{U}(\bar{x},\bar{r},\bar{c}) + V(\bar{c},\bar{C})\} \leq 0, \tag{5.8}$$

  for all $(\bar{x},\bar{r},\bar{c}) \in [0,T] \times \mathscr{R}$ and for all $\bar{U} \in \mathscr{C}^{2,2}([0,T] \times \mathscr{R})$ such that $(\bar{x},\bar{r},\bar{c})$ gives the minimum point of $\nu - \bar{U}$;
- *ν as a viscosity solution of (5.6), if it is both a viscosity subsolution and a viscosity supersolution of (5.6).*

Theorem 5.12. *Our value function U given in (5.4) is a viscosity solution of our HJB equation (5.6).*

Proof. see Appendix 6. □

Next, we give the Verification Theorem based on the classical solutions of the HJB equation of (5.6).

Theorem 5.13. *(Verification theorem): Let $\nu : [0,T] \times \mathscr{R} \to \mathscr{R}$ be a function in $C^{2,2}([0,T] \times \mathscr{R})$ and suppose there exists a constant $N > 0$ such that*

$$\nu(x,r,c) \leq N(1 + \alpha^m x^m) \ \forall (x,r,c) \in [0,T] \times \mathscr{R}.$$

Suppose

$$\max_u \{\mathscr{L}^u \nu(x,r,c) + V(c,\alpha x; u)\} \leq 0 \ \forall (x,r,c) \in [0,T] \times \mathscr{R}, \tag{5.9}$$

then $\nu(x,r,c) \leq U(x,r,c)$ $\forall (x,r,c) \in [0,T] \times \mathscr{R}$. Again, suppose that there exists a $\mathscr{A}-$valued measurable function $u^*(x,r,c)$ such that

$$\max_u \{\mathscr{L}^u \nu(x,r,c) + V(c,\alpha x; u)\} = \mathscr{L}^{u^*} \nu(x^*,r,c) + V(c,\alpha^* x^*; u^*) = 0 \ \forall (x,r,c) \in [0,T] \times \mathscr{R}.$$

Then, $\nu(x,r,c) = U(x,r,c)$ $\forall (x,r,c) \in [0,T] \times \mathscr{R}$ and $u^* = \{u^*(X^{u^*}(t), r(t), c(t))\}_{t \in [0,T]}$ is an optimal control of our problem.

Proof. see Appendix 7. □

6. Optimal investment, optimal consumption of a PPM and some numerical examples

In this section, we consider the optimal investment and optimal consumption rate of a PPM as well as some numerical examples.

6.1. Optimal investment and optimal consumption of a PPM

Considering the two control parameters π and c. We have that (5.6) can be rewritten as

$$\begin{aligned}&\max_u\{V(c,C)+\kappa(\theta-r)U_r+\frac{1}{2}\sigma_r\sigma_r'U_{rr}+\psi(c,X,Y)U_c\\&+\frac{1}{2}\sigma_c(c,X,Y)\sigma_c(c,X,Y)'U_{cc}+rxU_x+x\pi(\mu-r)U_x+x\beta cyU_x-x\alpha U_x\\&+\frac{1}{2}x^2\pi^2\sigma_S\sigma_S'U_{xx}+\frac{1}{2}x^2cy\pi\sigma_Y\sigma_S'U_{xx}+\frac{1}{2}x^2cy\pi\sigma_S\sigma_Y'U_{xx}\\&+\frac{1}{2}x^2c^2y^2\sigma_Y\sigma_Y'U_{xx}+\sigma_r\sigma_c(c,X,Y)'U_{rc}+x\pi\sigma_S\sigma_r'U_{rx}+xcy\sigma_Y\sigma_r'U_{rx}\\&+x\pi\sigma_S\sigma_c(c,X,Y)'U_{cx}+xcy\sigma_Y\sigma_c(c,X,Y)'U_{cx}\}=0.\end{aligned} \tag{6.1}$$

Remark 6.1. *We give the following remarks:*

- *Here, we take $V(c,C)=L(c)C^n$, it implies that $V(c,C)=\frac{C^{1-\gamma}}{1-\gamma}e^{g(c)}$ which is the utility function of a PPM with respect to consumption, for $\gamma>0$, such that $\gamma\in(0,1)\cup(1,\infty)$, where $n=1-\gamma$ is the constant relative risk aversion parameter and $L(c)=\frac{e^{g(c)}}{1-\gamma}$.*
- *Note that $V(c,C)=V(c,\alpha x)$ which implies that $V(c,\alpha x)=L(c)(\alpha x)^n$ and $V(c,\alpha x)=\frac{(\alpha x)^{1-\gamma}}{1-\gamma}e^{g(c)}$.*
- *We take $U(x,r,c)=M(r,c)x^n$, it then follows that $M(r,c)x^n=\frac{x^{1-\gamma}}{1-\gamma}e^{h(r,c)}$, where $M(r,c)=\frac{e^{h(r,c)}}{1-\gamma}$.*
- *We now have a solution of $U(x,r,c)$ to be of the form*

$$U(x,r,c)=\frac{x^{1-\gamma}}{1-\gamma}e^{h(r,c)}. \tag{6.2}$$

Finding the partial derivatives of (6.2) with respect to x, xx, c, cx and cc, r, rc, rx, rr, we have the following: $U_r=h_rU$, $U_{rc}=(h_{rc}+h_ch_r)U$, $U_{rr}=(h_r^2+h_{rr})U$, $U_{rx}=\frac{1-\gamma}{x}h_rU$, $U_c=h_cU$, $U_{cc}=(h_c^2+h_{cc})U$, $U_{cx}=\frac{1-\gamma}{x}h_cU$, $U_x=\frac{1-\gamma}{x}U$, $U_{xx}=\frac{-\gamma(1-\gamma)}{x^2}U$. Then substituting into (6.1), we have the following:

$$\begin{aligned}&\max_u\{\left(\frac{C}{x}\right)^{1-\gamma}e^{(g(c)-h(r,c))}+\kappa(\theta-r)h_r+\frac{1}{2}\sigma_r\sigma_r'(h_r^2+h_{rr})\\&+\psi(c,X,Y)h_c+\frac{1}{2}\sigma_c(c,X,Y)\sigma_c(c,X,Y)'(h_c^2+h_{cc})+r(1-\gamma)\\&+\pi(\mu-r)(1-\gamma)+\beta cy(1-\gamma)-\alpha(1-\gamma)-\frac{1}{2}\pi^2\sigma_S\sigma_S'\gamma(1-\gamma)\\&-cy\pi\sigma_Y\sigma_S'\gamma(1-\gamma)-\frac{1}{2}c^2y^2\sigma_Y\sigma_Y'\gamma(1-\gamma)\\&+\sigma_r\sigma_c(c,X,Y)'(h_{rc}+h_ch_r)+\pi\sigma_S\sigma_r'(1-\gamma)h_r+cy\sigma_Y\sigma_r'(1-\gamma)h_r\\&+\pi\sigma_S\sigma_c(c,X,Y)'(1-\gamma)h_c+cy\sigma_Y\sigma_c(c,X,Y)'(1-\gamma)h_c\}=0.\end{aligned} \tag{6.3}$$

Then, using the fact that $C=\alpha x$, (6.3) becomes

$$\begin{aligned}&\max_\alpha\{\frac{\alpha^{1-\gamma}}{1-\gamma}e^{(g(c)-h(r,c))}-\alpha\}+\max_\pi\{\pi(\mu-r)-\frac{1}{2}\pi^2\sigma_S\sigma_S'\gamma\\&-cy\pi\sigma_S\sigma_Y'\gamma+\pi\sigma_S\sigma_r'h_r+\pi\sigma_S\sigma_c(c,X,Y)'h_c\}+f(c,X,Y)=0,\end{aligned} \tag{6.4}$$

where

$$\begin{aligned}f(c,X,Y)=&\frac{\kappa(\theta-r)h_r}{1-\gamma}+r+\beta cy\\&+\frac{1}{2(1-\gamma)}\sigma_r\sigma_r'(h_r^2+h_{rr})+\frac{\psi(c,X,Y)h_c}{1-\gamma}+\frac{1}{2(1-\gamma)}\sigma_c(c,X,Y)\sigma_c(c,X,Y)'(h_c^2+h_{cc})\\&-\frac{1}{2}c^2y^2\sigma_Y\sigma_Y'\gamma+\frac{1}{1-\gamma}\sigma_r\sigma_c(c,X,Y)'(h_{rc}+h_ch_r)+cy\sigma_Y\sigma_r'h_r+cy\sigma_Y\sigma_c(c,X,Y)'h_c.\end{aligned}$$

By first order condition, we have that

$$\frac{\alpha^{1-\gamma}}{1-\gamma}e^{(g(c)-h(r,c))}-\alpha=0,$$

$$\pi(\mu-r)-\frac{1}{2}\pi^2\sigma_S\sigma_S'\gamma-cy\pi\sigma_S\sigma_Y'\gamma+\pi\sigma_S\sigma_r'h_r+\pi\sigma_S\sigma_c(c,X,Y)'h_c=0$$

and

$$f(c,X,Y)=0.$$

Proposition 6.2. *The CGR of a PPM contribution rate is*

$$\begin{aligned}\psi(c,X,Y) &= \frac{1-\gamma}{2h_c}c^2y^2\sigma_Y\sigma_Y'\gamma-\kappa(\theta-r)\frac{h_r}{h_c}-\frac{1-\gamma}{h_c}(r+\beta cy+cy\sigma_Y\sigma_r'h_r\\ &+cy\sigma_Y\sigma_c'h_c)-\frac{1}{2h_c}\sigma_r\sigma_r'(h_r^2+h_{rr})-\frac{1}{2h_c}\sigma_c(c,X,Y)\sigma_c(c,X,Y)'(h_c^2+h_{cc})\\ &-\sigma_r\sigma_c(c,X,Y)'\frac{h_{rc}+h_ch_r}{h_c}.\end{aligned} \tag{6.5}$$

Proof. Setting $f(c,X,Y)=0$ and solve for $\psi(c,X,Y)$, the result follows. □

(6.5) gives the drift term of the contribution rate of a PPM over time.
We now have the following proposition for optimal investment and consumption processes of a PPM over time.

Proposition 6.3. *The optimal investment strategy in a risky and riskless assets, and optimal consumption rate of a PPM are*

$$\pi^*=\frac{\mu-r}{\sigma_S\sigma_S'\gamma}-\frac{cy\sigma_S\sigma_Y'}{\sigma_S\sigma_S'}+\frac{\sigma_S\sigma_r'h_r}{\sigma_S\sigma_S'\gamma}+\frac{\sigma_S\sigma_c(c,X^*,Y)'h_c}{\sigma_S\sigma_S'\gamma}, \tag{6.6}$$

$$\pi_0^*=1-\frac{\mu-r}{\sigma_S\sigma_S'\gamma}+\frac{cy\sigma_S\sigma_Y'}{\sigma_S\sigma_S'}-\frac{\sigma_S\sigma_r'h_r}{\sigma_S\sigma_S'\gamma}-\frac{\sigma_S\sigma_c(c,X^*,Y)'h_c}{\sigma_S\sigma_S'\gamma}, \tag{6.7}$$

and

$$\alpha^*=e^{-\frac{1}{\gamma}(h(r,c)-g(c))} \tag{6.8}$$

respectively.

Proof. From (6.4), we apply the first order conditions and then solve for the control parameters, the results follow. □
Now, solving the following ODE $h_{rc}=h_rh_c$, we have a solution of the form: $h(r,c)=-\log_e(rc)$. It then follow that $h_c=-\frac{1}{c}$ and $h_{cc}=\frac{1}{c^2}$, $h_r=-\frac{1}{r}$ and $h_{rr}=\frac{1}{r^2}$. Here, we assuming a solution of the form $\sigma_c(c,X^*,Y):=-\log_e\left(\frac{c\sigma_c}{X^*Y}\right)$, where $\sigma_c\in\mathscr{R}^4$. Therefore, the optimal investment strategy in a risky asset, optimal consumption rate and the drift of the contribution rate of a PPM will now become

$$\pi^*=\frac{\mu-r}{\sigma_S\sigma_S'\gamma}-\frac{c^*Y\sigma_S\sigma_Y'}{\sigma_S\sigma_S'X^*}-\frac{\sigma_S\sigma_r'}{r\sigma_S\sigma_S'\gamma}-\frac{\sigma_S(\ln(\frac{c^*\sigma_c}{X^*Y}))'}{c^*\sigma_S\sigma_S'\gamma}, \tag{6.9}$$

$$\alpha^*=(rc^*)^{\frac{1}{\gamma}}e^{\frac{1}{\gamma}g(c)}, \tag{6.10}$$

and

$$\begin{aligned}\psi(c^*,X^*,Y) &= -\frac{1-\gamma}{2}c^{*3}y^2\sigma_Y\sigma_Y'\gamma-\frac{c^*\kappa(\theta-r)}{r}+c^*(1-\gamma)(r+\beta c^*y-\frac{c^*y\sigma_Y\sigma_r'}{r}\\ &-y\sigma_Y\sigma_c(c^*,X^*,Y)')+\frac{c\sigma_r\sigma_r'}{r^2}+\frac{\sigma_c(c^*,X^*,Y)\sigma_c(c^*,X^*,Y)'}{c^*}-\frac{2\sigma_r\sigma_c(c^*,X^*,Y)'}{r}.\end{aligned} \tag{6.11}$$

It then follows that

$$\begin{aligned}\psi(c^*,X^*,Y) &= -\frac{1-\gamma}{2X^{*2}}c^{*3}Y^2\sigma_Y\sigma_Y'\gamma-\frac{c^*\kappa(\theta-r)}{r}+c^*(1-\gamma)(r+\frac{\beta c^*Y}{X^*}\\ &-\frac{c^*Y\sigma_Y\sigma_r'}{rX^*}+\frac{Y\sigma_Y\left(\log_e\left(\frac{c^*\sigma_c}{X^*Y}\right)\right)'}{X^*})+\frac{c^*\sigma_r\sigma_r'}{r^2}+\frac{\left(\log_e\left(\frac{c^*\sigma_c}{X^*Y}\right)\right)\left(\log_e\left(\frac{c^*\sigma_c}{X^*Y}\right)\right)'}{c^*}\\ &-\frac{2\sigma_r\left(\log_e\left(\frac{c^*\sigma_c}{X^*Y}\right)\right)'}{r}.\end{aligned} \tag{6.12}$$

Observe (6.9) that the optimal investment of a PPM depend on the contribution rate, contribution process, market price of stock risks, contribution rate risks, salary process, salary shocks, interest rate, interest rate risks and coefficient of risk averse. All these are factors that may influence the contribution rate of a PPM contribution process over time. We also observe that the optimal investment of a PPM depend directly on the contribution of a PPM and inversely on the short-term interest rate and the contribution rate process of a PPM overtime.
From (6.9), it shows that it is optimal to invest in a portfolio that comprises of four components:

1. a speculative portfolio $\frac{\mu-r}{\sigma_S\sigma_S'\gamma}$ proportional to the market price of risks corresponding to the risky asset through the relative risk averse index $\frac{1}{\gamma}$,
2. a contribution rate risks hedging portfolio $\frac{c^*Y\sigma_S\sigma_Y'}{\sigma_S\sigma_S'X^*}$ proportional to the contribution process of a PPM and diffusion term of the salary process, and inversely proportional to the risks arising from the risky asset and the optimal wealth process of s PPM over time,
3. an inter-temporary short-term interest rate hedging portfolio $\frac{\sigma_S\sigma_r'}{r\sigma_S\sigma_S'\gamma}$ proportional to the risks arising from the short-term interest rate and inversely proportional to the short-term interest rate and the relative risk averse index γ of a PPM over time.

4. an inter-temporary contribution rate hedging portfolio $\frac{\sigma_S[\ln(\frac{c^*\sigma_c}{X^*Y})]'}{c^*\sigma_S\sigma_S'\gamma}$ proportional to the risks arising from the contribution rate, salary and wealth processes of a PPM and inversely proportional to the contribution rate, risks arising from the risky asset and the relative risk averse index $\frac{1}{\gamma}$ of a PPM over time.

(6.10) shows the optimal consumption rate of a PPM over time. It is found to depend on the short-term interest rate, contribution rate and the relative risk averse index of a PPM over time.
From (6.12), we found that the contribution rate of a PPM depend on the optimal wealth, salary, contribution, contribution rate processes, short-term interest rate and coefficient of risk averse of a PPM over time. This makes sense since: if the wealth of a PPM is growing positively, the PPM will want to make more contributions into the scheme over time, and contributes less if the wealth is going down. Another factor that can influence the contribution rate of a PPM, is the salary process. If a PPM salary is not enough to take care of his or her needs, the contribution rate will be affected negatively. If the salary is more than enough for a PPM, he or she may like to contribute more into the scheme especially if the investment portfolio is encouraging. The short-term interest rate is one of the factors that can influence the contribution rate of a PPM over time. Finally, the risk averse coefficient of a PPM can also affect the contribution rate of a PPM. If a PPM is risk loving, he or she may even contribute more of the salary into the scheme irrespective of the level of risks involve. If a PPM is risk averse, he or she may contribute less or nothing of his or her salary into the scheme, if the risks involve is high.

6.2. Optimal contribution rate of a PPM

Now, using (6.12) on (3.3), we have the optimal contribution rate of a PPM at time t as follows:

$$c^*(t) = c_0 + \int_0^t [\psi(s,c^*,X^*,Y)ds + \sigma_c(s,c^*,X^*,Y)dW(s)]. \tag{6.13}$$

It therefore follows that the optimal AVC rate of a PPM at time t is

$$c_A^*(t) = \int_0^t [\psi(s,c^*,X^*,Y)ds + \sigma_c(s,c^*,X^*,Y)dW(s)].$$

At $t = T$, we have the optimal terminal contribution rate of a PPM to be

$$c^*(T) = c_0 + \int_0^T [\psi(t,c^*,X^*,Y)dt + \sigma_c(t,c^*,X^*,Y)dW(t)].$$

At $t = T$, the optimal terminal AVC rate of a PPM is obtained as

$$c_A^*(T) = \int_0^T [\psi(t,c^*,X^*,Y)dt + \sigma_c(t,c^*,X^*,Y)dW(t)].$$

It is imperative for a PPM to know what to contribute when the control strategy is of class u_1 or u_2. We now have the following proposition

Proposition 6.4. *Suppose that the investment strategy is in class*

- u_1, *then for* $\sigma_c = \frac{1}{c^*}\mathrm{I}$,

$$c^* > \frac{zX^* + \sqrt{z^2X^{*2} - \frac{4X^*Yr^2\gamma\varpi}{\sigma_S\sigma_Y'}}}{2r\gamma Y} \text{ or } c^* > \frac{zX^* - \sqrt{z^2X^{*2} - \frac{4X^*Yr^2\gamma\varpi}{\sigma_S\sigma_Y'}}}{2r\gamma Y}, \tag{6.14}$$

- u_2, *then for* $\sigma_c = \frac{1}{c^*}\mathrm{I}$,

$$c^* < \frac{zX^* + \sqrt{z^2X^{*2} - \frac{4X^*Yr^2\gamma\varpi}{\sigma_S\sigma_Y'}}}{2r\gamma Y} \text{ or } c^* < \frac{zX^* - \sqrt{z^2X^{*2} - \frac{4X^*Yr^2\gamma\varpi}{\sigma_S\sigma_Y'}}}{2r\gamma Y}, \tag{6.15}$$

where $z = \frac{r(\mu-r)-\sigma_S\sigma_r'}{\sigma_S\sigma_Y'}$, $\varpi = \sigma_S(\log_e(\frac{1}{X^*Y}\mathrm{I}))'$ *and* $\mathrm{I} = [1,1,1,1]$.

Proof. When the investment strategy is in class u_1, it then follows that

$$\pi^* = \frac{\mu - r}{\sigma_S\sigma_S'\gamma} - \frac{c^*Y\sigma_S\sigma_Y'}{\sigma_S\sigma_S'X^*} - \frac{\sigma_S\sigma_r'}{r\sigma_S\sigma_S'\gamma} - \frac{\sigma_S(\ln(\frac{c^*\sigma_c}{X^*Y}))'}{c^*\sigma_S\sigma_S'\gamma} = \pi_1^* < 0 \tag{6.16}$$

It then implies that

$$r\sigma_S\sigma_S'\gamma^2Y\sigma_S\sigma_Y'c^{*2} + \gamma\sigma_S\sigma_S'X^*[\sigma_S\sigma_r' - r(\mu - r)]c^* + r\sigma_S\sigma_S'\gamma\varpi X^* > 0. \tag{6.17}$$

Solving the quadratic equation (6.17), we have

$$c^* > \frac{zX^* + \sqrt{z^2X^{*2} - \frac{4X^*Yr^2\gamma\varpi}{\sigma_S\sigma_Y'}}}{2r\gamma Y} \text{ or } c^* > \frac{zX^* - \sqrt{z^2X^{*2} - \frac{4X^*Yr^2\gamma\varpi}{\sigma_S\sigma_Y'}}}{2r\gamma Y}. \tag{6.18}$$

In a similar manner, when the investment strategy is in class u_2, then

$$c^* < \frac{zX^* + \sqrt{z^2X^{*2} - \frac{4X^*Yr^2\gamma\varpi}{\sigma_S\sigma_Y'}}}{2r\gamma Y} \text{ or } c^* < \frac{zX^* - \sqrt{z^2X^{*2} - \frac{4X^*Yr^2\gamma\varpi}{\sigma_S\sigma_Y'}}}{2r\gamma Y}. \tag{6.19}$$

□

Corollary 6.5. *The contribution rate c is real if* $z > 2r\sqrt{\frac{Y\gamma\varpi}{\sigma_S\sigma_Y'X^*}}$ *and complex if* $z < 2r\sqrt{\frac{Y\gamma\varpi}{\sigma_S\sigma_Y'X^*}}$.

6.3. Some numerical examples

We now verify our optimal investment and consumption rate in (6.6) and (6.8) respectively, using numerical examples. Figure 6.1 shows the optimal consumption rate for varying $h(r,c)$ and $g(c)$, and $\gamma = 20$.

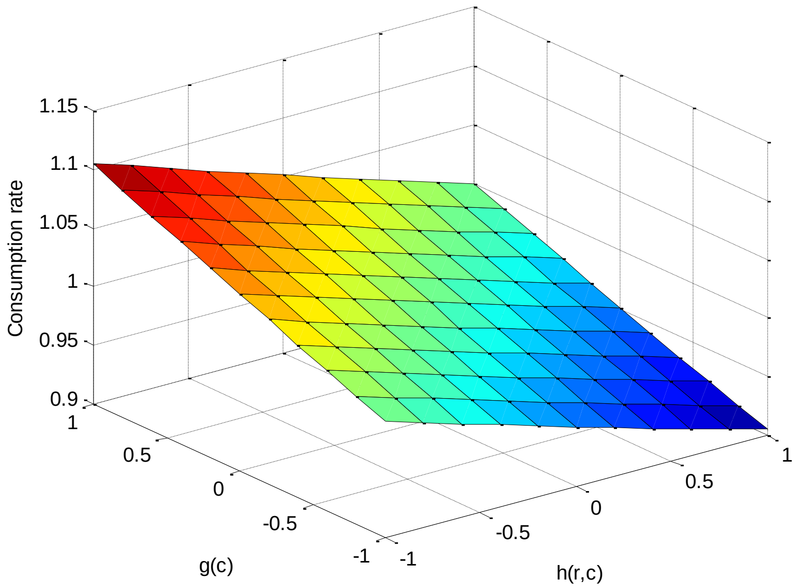

Figure 6.1: Optimal consumption rate for $\gamma = 20$

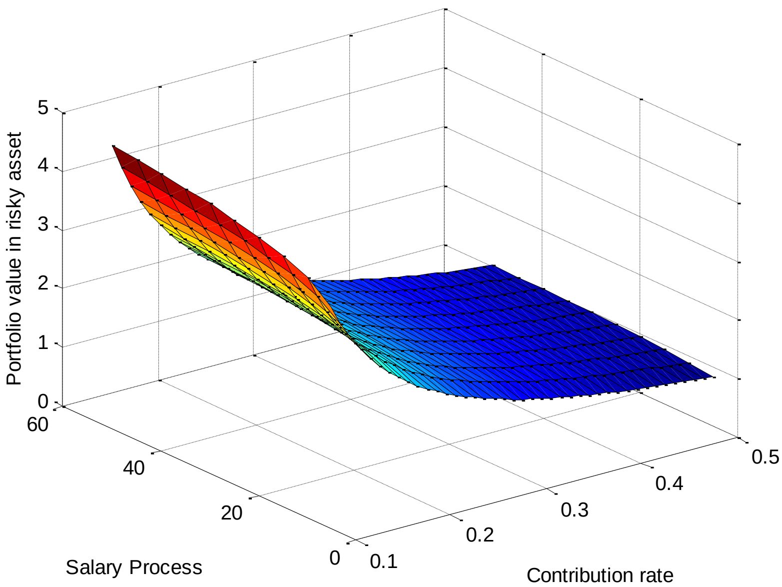

Figure 6.2: Optimal portfoilo value in stock for $\mu = 0.28$, $r = 0.15$, $\gamma = 20$, $\sigma_S = [0.35, 0.27, 0.3, 0.28]$, $\sigma_Y = [0.12, 0.20, 0.16, 0.06]$, $\sigma_r = [0.08, 0.07, 0.15, 0.06]$, $\sigma_c(c, X^*, Y) = [0.2, 0.3, 0.1, 0.35]$, $h_r = -6.7$, $h_c = -6.7$

Figure 6.2 shows the optimal investment portfolio value in the risky asset for varying salary process and contribution rate of a PPM over time, for all other parameters remain fixed. It is observe that the investment portfolio in the risky asset is positive. This shows that more of the investment should remain in stock, which implies that the portfolio value is in class u_2. Figure 6.3 shows the optimal investment portfolio value in the riskless asset for varying salary process and contribution rate of a PPM over time, for all other parameters remain fixed. It is also observe that the investment portfolio in the riskless asset is negative. This shows that little or no investment should be in riskless asset, since the portfolio value is in class u_2. In otherwords, the lower the contribution rate, the lower the investment in the riskless asset and vice versa. The negative value indicates that some fund need to be moved from the riskless asset to fund the risky assets over time, since the risky assets are performing better than the riskless asset.

Figure 6.4 shows the optimal investment portfolio value in the risky asset for varying salary process and optimal wealth of a PPM over time, for all other parameters remain fixed. It is observe that the investment portfolio in the risky asset is positive. This also shows that more of the

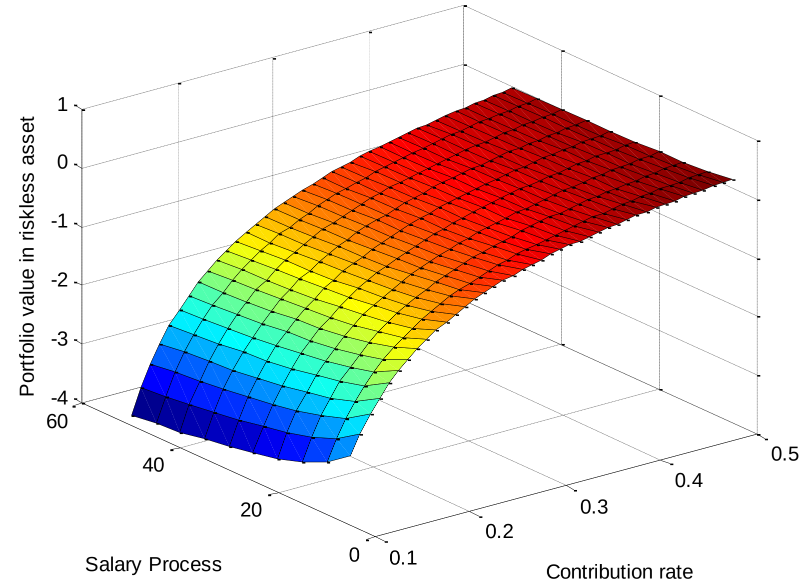

Figure 6.3: Optimal portfoilo value in riskless asset for $\mu = 0.28$, $r = 0.15$, $\gamma = 20$, $\sigma_S = [0.35, 0.27, 0.3, 0.28]$, $\sigma_Y = [0.12, 0.20, 0.16, 0.06]$, $\sigma_r = [0.08, 0.07, 0.15, 0.06]$, $\sigma_c(c, X^*, Y) = [0.2, 0.3, 0.1, 0.35]$, $h_r = -6.7$, $h_c = -6.7$

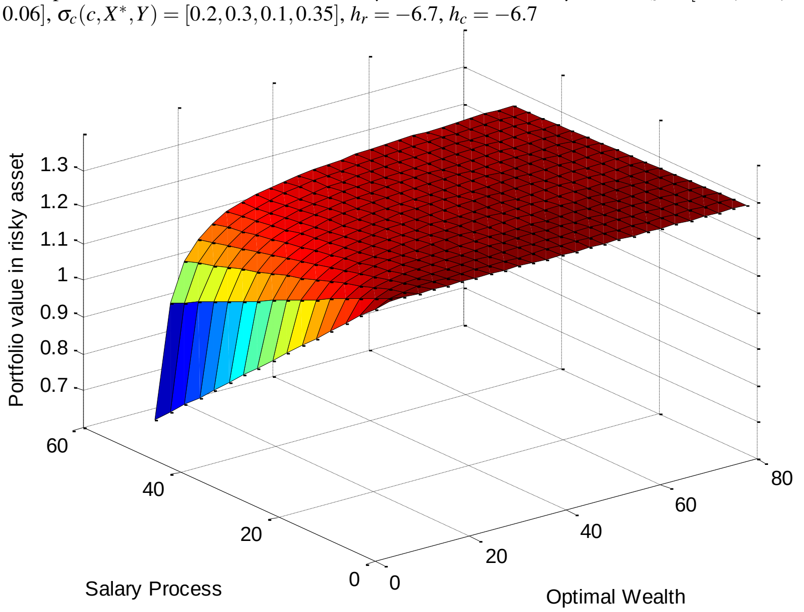

Figure 6.4: Optimal portfoilo value in risky asset for $\mu = 0.28$, $r = 0.15$, $c = 0.15$, $\gamma = 20$, $\sigma_S = [0.35, 0.27, 0.3, 0.28]$, $\sigma_Y = [0.12, 0.20, 0.16, 0.06]$, $\sigma_r = [0.08, 0.07, 0.15, 0.06]$, $\sigma_c = [7.33, 7.33, 6.67, 6.67]$, $\sigma_c(c, X^*, Y) = -\log_e(\frac{c\sigma_c}{X^*Y})$

investment should remain in stock, which implies that the portfolio value is in class u_2. Figure 6.5 shows the optimal investment portfolio value in the riskless asset for varying salary process and optimal wealth of a PPM over time, for all other parameters remain fixed. Therefore, from figure 6.4 and figure 6.5, we have that more the investment portfolio should remain in the risky asset than the riskless asset over time.

6.4. The explicit form of our HJB equation

The explicit form of our HJB equation is given by the following: substituting (6.9) and (6.10) into (6.4), we have

$$\begin{aligned}
&\frac{\gamma}{1-\gamma}e^{-\frac{1}{\gamma}(h(r,c)-g(c))} + \frac{(\mu-r)^2}{2\sigma_S\sigma_S'\gamma} + \frac{c^2y^2(\sigma_S\sigma_Y')^2\gamma}{2\sigma_S\sigma_S'} + \frac{(\sigma_S\sigma_r')^2h_r^2}{2\sigma_S\sigma_S'\gamma} \\
&+\frac{(\sigma_S\sigma_c(c,X^*,Y)')^2h_c^2}{2\sigma_S\sigma_S'\gamma} - \frac{(\mu-r)\sigma_S\sigma_c(c,X^*,Y)'h_r}{\sigma_S\sigma_S'\gamma} + 2\frac{(\mu-r)\sigma_S\sigma_c(c,X^*,Y)'h_c}{\sigma_S\sigma_S'\gamma} \\
&-\frac{cy(\mu-r)\sigma_S\sigma_Y'}{\sigma_S\sigma_S'} - \frac{cy(\sigma_S\sigma_Y')(\sigma_S\sigma_c(c,X^*,Y)')h_c}{\sigma_S\sigma_S'} + \frac{(\mu-r)\sigma_S\sigma_r'h_r}{2\sigma_S\sigma_S'\gamma} \\
&+\frac{(\sigma_S\sigma_r')(\sigma_S\sigma_c(c,X^*,Y)')h_ch_r}{\sigma_S\sigma_S'\gamma} - \frac{cy(\sigma_S\sigma_Y')(\sigma_S\sigma_r')h_r}{\sigma_S\sigma_S'} + f(c,X^*,Y) = 0,
\end{aligned} \tag{6.20}$$

Now, using the fact that $h(r,c) := -\log_e(rc)$, we have

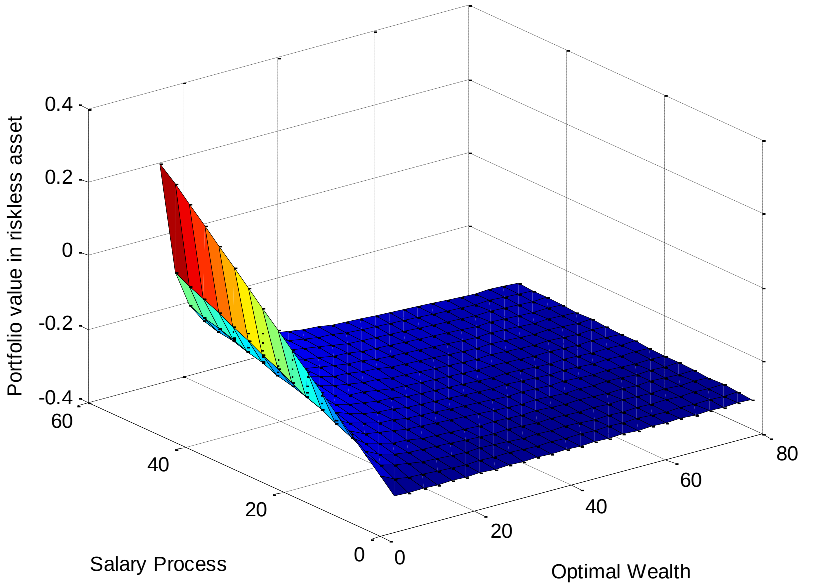

Figure 6.5: Optimal portfoilo value in riskless asset for $\mu = 0.28$, $r = 0.15$, $c = 0.15$, $\gamma = 20$, $\sigma_S = [0.35, 0.27, 0.3, 0.28]$, $\sigma_Y = [0.12, 0.20, 0.16, 0.06]$, $\sigma_r = [0.08, 0.07, 0.15, 0.06]$, $\sigma_c = [7.33, 7.33, 6.67, 6.67]$, $\sigma_c(c, X^*, Y) = -\log_e(\frac{c\sigma_c}{X^*Y})$

$$\begin{aligned}
&\frac{\gamma(rc)^{\frac{1}{\gamma}}}{1-\gamma}e^{\frac{1}{\gamma}g(c)} + \frac{(\mu-r)^2}{2\sigma_S\sigma_S'\gamma} + \frac{c^2y^2(\sigma_S\sigma_Y')^2\gamma}{2\sigma_S\sigma_S'} + \frac{(\sigma_S\sigma_r')^2}{2r^2\sigma_S\sigma_S'\gamma}\\
&+\frac{(\sigma_S\sigma_c(c,X^*,Y)')^2}{2c^2\sigma_S\sigma_S'\gamma} + \frac{(\mu-r)\sigma_S\sigma_c(c,X^*,Y)'}{r\sigma_S\sigma_S'\gamma} - 2\frac{(\mu-r)\sigma_S\sigma_c(c,X^*,Y)'}{c\sigma_S\sigma_S'\gamma}\\
&-\frac{cy(\mu-r)\sigma_S\sigma_Y'}{\sigma_S\sigma_S'} + \frac{y(\sigma_S\sigma_Y')(\sigma_S\sigma_c(c,X^*,Y)')}{\sigma_S\sigma_S'} - \frac{(\mu-r)\sigma_S\sigma_r'}{2r\sigma_S\sigma_S'\gamma}\\
&+\frac{(\sigma_S\sigma_r')(\sigma_S\sigma_c(c,X^*,Y)')}{rc\sigma_S\sigma_S'\gamma} + \frac{cy(\sigma_S\sigma_Y')(\sigma_S\sigma_r')}{r\sigma_S\sigma_S'} + f(c,X^*,Y) = 0,
\end{aligned} \tag{6.21}$$

Using the fact that $\sigma_c(c,X^*,Y) := -\log_e(\dfrac{c\sigma_c}{X^*Y})$, we have

$$\begin{aligned}
&\frac{\gamma(rc)^{\frac{1}{\gamma}}}{1-\gamma}e^{\frac{1}{\gamma}g(c)} + \frac{(\mu-r)^2}{2\sigma_S\sigma_S'\gamma} + \frac{c^2y^2(\sigma_S\sigma_Y')^2\gamma}{2\sigma_S\sigma_S'} + \frac{(\sigma_S\sigma_r')^2}{2r^2\sigma_S\sigma_S'\gamma}\\
&+\frac{(\sigma_S(\log_e(\frac{c\sigma_c}{X^*Y}))')^2}{2c^2\sigma_S\sigma_S'\gamma} - \frac{(\mu-r)\sigma_S(\log_e(\frac{c\sigma_c}{X^*Y}))'}{r\sigma_S\sigma_S'\gamma} + 2\frac{(\mu-r)\sigma_S(\log_e(\frac{c\sigma_c}{X^*Y}))'}{c\sigma_S\sigma_S'\gamma}\\
&-\frac{cy(\mu-r)\sigma_S\sigma_Y'}{\sigma_S\sigma_S'} - \frac{y(\sigma_S\sigma_Y')(\sigma_S(\log_e(\frac{c\sigma_c}{X^*Y}))')}{\sigma_S\sigma_S'} - \frac{(\mu-r)\sigma_S\sigma_r'}{2r\sigma_S\sigma_S'\gamma}\\
&-\frac{(\sigma_S\sigma_r')(\sigma_S(\log_e(\frac{c\sigma_c}{X^*Y}))')}{rc\sigma_S\sigma_S'\gamma} + \frac{cy(\sigma_S\sigma_Y')(\sigma_S\sigma_r')}{r\sigma_S\sigma_S'} + f(c,X^*,Y) = 0.
\end{aligned} \tag{6.22}$$

(6.22) gives the explicit form of our HJB equation.

7. Sensitivity Analyses of $\psi(c, X^*, Y)$

In this section, we give the sensitivity analyses of the drift term of a PPM contribution rate.
First, we consider the change in the drift term with respect to the wealth process of a PPM. It is given by

$$\begin{aligned}
\frac{\partial\psi(c,X^*,Y)}{\partial X^*} &= \frac{1}{c}\left(\sigma_c(c,X^*,Y)\frac{\partial\sigma_c(c,X^*,Y)'}{\partial X^*} + \frac{\partial\sigma_c(c,X^*,Y)}{\partial X^*}\sigma_c(c,X^*,Y)'\right)\\
&+c(1-\gamma)\left(\frac{cY\sigma_Y\sigma_r'}{rX^{*2}} - \frac{\beta cY}{X^{*2}} - y\sigma_Y\frac{\partial\sigma_c(c,X^*,Y)'}{\partial X^*} + \frac{Y\sigma_Y\sigma_c(c,X^*,Y)'}{X^{*2}}\right)\\
&-\frac{2}{r}\sigma_r\frac{\partial\sigma_c(c,X^*,Y)'}{\partial X^*} + \frac{(1-\gamma)c^3Y^2\sigma_Y\sigma_Y'\gamma}{X^{*3}}.
\end{aligned} \tag{7.1}$$

Assuming a solution of the form

$$\sigma_c(c,X^*,Y) := -\log_e\left(\frac{c\sigma_c}{X^*Y}\right), \tag{7.2}$$

where $\sigma_c \in \mathscr{R}^4$, we now have that

$$\frac{\partial\sigma_c(c,X^*,Y)}{\partial X^*} = \frac{1}{X^*}\mathrm{I}, \quad \frac{\partial\sigma_c(c,X^*,Y)}{\partial Y} = \frac{1}{Y}\mathrm{I}, \tag{7.3}$$

where $\mathrm{I} = [1,1,1,1]$. It then follows using (7.2) and (7.3) on (7.1) that

$$\frac{\partial\psi(c,X^*,Y)}{\partial X^*} = -\frac{2}{cX^*}\log_e\left(\frac{c\sigma_c}{X^*Y}\right)\mathrm{I}' - \frac{2}{rX^*}\sigma_r\mathrm{I}' + \frac{(1-\gamma)c^3Y^2\sigma_Y\sigma_Y'\gamma}{X^{*3}} + c(1-\gamma)\left(\frac{cY\sigma_Y\sigma_r'}{rX^{*2}} - \frac{\beta cY}{X^{*2}} - \frac{Y}{X^*}\sigma_Y\mathrm{I}' - \frac{Y\sigma_Y[\log_e(\frac{c\sigma_c}{X^*Y})]'}{X^{*2}}\right). \quad (7.4)$$

Next, we consider the change in the drift term with respect to the salary process of a PPM, and it is given by

$$\frac{\partial\psi(c,X^*,Y)}{\partial Y} = \frac{1}{c}\left(\sigma_c(c,X^*,Y)\frac{\partial\sigma_c(c,X^*,Y)'}{\partial Y} + \frac{\partial\sigma_c(c,X^*,Y)}{\partial Y}\sigma_c(c,X^*,Y)'\right) + c(1-\gamma)\left(-\frac{c\sigma_Y\sigma_r'}{rX^*} + \frac{\beta c}{X^*} - Y\sigma_Y\frac{\partial\sigma_c(c,X^*,Y)'}{\partial Y} - \frac{\sigma_Y\sigma_c(c,X^*,Y)'}{X^*}\right) - \frac{(1-\gamma)c^3Y\sigma_Y\sigma_Y'\gamma}{X^{*2}} - \frac{2}{r}\sigma_r\frac{\partial\sigma_c(c,X^*,Y)'}{\partial Y}. \quad (7.5)$$

Using (7.2) and (7.3) on (7.5), we have

$$\frac{\partial\psi(c,X^*,Y)}{\partial Y} = \frac{2}{cY}\log_e\left(\frac{c\sigma_c}{X^*Y}\right)\mathrm{I}' - \frac{(1-\gamma)c^3Y\sigma_Y\sigma_Y'\gamma}{X^{*2}} - \frac{2}{rY}\sigma_r\mathrm{I}' + c(1-\gamma)\left(-\frac{c\sigma_Y\sigma_r'}{rX^*} + \frac{\beta c}{X^*} - \sigma_Y\mathrm{I}' + \frac{\sigma_Y[\log_e(\frac{c\sigma_c}{X^*Y})]'}{X}\right). \quad (7.6)$$

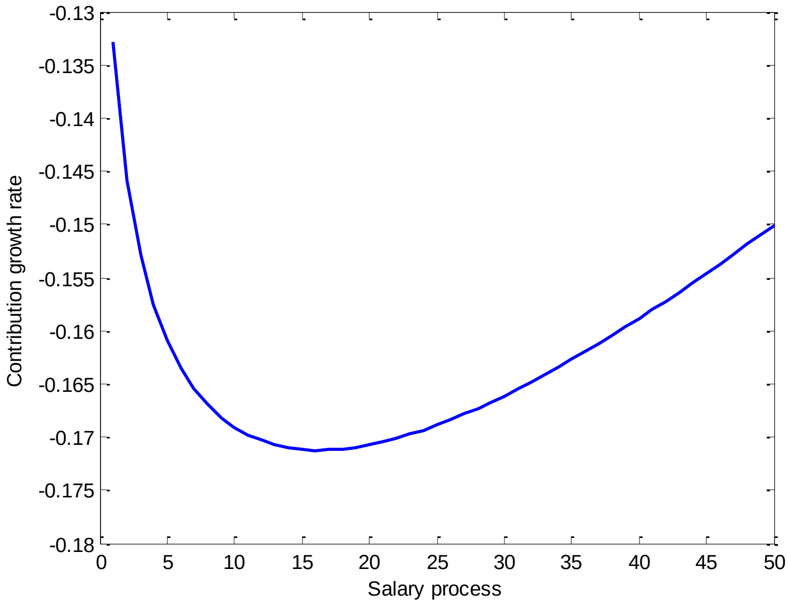

Figure 7.1: Contribution growth rate vs Salary process for $\mu = 0.28$, $r = 0.15$, $c = 0.15$, $\gamma = 20$, $X^* = 80$, $\sigma_S = [0.35, 0.27, 0.3, 0.28]$, $\sigma_Y = [0.12, 0.20, 0.16, 0.06]$, $\sigma_r = [0.08, 0.07, 0.15, 0.06]$, $\sigma_c(c,X^*,Y) = [0.2, 0.3, 0.1, 0.35]$

Figure 7.1 shows the contribution growth rate against the salary process of a PPM over time by taking $\sigma_c(c,X^*,Y) = [0.2, 0.3, 0.1, 0.35]$. It is observed that salary process of a PPM has a great effect on the contribution growth rate of a PPM. Hence, salary increase will lead to a high proportionate increase in the contribution growth rate of a PPM, for all other parameters remain fixed. In figure 7.2, we take $\sigma_c(c,X^*,Y) = -\log_e(\frac{c\sigma_c}{X^*Y})$ and varied the salary process of a PPM. We observed that the contribution growth rate declined when the salary process is between 0 and 8.5 and continue to rise thereafter. It simply implies that when the salary process is between 0 to about 8.5, PPMs may not make AVCs into the pension scheme. The AVCs will be encouraged if the salary process of a PPM is above 8.5.

The change in the drift term with respect to the contribution rate is given by the following:

$$\frac{\partial\psi(c,X^*,Y)}{\partial c} = -\frac{3(1-\gamma)}{2}c^2y^2\sigma_Y\sigma_Y'\gamma - \frac{\kappa(\theta-r)}{r} + \frac{\sigma_r\sigma_r'}{r^2} + (1-\gamma)\left(r + 2\beta cy - 2\frac{cy\sigma_Y\sigma_r'}{r} - y\sigma_Y\sigma_c(c,X^*,Y)'\right) - \frac{\sigma_c(c,X^*,Y)\sigma_c(c,X^*,Y)'}{c^2}. \quad (7.7)$$

Using (7.2) on (7.7), we have

$$\frac{\partial\psi(c,X^*,Y)}{\partial c} = -\frac{3(1-\gamma)}{2X^{*2}}c^2Y^2\sigma_Y\sigma_Y'\gamma - \frac{\kappa(\theta-r)}{r} + \frac{\sigma_r\sigma_r'}{r^2} + (1-\gamma)\left(r + \frac{2\beta cY}{X^*} - \frac{2cY\sigma_Y\sigma_r'}{rX^*} + Y\sigma_Y\left(\mathrm{I} + [\log_e(\frac{c\sigma_c}{X^*Y})]\right)'\right) + \frac{[\log_e(\frac{c\sigma_c}{X^*Y})](2\mathrm{I} - [\log_e(\frac{c\sigma_c}{X^*Y})])'}{c^2}. \quad (7.8)$$

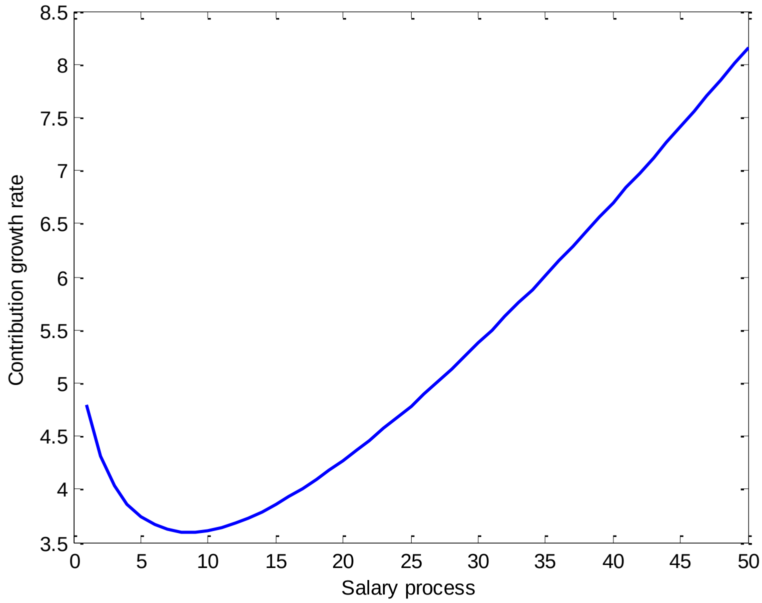

Figure 7.2: Contribution growth rate vs Salary process for $\mu = 0.28$, $r = 0.15$, $c = 0.15$, $\gamma = 20$, $X^* = 80$, $\sigma_S = [0.35, 0.27, 0.3, 0.28]$, $\sigma_Y = [0.12, 0.20, 0.16, 0.06]$, $\sigma_r = [0.08, 0.07, 0.15, 0.06]$, $\sigma_c = [7.33, 7.33, 6.67, 6.67]$, $\mathrm{I} = [1, 1, 1, 1]$, $\sigma_c(c, X^*, Y) = -\log_e(\frac{c\sigma_c}{X^*Y})$

The change in the drift term with respect to the short term interest rate is given by the following:

$$\frac{\partial\psi(c,X^*,Y)}{\partial r} = \frac{c\kappa\theta}{r^2} + c(1-\gamma) - \frac{2c\sigma_r\sigma_r'}{r^3} + \frac{2\sigma_r\sigma_c(c,X^*,Y)'}{r^2}. \tag{7.9}$$

Using (7.2) on (7.9), we have

$$\frac{\partial\psi(c,X^*,Y)}{\partial r} = \frac{c\kappa\theta}{r^2} + c(1-\gamma) - \frac{2c\sigma_r\sigma_r'}{r^3} - \frac{2\sigma_r[\log_e(\frac{c\sigma_c}{X^*Y})]'}{r^2}. \tag{7.10}$$

The change in the drift term with respect to the coefficient of risk averse is given by the following:

$$\frac{\partial\psi(c,X^*,Y)}{\partial\gamma} = -\frac{1-2\gamma}{2}c^3y^2\sigma_Y\sigma_Y' - c(r+\beta cy - \frac{cy\sigma_Y\sigma_r'}{r} - y\sigma_Y\sigma_c(c,X^*,Y)'). \tag{7.11}$$

Using (7.2) on (7.11), we have

$$\frac{\partial\psi(c,X^*,Y)}{\partial\gamma} = -\frac{1-2\gamma}{2X^{*2}}c^3Y^2\sigma_Y\sigma_Y' - c\left(r + \frac{\beta cY}{X^*} - \frac{cY\sigma_Y\sigma_r'}{rX^*} + \frac{Y\sigma_Y[\log_e(\frac{c\sigma_c}{X^*Y})]'}{X^*}\right). \tag{7.12}$$

The change in the drift term with respect to the PPM's expected salary growth rate is given by the following:

$$\frac{\partial\psi(c,X^*,Y)}{\partial\beta} = c^2(1-\gamma)y. \tag{7.13}$$

Using (7.2) on (7.13), we have

$$\frac{\partial\psi(c,X^*,Y)}{\partial\beta} = \frac{c^2(1-\gamma)Y}{X^*}. \tag{7.14}$$

The change in the drift term with respect to the volatility of a PPM's salary process is given by the following:

$$\frac{\partial\psi(c,X^*,Y)}{\partial\sigma_Y} = \left(-\gamma(1-\gamma)c^3y^2\sigma_Y - c(1-\gamma)\left(\frac{cy\sigma_r}{r} + y\sigma_c(c,X^*,Y)\right)\right)\mathrm{I}'. \tag{7.15}$$

Using (7.2) on (7.15), we have

$$\frac{\partial\psi(c,X^*,Y)}{\partial\sigma_Y} = -\left(\frac{\gamma(1-\gamma)c^3Y^2\sigma_Y}{X^{*2}} + c(1-\gamma)\left(\frac{cY\sigma_r}{rX^*} + \frac{Y[\log_e(\frac{c\sigma_c}{X^*Y})]'}{X^*}\right)\right)\mathrm{I}'. \tag{7.16}$$

We observe, for all other parameters remain fixed that

for $\left[\frac{\partial\psi(c,X^*,Y)}{\partial\sigma_Y}\right]_{\sigma_Y=[0.12,0.20,0.16,0.06]} = -57.8438,$

for $\left[\frac{\partial\psi(c,X^*,Y)}{\partial\sigma_Y}\right]_{\sigma_Y=[0.22,0.220,0.26,0.206]} = -57.6604,$

for $\left[\frac{\partial\psi(c,X^*,Y)}{\partial\sigma_Y}\right]_{\sigma_Y=[0.72,0.620,0.66,0.506]} = -56.8589.$

It simply implies that as the volatility of the salary process of a PPM increases, the contribution growth rate of a PPM increases as well. The change in the drift term with respect to the volatility of the short term interest rate is given by the following:

$$\frac{\partial\psi(c,X^*,Y)}{\partial\sigma_r} = \left(\frac{2c\sigma_r}{r^2} - \frac{c^2(1-\gamma)y\sigma_Y}{r} - \frac{2\sigma_c(c,X^*,Y)}{r}\right)\mathrm{I}'. \tag{7.17}$$

Using (7.2) on (7.17), we have

$$\frac{\partial\psi(c,X^*,Y)}{\partial\sigma_r} = \left(\frac{2c\sigma_r}{r^2} - \frac{c^2(1-\gamma)Y\sigma_Y}{rX^*} + \frac{2[\log_e(\frac{c\sigma_c}{X^*Y})]}{r}\right)\mathrm{I}'. \tag{7.18}$$

We observe, for all other parameters remain fixed that

for $\left[\frac{\partial\psi(c,X^*,Y)}{\partial\sigma_r}\right]_{\sigma_r=[0.08,0.07,0.15,0.06]} = -434.0458,$

for $\left[\frac{\partial\psi(c,X^*,Y)}{\partial\sigma_r}\right]_{\sigma_r=[0.408,0.407,0.415,0.406]} = -417.0325,$

for $\left[\frac{\partial\psi(c,X^*,Y)}{\partial\sigma_r}\right]_{\sigma_r=[0.908,0.907,0.915,0.906]} = -390.3658.$

This also implies that as the volatility of the short term interest rate increases, the contribution growth rate of a PPM increases as well. The change in the drift term with respect to the volatility of a PPM's contribution rate process is given by the following:

$$\frac{\partial\psi(c,X^*,Y)}{\partial\sigma_c(c,X^*,Y)} = \left(\frac{2\sigma_c(c,X^*,Y)}{c} - c(1-\gamma)y\sigma_Y - \frac{2\sigma_r}{r}\right)\mathrm{I}'. \tag{7.19}$$

Using (7.2) on (7.19), we have

$$\frac{\partial\psi(c,X^*,Y)}{\partial\sigma_c(c,X^*,Y)} = -\left(\frac{2[\log_e(\frac{c\sigma_c}{X^*Y})]}{c} + \frac{c(1-\gamma)Y\sigma_Y}{X^*} + \frac{2\sigma_r}{r}\right)\mathrm{I}'. \tag{7.20}$$

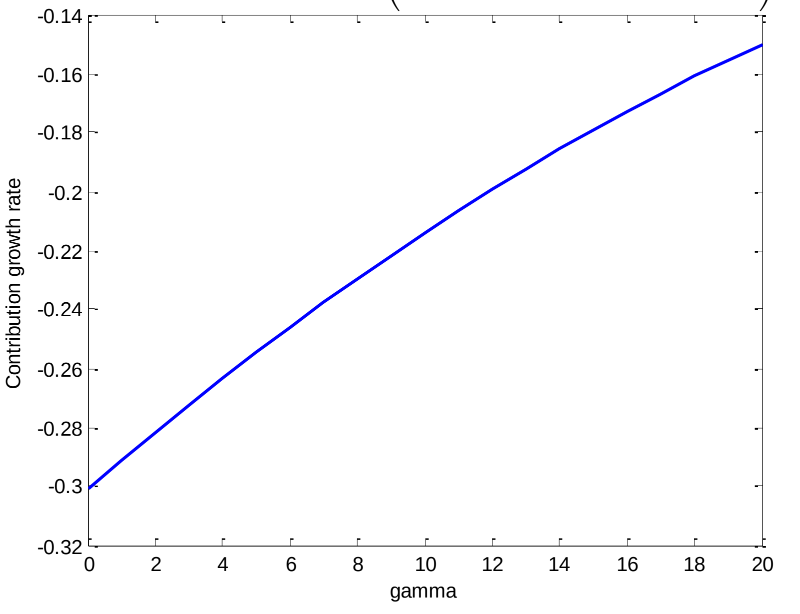

Figure 7.3: Contribution growth rate vs the coefficient of risk averse (γ) for $\mu = 0.28$, $r = 0.15$, $c = 0.15$, $Y = 50$, $X = 80$, $\sigma_S = [0.35, 0.27, 0.3, 0.28]$, $\sigma_Y = [0.12, 0.20, 0.16, 0.06]$, $\sigma_r = [0.08, 0.07, 0.15, 0.06]$, $\sigma_c = [7.33, 7.33, 6.67, 6.67]$, $\mathrm{I} = [1,1,1,1]$, $\sigma_c(c,X^*,Y) = -\log_e(\frac{c\sigma_c}{X^*Y})$

Figure 7.3 shows the contribution growth rate against the coefficient of risk averse, for all other parameters remain fixed. It is observed that as coefficient of risk averse increases, the contribution growth rate increases as well.

Table 1: The change in CGR with respect to r, c, β and X^*

r	$\psi(c,X^*,Y)$	c	$\psi(c,X^*,Y)$	β	$\psi(c,X^*,Y)$	X^*	$\psi(c,X^*,Y)$
0.01	−489.0670	0.10	25.4743	0.010	8.1589	5	94.4791
0.02	−234.6684	0.15	8.1539	0.011	8.1586	10	54.2936
0.03	−143.6545	0.20	0.8475	0.012	8.1583	15	37.1162
0.04	−96.9931	0.25	−2.4289	0.013	8.1581	20	29.0671
0.05	−68.6336	0.30	−3.6673	0.014	8.1578	30	20.4873
0.10	−11.1587	0.35	−3.7185	0.015	8.1575	40	15.8770
0.15	8.1539	0.40	−3.0055	0.020	8.1562	50	12.9401
0.20	17.7858	0.45	−1.7615	0.025	8.1549	60	10.8847
0.25	23.5228	0.50	−0.1243	0.030	8.1535	70	9.3508
0.30	27.3062	0.55	1.8201	0.035	8.1522	80	8.1539
0.35	29.9710	0.60	4.0156	0.040	8.1509	100	6.3897
0.40	31.9357	0.65	6.4252	0.100	8.1348	200	2.3139

Table 1 show the change in contribution growth rate of a PPM with respect to the short term interest rate r, contribution rate c, salary growth rate β and the optimal wealth process of a PPM. It is observed that increase in the short term interest rate, will lead to increase in the contribution growth rate of a PPM. We also observed that the contribution growth rate of a PPM decreases when the contribution rate increases from 0.10 to 0.37185, and continue to increase thereafter. For salary step increase β increases, the contribution growth rate decreases, which is unexpected. It then implies that salary increase have the tendency of affecting the contribution growth rate positively while the salary step increase has a negative effect on the contribution growth rate process. Also, in table 1, we observed that the increase in optimal wealth, decreases the contribution growth rate of PPM. This is expected, since our results shows that all investment should remain in the riskless asset over time, hence the investment strategy is in class u_1. Hence, an increase in investment portfolio in the risky asset will bring about an increase in the contribution growth rate, and vice versa.

8. Conclusion

A PPM may want to pay more to improve on his or her retirement benefits. In this paper, we considered an optimal pension fund management with stochastic contribution rate that comprises of both mandatory and additional voluntary contribution rates of a plan member's stochastic salary. The contribution growth rate and volatility of the stochastic contribution rate are assumed to depend on the contribution rate, the salary and the wealth processes of a PPM over time. The wealth process of a PPM was generated from an investment in a riskless and a risky assets, and consumption was assumed to be made from this wealth continuously over time by the PPM. The riskless asset was assumed to pay a stochastic interest rate that follows uncontrolled Vasisek model. The resulting wealth process was solved under a given control strategy, using dynamic programming approach for a stochastic differential equations. The control strategy was classified into two: a class of strategy that contained maximum portfolio value loss and a class that contained maximum portfolio value gain. It is assumed that a PPM may not make any additional voluntary contribution into the plan if the investment is experiencing portfolio value loss and contributes more if the investment is booming. After carrying out the optimization process, the optimal investment and optimal consumption strategies under a stochastic interest rate and a stochastic contribution rate are obtained. An explicit expression of the contribution growth rate of a PPM's stochastic contribution rate are obtained. Some factors that may influence the contribution by a PPM into the pension plan were studied in this paper. We found in this paper that the investment portfolio depends inversely on the contribution rate and the interest rate, and directly on the contribution of a PPM. The inter-temporary hedging portfolio strategies with respect to contribution rate, contribution process and interest rate are obtained.

Appendices

Appendix 1:

The proof:
Using (5.3) and Assumption 6, we have

$$\begin{aligned} G(x,r,c;u) &= E\int_0^T e^{-\delta t}V(c(t),\alpha(t)X^u(t))dt \\ &\leq LE\int_0^T e^{-\delta t}(1+(\alpha(t))^n(X^u(t))^n)dt \\ &\leq LE\int_0^T e^{-\delta t}dt + LE\int_0^T e^{-\delta t}(\alpha(t))^n(X^u(t))^n)dt \\ &\leq \frac{L}{\delta}(1-e^{-\delta T}) + LH^nx^nE\int_0^T e^{(\delta_2-\delta)t}dt \\ &\leq \frac{L}{\delta}(1-e^{-\delta T}) + \frac{LH^nx^n}{\delta_2-\delta}(e^{(\delta_2-\delta)T}-1) < +\infty. \end{aligned}$$

Appendix 2:

The proof:
Applying Assumption 6 and Proposition 5.2 , we have that $\forall\, u \in \mathscr{A}$, and $(x,r,c) \in [0,T]\times\mathscr{R}$

$$\begin{aligned} G(x,r,c;u) &\leq \frac{L}{\delta}(1-e^{-\delta T}) + \frac{LH^nx^n}{\delta_2-\delta}(e^{(\delta_2-\delta)T}-1) \\ &\leq \frac{L}{\delta}(1-e^{-\delta T}) + \frac{LH^nx^n}{\delta-\delta_2}(1-e^{-\delta T}) \\ &\leq \frac{L(1-e^{-\delta T})}{\delta}\left(1+\frac{\delta H^nx^n}{\delta-\delta_2}\right) \\ &\leq \frac{L(1-e^{-\delta T})}{\delta-\delta_2}\left(\frac{\delta-\delta_2}{\delta}+H^nx^n\right) \\ &\leq \frac{L(1-e^{-\delta T})}{\delta-\delta_2}(1+H^nx^n). \end{aligned}$$

Therefore, $U(x,r,c) \leq M(1+H^nx^n)\ \forall (x,r,c) \in [0,T]\times\mathscr{R}$ with $M = \dfrac{L(1-e^{-\delta T})}{\delta-\delta_2}$.

Appendix 3:

The proof:
Let $0 < x \le x^*$. From (4.8), we observe that for all $u \in \mathscr{A}$, $X^u(t) \le X^*(t)$ for all $t \in [0,T]$ $\mathbb{P}-a.s.$. It then follows that

$$E\int_0^T e^{-\delta t}V(c(t),\alpha(t)X^u(t);u_t)dt \le E\int_0^T e^{-\delta t}V(c(t),\alpha^*(t)X^*(t);u_t^*)dt, \forall u \in \mathscr{A}.$$

Hence,

$$\max_{u\in\mathscr{A}} E\int_0^T e^{-\delta t}V(c(t),\alpha(t)X^u(t);u_t)dt = E\int_0^T e^{-\delta t}V(c(t),\alpha^*(t)X^*(t);u_t^*)dt.$$

It implies that $\max_{u\in\mathscr{A}} U(x,r,c) = U(x^*,r,c)\ \forall (x^*,r,c) \in [0,T]\times\mathscr{R}$.

Appendix 4:

The proof:
Applying Proposition 5.5 and (4.11), we have

$$\begin{aligned} e^{-\delta T}E[U(X^u(T),r(T),c(T))] &\le e^{-\delta T}M(1+E[(\alpha(T))^n(X^u(T))^n]) \\ &\le e^{-\delta T}M(1+H^n x_0^n e^{\delta_2 T}). \end{aligned}$$

We now take the limit as T tends to infinity as follows.

$$\lim_{T\to+\infty} e^{-\delta T}E[U(X^u(T),r(T),c(T))] \le \lim_{T\to+\infty} e^{-\delta T}M(1+H^n x_0^n e^{\delta_2 T}) = 0.$$

It follows that

$$\lim_{T\to+\infty} e^{-\delta T}E[U(X^u(T),r(T),c(T))] \le 0.$$

Taking the maximum over u, we have

$$\max_u \left\{ \lim_{T\to+\infty} e^{-\delta T}E[U(X^u(T),r(T),c(T))] \le 0 \right\} = 0.$$

Appendix 5:

The proof:
Let $(X^u,r,c) = (X^u(t),r(t),c(t))$ be the solution to (2.2), (3.2) and (4.7) with initial values $(x_0,r_0,c_0) \in \mathscr{R}$. Now, by (4.8), we have that for any $u \in \mathscr{A}$

$$X^u(t) = x_0 e^{\int_0^t (A_1(s)-\frac{1}{2}A_2(s))ds} e^{\int_0^t A_3(s)dW(s)}, \forall t \in [0,T], \mathbb{P}-a.s., \tag{8.1}$$

where $A_1(s) = (r(s)+\pi(s)(\mu(s)-r(s))+\beta(s)c(s)y(s)-\alpha(s))$, $A_2(s) = (\pi(s)\sigma_S(s)+c(s)y(s)\sigma_Y(s))(\pi(s)\sigma_S(s)+c(s)y(s)\sigma_Y(s))'$, $A_3(s) = (\pi(s)\sigma_S(s)+c(s)y(s)\sigma_Y(s))$.
Let $\{(x_{0m},r_{0m},c_{0m})\}_{m\ge0}$ be a sequence in $\mathscr{R}$ that converges to $(x_0,r_0,c_0) \in \mathscr{R}$ as $m\to+\infty$, then $r_m(t)\to r(t)$ and $c_m(t)\to c(t)$ as $m\to+\infty$. By the dominated convergence theorem, it then follows that

$$X_m^u(t) \to X^u(t)$$

uniformly on $u \in \mathscr{A}$, as $m\to+\infty$. It then implies that

$$G(x_m,r_m,c_m;u) \to G(x,r,c;u)$$

uniformly on $u \in \mathscr{A}$, as $m\to+\infty$. It then follows that $U(x,r,c)$ is continuous and $U(x,r,c) = \max_{u\in\mathscr{A}} G(x,r,c;u)$ in $(x,r.c) \in [0,T]\times\mathscr{R}$.

Appendix 6:

The proof:
Using Proposition 5.10, we have that U is continuous. Suppose $(\bar{x},\bar{r},\bar{c}) \in [0,T]\times\mathscr{R}$ and $\bar{U} \in \mathscr{C}^{2,2}([0,T]\times\mathscr{R}$ such that

$$(U-\bar{U})(\bar{x},\bar{r},\bar{c}) = \max_{(x,r,c)\in[0,T]\times\mathscr{R}} (U-\bar{U})(x,r,c) = 0.$$

Suppose that $U \le \bar{U}$ and U is continuous, then there exists a sequence of solution $\{(x_m,r_m,c_m)\}_{m\ge1}$ such that $(x_m,r_m,c_m)\to(\bar{x},\bar{r},\bar{c})$ and $U(x_m,r_m,c_m)\to U(\bar{x},\bar{r},\bar{c})$ as $m\to+\infty$. Consequently, it follows that $\xi_m = U(x_m,r_m,c_m)-\bar{U}(x_m,r_m,c_m)\to0$ as $m\to+\infty$. It then follows that $\xi = U(\bar{x},\bar{r},\bar{c})-\bar{U}(\bar{x},\bar{r},\bar{c})\to0$ as well. Let $\bar{u}=\bar{u}_t$ for all $t\in[0,T]$ and $\bar{u}\in\mathscr{A}$. Let $\{\tau_m\}_{m\ge0}$ be a sequence of stopping times, a fixed number $\varepsilon>0$ and $\{h_m\}_{m\ge1}$ such that

$$\tau_m = \inf\{s\ge0|\max\{|X_m^{\bar{u}}(s)-x_m|,|r^m(s)-r_m|,|c^m(s)-c_m|\}>\varepsilon\}\wedge h_m, m\ge1.$$

Now for $h_m\to0$, $\dfrac{\xi_m}{h_m}\to0$ as $m\to+\infty$.

Hence, $\{r^m(t)\}_{t\geq 0}$ is a solution of (2.2) with initial condition $r^m(0)=r_m$, $\{c^m(t)\}_{t\geq 0}$ is a solution of (3.2) with initial condition $c^m(0)=c_m$ and $\{X_m^{\bar{u}}(t)\}_{t\geq 0}$ is a solution of (4.7) with initial condition $X_m^{\bar{u}}(0)=x_m$. Applying the principle of dynamic programming for a stochastic differential equation, we have

$$U(x_m,r_m,c_m)\leq E\begin{bmatrix}\int_0^{\tau_m} e^{-\delta t}V(c^m(t),\alpha(t)X_m^{\bar{u}}(t);\bar{u})dt\\ +e^{-\delta\tau_m}U(X_m^{\bar{u}}(\tau_m),r^m(\tau_m),c^m(\tau_m))\end{bmatrix}.$$

It then follows that

$$\begin{aligned} U(x_m,r_m,c_m) &\leq E\left[\int_0^{\tau_m} e^{-\delta t}V(c^m(t),\alpha_m(t)X_m^{\bar{u}}(t);\bar{u})dt\right]\\ &+ E\left[e^{-\delta\tau_m}U(X_m^{\bar{u}}(\tau_m),r^m(\tau_m),c^m(\tau_m))\right]. \end{aligned}$$

And

$$\bar{U}(x_m,r_m,c_m)+\xi_m\leq E\begin{bmatrix}\int_0^{\tau_m} e^{-\delta t}V(c^m(t),\alpha_m(t)X_m^{\bar{u}}(t);\bar{u})dt\\ +e^{-\delta\tau_m}\bar{U}(X_m^{\bar{u}}(\tau_m),r^m(\tau_m),c^m(\tau_m))\end{bmatrix},$$

it then follows that

$$\begin{aligned} \bar{U}(x_m,r_m,c_m)+\xi_m &\leq E\left[\int_0^{\tau_m} e^{-\delta t}V(c^m(t),\alpha_m(t)X_m^{\bar{u}}(t);\bar{u})dt\right]\\ &+ E\left[e^{-\delta\tau_m}\bar{U}(X_m^{\bar{u}}(\tau_m),r^m(\tau_m),c^m(\tau_m))\right]. \end{aligned}$$

We now have that

$$\begin{aligned} \bar{U}(x_m,r_m,c_m)+\xi_m &= \xi_m - E\left[\int_0^{\tau_m} e^{-\delta t}(\mathscr{L}^{\bar{u}}\bar{U}(X_m^{\bar{u}}(t),r^m(t),c^m(t)))dt\right]\\ &+ E\left[e^{-\delta\tau_m}\bar{U}(X_m^{\bar{u}}(\tau_m),r^m(\tau_m),c^m(\tau_m))\right]\\ &\leq E\left[\int_0^{\tau_m} e^{-\delta t}V(c^m(t),\alpha(t)X_m^{\bar{u}}(t);\bar{u})dt\right]\\ &+ E\left[e^{-\delta\tau_m}\bar{U}(X_m^{\bar{u}}(\tau_m),r^m(\tau_m),c^m(\tau_m))\right]. \end{aligned}$$

Equating the above two expressions, we have

$$\begin{aligned} \xi_m &- E\left[\int_0^{\tau_m} e^{-\delta t}(\mathscr{L}^{u}\bar{U}(X_m^{\bar{u}}(t),r^m(t),c^m(t)))dt\right]\\ &\leq E\left[\int_0^{\tau_m} e^{-\delta t}V(c^m(t),\alpha_m(t)X_m^{\bar{u}}(t);\bar{u})dt\right]. \end{aligned}$$

It therefore follows that

$$\xi_m\leq E\left[\int_0^{\tau_m} e^{-\delta t}\{\mathscr{L}^{\bar{u}}\bar{U}(X_m^{\bar{u}}(t),r^m(t),c^m(t))+V(c^m(t),\alpha_m(t)X_m^{\bar{u}}(t);\bar{u})\}dt\right]. \tag{8.2}$$

Dividing bothsides of (8.3) by h_m for $\tau_m\leq h_m$, we have

$$\frac{\xi_m}{h_m}\leq E\left[\frac{1}{h_m}\int_0^{\tau_m} e^{-\delta t}\{\mathscr{L}^{\bar{u}}\bar{U}(X_m^{\bar{u}}(t),r^m(t),c^m(t))+V(c^m(t),\alpha_m(t)X_m^{\bar{u}}(t);\bar{u})\}dt\right]. \tag{8.3}$$

Letting $m\to+\infty$, we have that $X_m^{\bar{u}}(t)\to X^{\bar{u}}(t)$, $r^m(t)\to r(t)$ and $c^m(t)\to c(t)$ for all $t\in[0,T]$ $\mathbb{P}$-a.s. and by mean-value theorem for integral, we have that $\mathscr{L}^{\bar{u}}\bar{U}(X_m^{\bar{u}}(t),r^m(t),c^m(t))+V(c^m(t),\alpha_m(t)X^{\bar{u}}(t);\bar{u})\to\mathscr{L}^{\bar{u}}\bar{U}(\bar{x},\bar{r},\bar{c})+V(\bar{c},\bar{\alpha}\bar{x};\bar{u})$. It then follows that

$$\mathscr{L}^{\bar{u}}\bar{U}(\bar{x},\bar{r},\bar{c})+V(\bar{c},\bar{\alpha}\bar{x};\bar{u})\geq 0. \tag{8.4}$$

Taking the minimum over the arbitrary control parameter $\bar{u}$, that is,

$$\min_{\bar{u}}\{\mathscr{L}^{\bar{u}}\bar{U}(\bar{x},\bar{r},\bar{c})+V(\bar{c},\bar{\alpha}\bar{x};\bar{u})\}=0, \tag{8.5}$$

which shows that U is indeed a viscosity subsolution of (5.6). Next, we show that U is a viscosity supersolution of (5.6). In a similar way, we suppose that $(\bar{x},\bar{r},\bar{c})\in[0,T]\times\mathscr{R}$ and $\bar{U}\in\mathscr{C}^{2,2}([0,T]\times\mathscr{R})$ such that

$$(U-\bar{U})(\bar{x},\bar{r},\bar{c})=\min_{(x,r,c)\in[0,T]\times\mathscr{R}}(U-\bar{U})(x,r,c)=0.$$

We now find a suitable strategy $\{v_t\}_{t\geq 0}\in\mathscr{A}$ such that

$$U(x_m,r_m,c_m)+h_m^2 \geq E\left[\int_0^{\tau_m} e^{-\delta t}V(c^m(t),\alpha_m(t)X_m^v(t);v_t)dt\right] \\ +E\left[e^{-\delta\tau_m}U(X_m^v(\tau_m),r^m(\tau_m),c^m(\tau_m))\right].$$

And

$$\bar{U}(x_m,r_m,c_m)+\xi_m+h_m^2 \geq E\left[\begin{array}{l}\int_0^{\tau_m} e^{-\delta t}V(c^m(t),\alpha_m(t)X_m^v(t);v_t)dt \\ +e^{-\delta\tau_m}\bar{U}(X_m^v(\tau_m),r^m(\tau_m),c^m(\tau_m))\end{array}\right].$$

Hence,

$$\begin{aligned}\xi_m+h_m^2 &+ E\left[e^{-\delta\tau_m}\bar{U}(X_m^v(\tau_m),r^m(\tau_m),c^m(\tau_m))\right] \\ &- E\left[\int_0^{\tau_m} e^{-\delta t}(\mathscr{L}^v\bar{U}(X_m^v(t),r^m(t),c^m(t)))dt\right] \\ &\geq E\left[\int_0^{\tau_m} e^{-\delta t}V(c^m(t),\alpha_m(t)X_m^v(t);v_t)dt\right] \\ &+ E\left[e^{-\delta\tau_m}\bar{U}(X_m^v(\tau_m),r^m(\tau_m),c^m(\tau_m))\right].\end{aligned}$$

It then follows that

$$\begin{aligned}\xi_m+h_m^2 &\geq E\left[\int_0^{\tau_m} e^{-\delta t}(\mathscr{L}^v\bar{U}(X_m^v(t),r^m(t),c^m(t))+V(c^m(t),\alpha_m(t)X_m^v(t);v_t))dt\right] \\ &\geq E\left[\int_0^{\tau_m}\min_{\bar{u}}\{\mathscr{L}^{\bar{u}}\bar{U}(X_m^{\bar{u}}(t),r^m(t),c^m(t))+V(c^m(t),\alpha(t)X_m^{\bar{u}}(t);\bar{u})\}dt\right] \\ &\geq E\left[\int_0^{\tau_m}\min_{u}\{\mathscr{L}^{u}\bar{U}(X_m^{u}(t),r^m(t),c^m(t))+V(c^m(t),\alpha_m(t)X_m^{u}(t);u)\}dt\right].\end{aligned}$$

Now, dividing bothsides by $\tau_m > h_m$, we have

$$\begin{aligned}\frac{\xi_m}{h_m}+h_m &\geq E\left[\frac{1}{h_m}\int_0^{\tau_m} e^{-\delta t}(\mathscr{L}^v\bar{U}(X^v(t),r^m(t),c^m(t))+V(c^m(t),\alpha_m(t)X_m^v(t);v_t))dt\right] \\ &\geq E\left[\frac{1}{h_m}\int_0^{\tau_m}[\mathscr{L}^{\bar{u}}\bar{U}(X_m^{\bar{u}}(t),r^m(t),c^m(t))+V(c^m(t),\alpha_m(t)X_m^{\bar{u}}(t);\bar{u})]dt\right] \\ &\geq E\left[\frac{1}{h_m}\int_0^{\tau_m}[\mathscr{L}^{u}\bar{U}(X_m^{u}(t),r^m(t),c^m(t))+V(c^m(t),\alpha_m(t)X_m^{u}(t);u)]dt\right].\end{aligned}$$

Both

$$\lim_{m\to+\infty}\frac{\xi_m}{h_m}=\min\left\{\lim_{m\to+\infty}\frac{\chi_m}{h_m},1\right\}=1,$$

where $\chi_m=\inf\{s\geq 0|\max\{|X_m^{\bar{u}}(s)-x_m|,|r^m(s)-r_m|,|c^m(s)-c_m|\}\geq\varepsilon\}$.
By the application of mean-value theorem for integral, we have that

$$\mathscr{L}^u\bar{U}(\bar{x},\bar{r},\bar{c})+V(\bar{c},\bar{\alpha}\bar{x};u)\leq 0. \tag{8.6}$$

Hence, taking the maximum over u, we have that

$$\max_u\{\mathscr{L}^u\bar{U}(\bar{x},\bar{r},\bar{c})+V(\bar{c},\bar{\alpha}\bar{x};u)\}=0. \tag{8.7}$$

This shows that U is a viscosity supersolution of (5.6).

Appendix 7:

The proof: Given that $\nu:[0,T]\times\mathscr{R}\to\mathscr{R}$, we then apply Itô Lemma on $e^{-\delta t}\nu(X^u(t),r(t),c(t))$ for a give arbitrary $u\in\mathscr{A}$, then take the mathematical expectation and thereafter use the inequality (5.9), we have
$E[e^{-\delta T\wedge\tau_m}\nu(X^u(T\wedge\tau_m),r(T\wedge\tau_m),c(T\wedge\tau_m))]\geq\nu(x,r,c)-$
$E\left[\int_0^{T\wedge\tau_m}e^{-\delta t}V(c(t),\alpha(t)X^u(t);u_t)dt\right]$.
Employing a uniform growth condition on ν and Proposition 5.2, we have by the dominated convergence theorem and by allowing $m\to+\infty$ that

$$E[e^{-\delta T}\nu(X^u(T),r(T),c(T))]\geq\nu(x,r,c) \tag{8.8}$$

$$-E\left[\int_0^T e^{-\delta t}V(c(t),\alpha(t)X^u(t);u_t)dt\right],u\in\mathscr{A}. \tag{8.9}$$

But, by (4.11), we have
$\lim_{T\to+\infty}\max_u e^{-\delta T}E[v(X^u(T),r(T),c(T))] \leq N\lim_{T\to+\infty}\max_u e^{-\delta T}E[1+(\alpha(T))^m(X^u(T))^m] \leq 0, \forall(x,r,c)\in[0,T]\times\mathscr{R}, u\in\mathscr{A}$.
It now follows from (8.8) that by taking $T\to+\infty$, we obtain

$$v(x,r,c) \leq E\left[\int_0^\infty e^{-\delta t}V(c(t),\alpha(t)X^u(t);u_t)dt\right], u\in\mathscr{A},$$

which implies the $v(x,r,c)\leq U(x,r,c)$ $\forall(x,r,c)\in[0,T]\times\mathscr{R}, u\in\mathscr{A}$, it then follows that $v(x,r,c)\leq U(x,r,c)$ $\forall(x,r,c)\in[0,T]\times\mathscr{R}, u^*\in\mathscr{A}$. Similarly, for $\{u_t^* = u^*(X^{u^*}(t),r(t),c(t))\}_{t\leq 0}$, we have that

$$E[e^{-\delta T}v(X^{u^*}(T),r(T),c(T))] = v(x^*,r,c) - E\left[\int_0^T e^{-\delta t}V(c(t),\alpha^*(t)X^{u^*}(t);u_t^*)dt\right], u^*\in\mathscr{A}. \tag{8.10}$$

Employing the following $\lim_{T\to+\infty}\inf[e^{-\delta T}E[v(X^{u^*}(T),r(T),c(T))]]\geq 0$ $\forall(x^*,r,c)\in[0,T]\times\mathscr{R}, u^*\in\mathscr{A}$ holds, then taking $T\to+\infty$, we therefore deduce that

$$v(x^*,r,c) \geq E\left[\int_0^T e^{-\delta t}V(c(t),\alpha^*(t)X^{u^*}(t);u_t^*)dt\right], u^*\in\mathscr{A}.$$

Hence, we have that $v(x^*,r,c)\geq U(x^*,r,c)$ $\forall(x^*,r,c)\in[0,T]\times\mathscr{R}, u^*\in\mathscr{A}$. It then follows that $v(x^*,r,c)=U(x^*,r,c)$ $\forall(x^*,r,c)\in[0,T]\times\mathscr{R}$ is the optimal solution, and $u^*\in\mathscr{A}$ is an optimal control of our problem.

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