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Research Article

Inner-gMP and gMP-inner inverses

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Abstract

Solving some systems of operator equations, new kinds of generalized inverses are introduced. Since these new inverses can be expressed by inner and gMP inverses, they are called inner-gMP and gMP-inner inverses. In this way, the concepts of gMP, 1MP and MP1 inverses are generalized. Various representations and characterizations of inner-gMP and gMP-inner inverses are presented. Using the inner and *gMP inverse, we define the inner-*gMP and *gMP-inner inverses which are new extensions of 1MP, MP1 and *gMP inverses. We apply inner-gMP and gMP-inner inverses as well as inner-*gMP and *gMP-inner inverses to solve several kinds of linear equations. Consequently, we obtain solvability of the normal equation which is connected to the least-squares solution. Numerical examples are given to illustrate our results.

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1. Introduction

Let $\mathcal{B}(H,K)$ be the set of all bounded linear operators from H to K, where H and K are arbitrary Hilbert spaces. If $H = K$, the notation $\mathcal{B}(H) = \mathcal{B}(H,H)$ will be used. The symbols A^*, $R(A)$, $N(A)$ and $\sigma(A)$ represent the adjoint, the range, the null space and the spectrum of $A \in \mathcal{B}(H,K)$, respectively. An operator $P \in \mathcal{B}(H)$ is a projector if $P^2 = P$, and it is the orthogonal projector if $P^2 = P = P^*$.

Recall that $A \in \mathcal{B}(H)$ is generalized Drazin invertible [10], if there is $X \in \mathcal{B}(H)$ such that

$$XAX = X, \qquad AX = XA, \qquad A - A^2X \text{ is quasinilpotent},$$

where $Q \in \mathcal{B}(H)$ is quasinilpotent if $\sigma(Q) = \{0\}$. If the generalized Drazin inverse X of A exists, it is unique and denoted by A^d [10]. Note that A^d exists if and only if $0 \notin \operatorname{acc} \sigma(A)$, where $\operatorname{acc} \sigma(A)$ is the set of all accumulation points of $\sigma(A)$. The symbol $\mathcal{B}(H)^d$ denotes the set of all generalized Drazin invertible operators of $\mathcal{B}(H)$.

In the case that $A - A^2X$ is nilpotent (or equivalently $A^{k+1}X = A^k$, for some non-negative integer k) in the definition of the generalized Drazin inverse, then $A^D = A^d$ is the Drazin inverse of A. The symbol $\operatorname{ind}(A)$ denotes the index of A, i.e. the smallest

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non-negative integer k such that $A^{k+1}X = A^k$ holds. If $\mathrm{ind}(A) \leq 1$, $A^{\#} = A^D$ is the group inverse of A. The sets of all Drazin invertible and group invertible operators of $\mathcal{B}(H)$ are denoted by $\mathcal{B}(H)^D$ and $\mathcal{B}(H)^{\#}$, respectively. Recent results about expressions for the Drazin inverse can be found in [19, 20].

An operator $X \in \mathcal{B}(K, H)$ is the Moore–Penrose inverse of $A \in \mathcal{B}(H, K)$ [3] if

$$AXA = A, \qquad XAX = X, \qquad (AX)^* = AX, \qquad (XA)^* = XA.$$

The Moore–Penrose inverse of A is unique (if it exists) and denoted by $A^{\dagger}$. If $AXA = A$ is satisfied, X is an inner inverse (or $\{1\}$-inverse) of A and the operator A is regular. Notice that the Moore–Penrose inverse of A exists if and only if A is regular if and only if $R(A)$ is closed in K. The set of all inner inverses of A will be denoted by $A\{1\}$. We use the following notations: $A\{1,3\} = \{X \in A\{1\} : (AX)^* = AX\}$, $\mathcal{B}(H)^{d,-} = \{A \in \mathcal{B}(H)^d : A \text{ is regular}\}$ and $\mathcal{B}(H)^{D,-} = \{A \in \mathcal{B}(H)^D : A \text{ is regular}\}$.

Let us recall that $X \in \mathcal{B}(K, H)$ is an outer inverse (or $\{2\}$-inverse) of $A \in \mathcal{B}(H, K)$ if $XAX = X \neq 0$. The outer inverse X of A with $R(X) = T$ and $N(A) = S$, where T and S are subspaces of H and K, respectively, is unique (if it exists) and denoted by $A^{(2)}_{T,S}$ [3]. When $A^{(2)}_{T,S}$ satisfies $AA^{(2)}_{T,S}A = A$, it will be denoted by $A^{(1,2)}_{T,S}$. The operator $A^{(2)}_{T,S}$ (or $A^{(1,2)}_{T,S}$) such that $(A^{(2)}_{T,S}A)^* = A^{(2)}_{T,S}A$ (or $(A^{(1,2)}_{T,S}A)^* = A^{(1,2)}_{T,S}A$) is marked by $A^{(2,4)}_{T,S}$ (or $A^{(1,2,4)}_{T,S}$).

The notion of the core–EP inverse firstly defined in [12] for a square matrix, was generalized in [14, 15] for a generalized Drazin invertible operator on a Hilbert space. Let $A \in \mathcal{B}(H)^d$. There is a core–EP inverse $X \in \mathcal{B}(H)$ of A for which

$$XAX = X, \qquad R(X) = R(X^*) = R(A^d).$$

As the dual core–EP inverse, there exists a *core–EP inverse $X \in \mathcal{B}(H)$ of A satisfing

$$XAX = X, \qquad R(X) = R(X^*) = R((A^d)^*).$$

The core–EP (or *core–EP) inverse of A is unique and denoted by $A^{\textcircled{\dagger}}$ ($A_{\textcircled{\dagger}}$) [14, 15]. Recall that, by [15, Theorem 6.1], $A^{\textcircled{\dagger}}$ satisfies

$$A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}, \ (AA^{\textcircled{\dagger}})^* = AA^{\textcircled{\dagger}}, \ A^{\textcircled{\dagger}}AA^d = A^d \text{ and } A^{\textcircled{\dagger}} = A^dAA^{\textcircled{\dagger}}.$$

Especially, if $A \in \mathcal{B}(H)^D$ and $k = \mathrm{ind}(A)$, according to [7], $A^{\textcircled{\dagger}} = A^{\textcircled{D}} = A^DA^k(A^k)^{\dagger}$ and $A_{\textcircled{\dagger}} = A_{\textcircled{D}} = (A^k)^{\dagger}A^kA^D$. Consequently, for $A \in \mathcal{B}(H)^{\#}$, the core–EP (or *core–EP) inverse of A coincides with the core (or dual core) inverse $A^{\textcircled{\#}} = A^{\#}AA^{\dagger}$ (or $A_{\textcircled{\#}} = A^{\dagger}AA^{\#}$) [1]. Significant results about the core and core–EP inverse are established in [2, 5, 6, 9, 11, 21–23].

The above mentioned generalized inverses are significant in various applications: the Drazin inverse gives the solution of a singular linear control system; the group inverse has applications in Markov chain theory; the Moore-Penrose inverse is applied to solve the least-squares problem; the core–EP inverse is used to solve some approximation problems [3, 16].

To extend the concept of the Moore-Penrose inverse from an operator with closed range to a generalized Drazin invertible operator, the generalized Moore-Penrose inverse was introduced in [18] as a new generalized inverse. For $A \in \mathcal{B}(H)^d$, the generalized Moore–Penorse (or gMP) inverse of A is defined as unique solution to the system

$$XAX = X, \qquad AX = A(A^{\textcircled{\dagger}}A)^{\dagger}A^{\textcircled{\dagger}} \qquad \text{and} \qquad XA = (A^{\textcircled{\dagger}}A)^{\dagger}A^{\textcircled{\dagger}}A.$$

The gMP inverse of A is denoted by $A^{\diamond}$ and it is expressed as

$$A^{\diamond} = (A^{\textcircled{\dagger}}A)^{\dagger}A^{\textcircled{\dagger}}.$$

Dually, the dual gMP (or *gMP) inverse of A [18] is represented by

$$A_{\diamond} = A_{Ⓓ}(AA_{Ⓓ})^{\dagger}.$$

When $A \in \mathcal{B}(H)^{\#}$, the gMP inverse $A^{\diamond}$ and the *gMP inverse $A_{\diamond}$ reduce to the Moore–Penrose inverse $A^{\dagger}$ [18]. Interesting properties of the gMP inverse can be found in [4, 18].

As a new type of generalized inverses, the 1MP inverse was presented recently in [8] for a rectangular complex matrix, based on the Moore–Penrose and inner inverses. We give the definition of the 1MP inverse for a regular operator $A \in \mathcal{B}(H)$: if $A^{-} \in A\{1\}$, a 1MP inverse of A is introduced in [8] as

$$A^{-,\dagger} = A^{-}AA^{\dagger}.$$

Also, a MP1 inverse of A is defined as

$$A^{\dagger,-} = A^{\dagger}AA^{-}.$$

Interesting results about 1MP and MP1 inverses, which are used in studying partial orders, were proposed in [8] for rectangular complex matrices and in [13, 17] for elements of a ring with involution.

Motivated by the fact that the gMP and *gMP inverses are generalizations of the Moore–Penrose inverse and by the recent researches about the 1MP and MP1 inverses, we further continue to study these topics and connect them. In particular, our intention is to introduce new generalized inverses for a bounded linear generalized Drazin invertible regular operator on a Hilbert space. Firstly, we use the gMP inverse instead of the Moore–Penrose inverse in the definitions of the 1MP and MP1 inverses and present the inner-gMP and gMP-inner inverses. The names of these inverses origin from the fact that we define them using the inner and gMP inverses. We will observe that 1MP, MP1 and gMP inverses are special cases of the inner-gMP and gMP-inner inverses and so we propose wider classes of generalized inverses. Some properties and characterizations of inner-gMP and gMP-inner inverses are shown. Since operator matrix forms find their use in various different fields, such as solving some linear equations, numerical computations involving matrix inversion etc., operator matrix forms of inner-gMP and gMP-inner inverses are computed. Utilizing the *gMP inverse, we also define the inner-*gMP and *gMP-inner inverses which are new extensions of 1MP, MP1 and *gMP inverses. Applying inner-gMP and gMP-inner inverses as well as inner-*gMP and *gMP-inner inverses, we obtain solvability of several kinds of linear equations. As a consequence of our result, we get solvability of the normal equation which is connected to the least–squares solution. We also give numerical examples to illustrate our results.

The content of this paper is organized as follows. Section 2 contains definitions, properties and characterizations of inner-gMP and gMP-inner inverses. Section 3 is devoted to inner-*gMP and *gMP-inner inverses. Applications of inner-gMP and gMP-inner inverses in solving some linear equations are given in Section 4. Section 5 involves linear equations which are solved in terms of inner-*gMP and *gMP-inner inverses.

2. Inner-gMP and gMP-inner inverses

To extend the concepts of 1MP and MP1 inverses, we define new kinds of generalized inverses based on the inner and gMP inverses.

Theorem 2.1. *For $A \in \mathcal{B}(H)^{d,-}$ and an arbitrary but fixed $A^{-} \in A\{1\}$, we have*

(a) *$X = A^{-}AA^{\diamond}$ represents the unique solution to the system*

$$XAX = X, \qquad AX = AA^{\diamond} \qquad \text{and} \qquad XA = A^{-}AA^{\diamond}A; \tag{2.1}$$

(b) *$X = A^{\diamond}AA^{-}$ represents the unique solution to the system*

$$XAX = X, \qquad AX = AA^{\diamond}AA^{-} \qquad \text{and} \qquad XA = A^{\diamond}A.$$

Proof. (a) Let $X = A^-AA^\diamond$. Then $AX = (AA^-A)A^\diamond = AA^\diamond$, $XA = A^-AA^\diamond A$ and

$$X(AX) = (XA)A^\diamond = A^-A(A^\diamond AA^\diamond) = A^-AA^\diamond = X.$$

Hence, the system (2.1) has a solution $X = A^-AA^\diamond$.

If X is a solution to (2.1), then

$$X = (XA)X = A^-AA^\diamond(AX) = A^-A(A^\diamond AA^\diamond) = A^-AA^\diamond.$$

Thus, $X = A^-AA^\diamond$ is the unique solution to (2.1).

(b) As part (a), we can show this part too. □

Definition 2.2. Let $A \in \mathcal{B}(H)^{d,-}$ and $A^- \in A\{1\}$ be arbitrary but fixed.

(a) The inner-gMP inverse of A is defined by

$$A^{-,\diamond} = A^-AA^\diamond.$$

(b) The gMP-inner inverse of A is defined by

$$A^{\diamond,-} = A^\diamond AA^-.$$

Several particular cases of inner-gMP and gMP-inner inverses are given:

- if $A \in \mathcal{B}(H)^\#$, we know that $A^\diamond = A^\dagger$ and thus $A^{-,\diamond} = A^-AA^\dagger = A^{-,\dagger}$ and $A^{\diamond,-} = A^\dagger AA^- = A^{\dagger,-}$, that is, inner-gMP and gMP-inner inverses become 1MP and MP1 inverses, respectively;
- for $A^- = A^\dagger$, notice that

$$\begin{aligned} A^{-,\diamond} &= A^\dagger AA^\diamond = A^\dagger A(A^{ⓓ}A)^\dagger A^{ⓓ} = A^\dagger A(A^{ⓓ}A)^\dagger A^{ⓓ}AA^{ⓓ} \\ &= ((A^{ⓓ}A)^\dagger A^{ⓓ}AA^\dagger A)^*A^{ⓓ} = ((A^{ⓓ}A)^\dagger A^{ⓓ}A)^*A^{ⓓ} \\ &= (A^{ⓓ}A)^\dagger A^{ⓓ}AA^{ⓓ} = (A^{ⓓ}A)^\dagger A^{ⓓ} \\ &= A^\diamond, \end{aligned}$$

  i.e. the inner-gMP inverse reduces to the gMP inverse.

By the definition of the gMP inverse and [18, Corollary 1], we get the next representations of inner-gMP and gMP-inner inverses.

Corollary 2.3. *For $A \in \mathcal{B}(H)^{d,-}$ and an arbitrary but fixed $A^- \in A\{1\}$, we have*

$$A^{-,\diamond} = A^-AA^\diamond = A^-A(A^{ⓓ}A)^\dagger A^{ⓓ} = A^-A(AA^{ⓓ}A)^\dagger$$

and

$$A^{\diamond,-} = A^\diamond AA^- = (A^{ⓓ}A)^\dagger A^{ⓓ}AA^- = (AA^{ⓓ}A)^\dagger AA^-.$$

If $A \in \mathcal{B}(H)^{D,-}$ in Corollary 2.3, notice, by [18, Corollary 2], that $A^{-,\diamond} = A^-A(A^k(A^k)^\dagger A)^\dagger$ and $A^{\diamond,-} = (A^k(A^k)^\dagger A)^\dagger AA^-$, where $\mathrm{ind}(A) = k$.

Example 2.4. Let A be a 3×3 complex matrix given by

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$A^{ⓓ} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A^\diamond = \begin{bmatrix} \frac{3}{10} & 0 & 0 \\ \frac{1}{10} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A^\dagger = \begin{bmatrix} \frac{3}{10} & 0 & 0 \\ \frac{1}{10} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad A^- = \begin{bmatrix} a & d & c \\ 1-3a & -3d & c_1 \\ 0 & 1 & c_2 \end{bmatrix},$$

where $a, d, c, c_1, c_2 \in \mathbb{C}$ are arbitrary. Notice that

$$A^{-,\diamond} = A^-AA^\diamond = \begin{bmatrix} a & 0 & 0 \\ 1-3a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A^{\diamond,-} = A^\diamond AA^- = \begin{bmatrix} \frac{3}{10} & 0 & \frac{9c+3c_1}{10} \\ \frac{1}{10} & 0 & \frac{c_2}{10} \\ 0 & 0 & 0 \end{bmatrix},$$

$$A^{-,\dagger} = A^-AA^\dagger = \begin{bmatrix} a & d & 0 \\ 1-3a & -3d & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad A^{\dagger,-} = A^\dagger AA^- = \begin{bmatrix} \frac{3}{10} & 0 & \frac{9c+3c_1}{10} \\ \frac{1}{10} & 0 & \frac{3c+c_1}{10} \\ 0 & 1 & c_2 \end{bmatrix}.$$

Comparing $A^{-,\diamond}$ and $A^{\diamond,-}$ with already known generalized inverses: $A^{\textcircled{d}}$, $A^\diamond$, $A^\dagger$, A^-, $A^{-,\dagger}$ and $A^{\dagger,-}$, we see that they are different and so inner-gMP and gMP-inner inverses present new types of generalized inverses.

Necessary and sufficient conditions for an operator to be the inner-gMP inverse, are studied now.

Theorem 2.5. *For $A \in \mathcal{B}(H)^{d,-}$, an arbitrary but fixed $A^- \in A\{1\}$ and $X \in \mathcal{B}(H)$, the following statements are equivalent:*

(i) $X = A^{-,\diamond}$;
(ii) $XAX = X$, $AXA = AA^\diamond A$, $AX = AA^\diamond$ *and* $XA = A^-AA^\diamond A$;
(iii) $XA = A^-AA^\diamond A$ *and* $XAA^\diamond = X$;
(iv) $XA = A^-AA^\diamond A$ *and* $XAA^{\textcircled{d}} = X$;
(v) $XAA^\dagger = A^-AA^\diamond AA^\dagger$ *and* $XAA^{\textcircled{d}} = X$;
(vi) $XAA^* = A^-AA^\diamond AA^*$ *and* $XAA^{\textcircled{d}} = X$;
(vii) $AX = AA^\diamond$ *and* $A^-AA^\diamond AX = X$;
(viii) $AX = AA^\diamond$ *and* $A^-AX = X$;
(ix) $A^\dagger AX = A^\diamond (= A^\dagger AA^\diamond)$ *and* $A^-AX = X$;
(x) $A^*AX = A^*AA^\diamond$ *and* $A^-AX = X$;
(xi) $XAA^\diamond AX = X$, $AA^\diamond AXAA^\diamond A = AA^\diamond A$, $AA^\diamond AX = AA^\diamond$ *and* $XAA^\diamond A = A^-AA^\diamond A$;
(xii) $XAA^\diamond AX = X$, $AA^\diamond AX = AA^\diamond$ *and* $XAA^\diamond A = A^-AA^\diamond A$.

Proof. (i) $\Rightarrow$ (ii): This implication follows by Theorem 2.1.

(ii) $\Rightarrow$ (iii): The conditions $XAX = X$ and $AX = AA^\diamond$ yield $X = X(AX) = XAA^\diamond$.

(iii) $\Rightarrow$ (iv): Using $XAA^\diamond = X$, we obtain

$$X = XAA^\diamond = XA(A^{\textcircled{d}}A)^\dagger A^{\textcircled{d}} = (XA(A^{\textcircled{d}}A)^\dagger A^{\textcircled{d}})AA^{\textcircled{d}} = XAA^{\textcircled{d}}.$$

(iv) $\Rightarrow$ (i): The assumptions $XA = A^-AA^\diamond A$ and $XAA^{\textcircled{d}} = X$ imply

$$X = (XA)A^{\textcircled{d}} = A^-AA^\diamond AA^{\textcircled{d}} = A^-A(A^{\textcircled{d}}A)^\dagger A^{\textcircled{d}}AA^{\textcircled{d}} = A^-AA^\diamond.$$

The rest can be verified in the analogy manner. □

The following characterizations of the gMP-inner inverse can be proved as Theorem 2.5.

Theorem 2.6. *For $A \in \mathcal{B}(H)^{d,-}$, an arbitrary but fixed $A^- \in A\{1\}$ and $X \in \mathcal{B}(H)$, the following statements are equivalent:*

(i) $X = A^{\diamond,-}$;
(ii) $XAX = X$, $AXA = AA^\diamond A$, $AX = AA^\diamond AA^-$ *and* $XA = A^\diamond A$;
(iii) $AX = AA^\diamond AA^-$ *and* $A^\diamond AX = X$;
(iv) $AX = AA^\diamond AA^-$ *and* $(A^{\textcircled{d}}A)^\dagger A^{\textcircled{d}}AX = X$;
(v) $A^\dagger AX = A^\dagger AA^\diamond AA^-$ *and* $A^\diamond AX = X$;
(vi) $A^*AX = A^*AA^\diamond AA^-$ *and* $A^\diamond AX = X$;
(vii) $XA = A^\diamond A$ *and* $X = XAA^\diamond AA^-$;
(viii) $XA = A^\diamond A$ *and* $XAA^- = X$;
(ix) $XAA^\dagger = A^\diamond (= A^\diamond AA^\dagger)$ *and* $XAA^- = X$;
(x) $XAA^* = A^\diamond AA^*$ *and* $XAA^- = X$;

(xi) $XAA^\diamond AX = X$, $AA^\diamond AXAA^\diamond A = AA^\diamond A$, $AA^\diamond AX = AA^\diamond AA^-$ *and* $XAA^\diamond A = A^\diamond A$;

(xii) $XAA^\diamond AX = X$, $AA^\diamond AX = AA^\diamond AA^-$ *and* $XAA^\diamond A = A^\diamond A$.

Remark that, replacing $A^\diamond$ with one of its representations $(A^{ⓓ}A)^\dagger A^{ⓓ}$ or $(AA^{ⓓ}A)^\dagger$, we can obtain more characterizations of $A^{-,\diamond}$ or $A^{\diamond,-}$.

Lemma 2.7. *If* $A \in \mathcal{B}(H)^{d,-}$ *and* $A^- \in A\{1\}$ *is arbitrary but fixed, then*

(i) $AA^{-,\diamond}$ *is a projection onto* $R(A(A^{ⓓ}A)^*)$ *along* $N(A^{ⓓ})$;

(ii) $A^{-,\diamond}A$ *is a projection onto* $R(A^-A(A^{ⓓ}A)^*)$ *along* $N(A^{ⓓ}A)$;

(iii) $A^{-,\diamond} = A^{(2)}_{R(A^-A(A^{ⓓ}A)^*),N(A^{ⓓ})} = (AA^\diamond A)^{(1,2)}_{R(A^-A(A^{ⓓ}A)^*),N(A^{ⓓ})}$;

(iv) $AA^{\diamond,-}$ *is a projection onto* $R(A(A^{ⓓ}A)^*)$ *along* $N(A^{ⓓ}AA^-)$;

(v) $A^{\diamond,-}A = (A^{ⓓ}A)^\dagger A^{ⓓ}A$ *is the orthogonal projection onto* $R((A^{ⓓ}A)^*)$;

(vi) $A^{\diamond,-} = A^{(2,4)}_{R((A^{ⓓ}A)^*),N(A^{ⓓ}AA^-)} = (AA^\diamond A)^{(1,2,4)}_{R((A^{ⓓ}A)^*),N(A^{ⓓ}AA^-)}$.

Proof. (i) By Theorem 2.7, we conclude that $AA^{-,\diamond}$ is a projector. Since $AA^{-,\diamond} = AA^\diamond = A(A^{ⓓ}A)^\dagger A^{ⓓ}$, we have $R(AA^{-,\diamond}) = R(A(A^{ⓓ}A)^\dagger) = R(A(A^{ⓓ}A)^*)$ and

$$N(AA^{-,\diamond}) = N(A(A^{ⓓ}A)^\dagger A^{ⓓ}AA^{ⓓ}) \subseteq N(A^{ⓓ}A(A^{ⓓ}A)^\dagger A^{ⓓ}AA^{ⓓ}) = N(A^{ⓓ}) \subseteq N(AA^{-,\diamond}).$$

Hence, $N(AA^{-,\diamond}) = N(A^{ⓓ})$.

(ii) The equality $A^{-,\diamond}A = A^-AA^\diamond A = A^-A(A^{ⓓ}A)^\dagger A^{ⓓ}A$ yields $A^{-,\diamond}A$ is a projection onto $R(A^{-,\diamond}A) = R(A^-A(A^{ⓓ}A)^*)$ along $N(A^{-,\diamond}A) = N(A^{ⓓ}A)$ by

$$N(A^{-,\diamond}A) \subseteq N(A^{ⓓ}AA^-A(A^{ⓓ}A)^\dagger A^{ⓓ}A) = N(A^{ⓓ}A) \subseteq N(A^{-,\diamond}A).$$

(iii) This part is clear by $R(A^{-,\diamond}A) = R(A^{-,\diamond})$ and $N(AA^{-,\diamond}) = N(A^{-,\diamond})$.

The proof can be finished similarly. □

The orthogonal projector onto a closed subspace V will be marked by P_V, and, for closed subspaces V and U of H satisfying $H = V \oplus U$, a projector onto the subspace V along U will be denoted by $P_{V,U}$. Inner-gMP and gMP-inner inverses can be considered as solutions of the following restricted equations.

Theorem 2.8. *If* $A \in \mathcal{B}(H)^{d,-}$ *and* $A^- \in A\{1\}$ *is arbitrary but fixed, then*

(i) $A^{-,\diamond}$ *is uniquely determined solution to*

$$AX = P_{R(A(A^{ⓓ}A)^*),N(A^{ⓓ})} \quad \text{and} \quad R(X) \subseteq R(A^-A); \tag{2.2}$$

(ii) $A^{\diamond,-}$ *is uniquely determined solution to*

$$AX = P_{R(A(A^{ⓓ}A)^*),N(A^{ⓓ}AA^-)} \quad \text{and} \quad R(X) \subseteq R((A^{ⓓ}A)^*).$$

Proof. (i) Remark that $A^{-,\diamond}$ is a solution to (2.2) by Lemma 2.7.

In the case that (2.2) has two solutions Y and X, by $A(Y - X) = 0$ and $R(Y - X) \subseteq R(A^-A)$, it follows $R(Y - X) \subseteq N(A^-A) \cap R(A^-A) = \{0\}$. So, $Y = X$ implies that (2.2) has the unique solution $A^{-,\diamond}$.

The part (ii) can be checked in an analogous way. □

Theorem 2.9. *If* $A \in \mathcal{B}(H)^{d,-}$ *and* $A^- \in A\{1\}$ *is arbitrary but fixed, then*

(i) $A^{-,\diamond}$ *is uniquely determined solution to*

$$XA = P_{R(A^-A(A^{ⓓ}A)^*),N(A^{ⓓ}A)} \quad \text{and} \quad R(X^*) \subseteq R((A^{ⓓ})^*); \tag{2.3}$$

(ii) $A^{\diamond,-}$ *is uniquely determined solution to*

$$XA = P_{R((A^{ⓓ}A)^*)} \quad \text{and} \quad R(X^*) \subseteq R((AA^-)^*).$$

Proof. (i) We see, by Lemma 2.7, that (2.3) has a solution $A^{-,\diamond}$.

For two solutions Y and X of (2.2), note that $A^*(Y^*-X^*)=0$ and $R(Y^*-X^*)\subseteq R((A^{ⓓ})^*)$ imply $R(Y^*-X^*)\subseteq N(A^*)\cap R((A^{ⓓ})^*)\subseteq N((A^{ⓓ})^*A^*)\cap R((A^{ⓓ})^*A^*)=\{0\}$. Therefore, $Y=X=A^{-,\diamond}$ is the uniquely determined solution to (2.3).

Similarly, we prove part (ii). □

As we know $A^{-,\diamond}$ and $A^{\diamond,-}$ are outer inverses of A, but it is interesting to study equivalent conditions for $A^{-,\diamond}$ (or $A^{\diamond,-}$) to be an inner inverse of A.

Theorem 2.10. *For $A\in\mathcal{B}(H)^{d,-}$ and an arbitrary but fixed $A^-\in A\{1\}$, the following statements are equivalent:*

(i) $A=AA^{-,\diamond}A$;

(ii) $A=AA^{\diamond}A$;

(iii) $A=AA^{\diamond,-}A$;

(iv) $N(A^{\diamond}A)=N(A)$ *(or equivalently* $N(A^{ⓓ}A)=N(A)$*);*

(v) $R(A)=R(AA^{\diamond})$ *(or equivalently* $R(A)=R(A(A^{ⓓ}A)^*)$*).*

Proof. (i) ⇔ (ii) ⇔ (iii): These equivalences follow by $AA^{-,\diamond}A=AA^{\diamond}A=AA^{\diamond,-}A$.

(ii) ⇔ (iv): It is clear by [18, Theorem 3] and $N(A)\subseteq N(A^{\diamond}A)=N(A^{ⓓ}A)$.

(ii) ⇔ (v): We have

$$\begin{aligned} A=AA^{\diamond}A &\Leftrightarrow (I-AA^{\diamond})A=0\\ &\Leftrightarrow R(A)\subseteq N(I-AA^{\diamond})\\ &\Leftrightarrow R(A)\subseteq R(AA^{\diamond})\\ &\Leftrightarrow R(A)=R(AA^{\diamond}). \end{aligned}$$

□

We can represent $A\in\mathcal{B}(H)^d$ with respect to the orthogonal sum $H=R(A^d)\oplus N((A^d)^*)$ as [14, Lemma 2.1 and Corollary 2.2]:

$$A=\begin{bmatrix} A_1 & A_2\\ 0 & A_3\end{bmatrix}:\begin{bmatrix} R(A^d)\\ N((A^d)^*)\end{bmatrix}\to\begin{bmatrix} R(A^d)\\ N((A^d)^*)\end{bmatrix}, \tag{2.4}$$

where $A_1\in\mathcal{B}(R(A^d))$ is invertible and $A_3\in\mathcal{B}[N((A^d)^*)]$ is quasinilpotent. The operator matrix forms for the inner-gMP and gMP-inner inverses are developed now.

Theorem 2.11. *If $A\in\mathcal{B}(H)^{d,-}$ is expressed by (2.4) and $A^-\in A\{1\}$ is arbitrary but fixed, then*

$$\begin{aligned} A^{-,\diamond} &= \begin{bmatrix} A_1^{-1}(I-A_2Y_3)+Y_2A_3A_2^*(A_1A_1^*+A_2A_2^*)^{-1} & 0\\ Y_3+Y_4A_3A_2^*(A_1A_1^*+A_2A_2^*)^{-1} & 0\end{bmatrix}:\\ &\quad \begin{bmatrix} R(A^d)\\ N((A^d)^*)\end{bmatrix}\to\begin{bmatrix} R(A^d)\\ N((A^d)^*)\end{bmatrix} \end{aligned}\tag{2.5}$$

and

$$\begin{aligned} A^{\diamond,-} = \begin{bmatrix} A_1^*(A_1A_1^*+A_2A_2^*)^{-1} & A_1^*(A_1A_1^*+A_2A_2^*)^{-1}(A_1Y_2+A_2Y_4)\\ A_2^*(A_1A_1^*+A_2A_2^*)^{-1} & A_2^*(A_1A_1^*+A_2A_2^*)^{-1}(A_1Y_2+A_2Y_4)\end{bmatrix}:\\ \begin{bmatrix} R(A^d)\\ N((A^d)^*)\end{bmatrix}\to\begin{bmatrix} R(A^d)\\ N((A^d)^*)\end{bmatrix}, \end{aligned}$$

where $(A_1Y_2+A_2Y_4)A_3=0$, $A_3Y_3=0$ and $A_3Y_4A_3=A_3$.

Proof. According to [18, Theorem 5], we have

$$A^{\diamond}=\begin{bmatrix} A_1^*(A_1A_1^*+A_2A_2^*)^{-1} & 0\\ A_2^*(A_1A_1^*+A_2A_2^*)^{-1} & 0\end{bmatrix}:\begin{bmatrix} R(A^d)\\ N((A^d)^*)\end{bmatrix}\to\begin{bmatrix} R(A^d)\\ N((A^d)^*)\end{bmatrix}. \tag{2.6}$$

Note that

$$A^- = \begin{bmatrix} A_1^{-1}(I - A_2Y_3) & Y_2 \\ Y_3 & Y_4 \end{bmatrix} : \begin{bmatrix} R(A^d) \\ N((A^d)^*) \end{bmatrix} \to \begin{bmatrix} R(A^d) \\ N((A^d)^*) \end{bmatrix},$$

where $(A_1Y_2 + A_2Y_4)A_3 = 0$, $A_3Y_3 = 0$ and $A_3Y_4A_3 = A_3$. Hence, (2.5) holds:

$$\begin{aligned} A^{-,\diamond} &= A^-AA^\diamond = A^- \begin{bmatrix} I & 0 \\ A_3A_2^*(A_1A_1^* + A_2A_2^*)^{-1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} A_1^{-1}(I - A_2Y_3) + Y_2A_3A_2^*(A_1A_1^* + A_2A_2^*)^{-1} & 0 \\ Y_3 + Y_4A_3A_2^*(A_1A_1^* + A_2A_2^*)^{-1} & 0 \end{bmatrix}. \end{aligned}$$

The formula for $A^{\diamond,-}$ follows similarly. □

We investigate equivalent conditions for the expression A^-AE to be equal to $A^{-,\diamond}$.

Theorem 2.12. *For $A \in \mathcal{B}(H)^{d,-}$, an arbitrary but fixed $A^- \in A\{1\}$ and $E \in \mathcal{B}(H)$, the following statements are equivalent:*

(i) $A^{-,\diamond} = A^-AE$;
(ii) $AA^\diamond = AE$;
(iii) $N(AA^\diamond) = N(AE)$ *and* $AA^\diamond A = AEA$;
(iv) $E = A^\diamond + (I - A^-A)F$, *for arbitrary* $F \in \mathcal{B}(H)$.

Proof. (i) ⇒ (ii): Notice that

$$AA^\diamond = A(A^-AA^\diamond) = AA^{-,\diamond} = (AA^-A)E = AE.$$

(ii) ⇒ (iii): It is clear.

(iii) ⇒ (i): Since $R(I - AA^\diamond) = N(AA^\diamond) = N(AE)$, we get $AE = AEAA^\diamond$. Thus, by $AA^\diamond A = AEA$,

$$A^-(AE) = A^-(AEA)A^\diamond = A^-A(A^\diamond AA^\diamond) = A^-AA^\diamond = A^{-,\diamond}.$$

(ii) ⇒ (iv): Since all solutions of equation $AA^\diamond = AE$ present a sum of a particular solution of this equation and the general solutions to the homogeneous equation $AE = 0$, by [3, p. 52], $AE = 0$ can be solved and the general solution form for $AA^\diamond = AE$ is equal to $E = A^\diamond + (I - A^-A)F$, for arbitrary $F \in \mathcal{B}(H)$.

(iv) ⇒ (ii): This implication follows by elementary computations. □

Similar result is true for $A^{\diamond,-}$.

Theorem 2.13. *For $A \in \mathcal{B}(H)^{d,-}$, an arbitrary but fixed $A^- \in A\{1\}$ and $E \in \mathcal{B}(H)$, the following statements are equivalent:*

(i) $A^{\diamond,-} = EAA^-$;
(ii) $A^\diamond A = EA$;
(iii) $R(A^\diamond A) = R(EA)$ *and* $AA^\diamond A = AEA$;
(iv) $E = A^\diamond + F(I - AA^-)$, *for arbitrary* $F \in \mathcal{B}(H)$.

The sets of all inner-gMP and gMP-inner inverses of A are marked by $A\{-,\diamond\}$ and $A\{\diamond,-\}$ and have the next properties.

Theorem 2.14. *If $A \in \mathcal{B}(H)^{d,-}$ and $A^- \in A\{1\}$ is arbitrary but fixed, then*

$$A\{-,\diamond\} = \{A^{-,\diamond} + (I - A^-A)MAA^\diamond : \ M \in \mathcal{B}(X)\}$$

and

$$A\{\diamond,-\} = \{A^{\diamond,-} + A^\diamond AM(I - AA^-) : \ M \in \mathcal{B}(X)\}.$$

Proof. This result follows by [3]

$$A\{1\} = \{A^- + M - A^-AMAA^- : \ M \in \mathcal{B}(X)\}.$$

□

3. Inner-*gMP and *gMP-inner inverses

The fact that the *gMP inverse is also an extension of the Moore–Penorse inverse motivated us to introduce inner-*gMP and *gMP-inner inverses as new types of generalized inverses which generalized the 1MP and MP1 inverses, respectively. We give the next results in this section without the proofs which are analogue to adequate results of Section 2.

Theorem 3.1. *For $A \in \mathcal{B}(H)^{d,-}$ and an arbitrary but fixed $A^- \in A\{1\}$, we have*

(a) $X = A^-AA_\diamond$ *represents the unique solution to the system*

$$XAX = X, \qquad AX = AA_\diamond \qquad \text{and} \qquad XA = A^-AA_\diamond A;$$

(b) $X = A_\diamond AA^-$ *represents the unique solution to*

$$XAX = X, \qquad AX = AA_\diamond AA^- \qquad \text{and} \qquad XA = A_\diamond A.$$

Definition 3.2. Let $A \in \mathcal{B}(H)^{d,-}$ and $A^- \in A\{1\}$ be arbitrary but fixed.

(a) The inner-*gMP inverse of A is defined by

$$A_{-,\diamond} = A^-AA_\diamond.$$

(b) The *gMP-inner inverse of A is defined by

$$A_{\diamond,-} = A_\diamond AA^-.$$

Remark that, for $A \in \mathcal{B}(H)^{\#}$, $A_\diamond = A^\dagger$ and so the inner-*gMP and *gMP-inner inverses reduce to the 1MP and MP1 inverses, respectively. In the case that $A^- = A^\dagger$, we can check that $A_{-,\diamond} = A_\diamond$.

We establish the following expressions for inner-*gMP and *gMP-inner inverses by [18, Corollary 4].

Corollary 3.3. *For $A \in \mathcal{B}(H)^{d,-}$ and an arbitrary but fixed $A^- \in A\{1\}$, we have*

$$A_{-,\diamond} = A^-AA_\diamond = A^-AA_{ⓓ}(AA_{ⓓ})^\dagger = A^-A(AA_{ⓓ}A)^\dagger$$

and

$$A_{\diamond,-} = A_\diamond AA^- = A_{ⓓ}(AA_{ⓓ})^\dagger AA^- = (AA_{ⓓ}A)^\dagger AA^-.$$

When $A \in \mathcal{B}(H)^{D,-}$ with $\text{ind}(A) = k$, Corollary 3.3 and [18, Corollary 5] give $A_{-,\diamond} = A^-A(A(A^k)^\dagger A^k)^\dagger$ and $A_{\diamond,-} = (A(A^k)^\dagger A^k)^\dagger AA^-$.

Theorem 3.4. *For $A \in \mathcal{B}(H)^{d,-}$, an arbitrary but fixed $A^- \in A\{1\}$ and $X \in \mathcal{B}(H)$, the following statements are equivalent:*

(i) $X = A_{-,\diamond}$;
(ii) $XAX = X$, $AXA = AA_\diamond A$, $AX = AA_\diamond$ *and* $XA = A^-AA_\diamond A$;
(iii) $XA = A^-AA_\diamond A$ *and* $XAA_\diamond = X$;
(iv) $XA = A^-AA_\diamond A$ *and* $XAA_{ⓓ}(AA_{ⓓ})^\dagger = X$;
(v) $XAA^\dagger = A^-AA_\diamond AA^\dagger$ *and* $XAA_\diamond = X$;
(vi) $XAA^* = A^-AA_\diamond AA^*$ *and* $XAA_\diamond = X$;
(vii) $AX = AA_\diamond$ *and* $A^-AA_\diamond AX = X$;
(viii) $AX = AA_\diamond$ *and* $A^-AX = X$;
(ix) $A^\dagger AX = A_\diamond (= A^\dagger AA_\diamond)$ *and* $A^-AX = X$;
(x) $A^*AX = A^*AA_\diamond$ *and* $A^-AX = X$;
(xi) $XAA_\diamond AX = X$, $AA_\diamond AXAA_\diamond A = AA_\diamond A$, $AA_\diamond AX = AA_\diamond$ *and* $XAA_\diamond A = A^-AA_\diamond A$;
(xii) $XAA_\diamond AX = X$, $AA_\diamond AX = AA_\diamond$ *and* $XAA_\diamond A = A^-AA_\diamond A$.

Analogously, we characterize the *gMP-inner inverse.

Theorem 3.5. *For $A \in \mathcal{B}(H)^{d,-}$, an arbitrary but fixed $A^- \in A\{1\}$ and $X \in \mathcal{B}(H)$, the following statements are equivalent:*

(i) $X = A_{\diamond,-}$;
(ii) $XAX = X$, $AXA = AA_\diamond A$, $AX = AA_\diamond AA^-$ *and* $XA = A_\diamond A$;
(iii) $AX = AA_\diamond AA^-$ *and* $A_\diamond AX = X$;
(iv) $AX = AA_\diamond AA^-$ *and* $A_{ⓓ}AX = X$;
(v) $A^\dagger AX = A^\dagger AA_\diamond AA^-$ *and* $A_{ⓓ}AX = X$;
(vi) $A^*AX = A^*AA_\diamond AA^-$ *and* $A_{ⓓ}AX = X$;
(vii) $XA = A_\diamond A$ *and* $X = XAA_\diamond AA^-$;
(viii) $XA = A_\diamond A$ *and* $XAA^- = X$;
(ix) $XAA^\dagger = A_\diamond (= A_\diamond AA^\dagger)$ *and* $XAA^- = X$;
(x) $XAA^* = A_\diamond AA^*$ *and* $XAA^- = X$;
(xi) $XAA_\diamond AX = X$, $AA_\diamond AXAA_\diamond A = AA_\diamond A$, $AA_\diamond AX = AA_\diamond AA^-$ *and* $XAA_\diamond A = A_\diamond A$;
(xii) $XAA_\diamond AX = X$, $AA_\diamond AX = AA_\diamond AA^-$ *and* $XAA_\diamond A = A_\diamond A$.

4. Applications of inner-gMP and gMP-inner inverses

Inner-gMP and gMP-inner inverses can be applied in solving some kinds of linear equations.

Theorem 4.1. *Let $b \in H$, $A \in \mathcal{B}(H)^{d,-}$ and $A^- \in A\{1\}$ be arbitrary but fixed. The general solution form of equation*

$$Ax = AA^\diamond b \tag{4.1}$$

is

$$x = A^{-,\diamond}b + (I - A^-A)z, \tag{4.2}$$

for arbitrary $z \in H$.

Proof. According to Theorem 2.1, recall that $AA^{-,\diamond} = AA^\diamond$. When x is given by (4.2), it is a solution of (4.1) by

$$Ax = AA^{-,\diamond}b + A(I - A^-A)z = AA^\diamond b.$$

Suppose that equation (4.1) has a solution x. We have that $A^-Ax = A^-AA^\diamond b = A^{-,\diamond}b$ gives

$$x = A^{-,\diamond}b + (I - A^-A)x.$$

So, (4.2) is the form of the solution x. □

In the case that $A \in \mathcal{B}(H)^\#$, by $A^\diamond = A^\dagger$, the equation (4.1) becomes $Ax = AA^\dagger b$ which is equivalent to $A^*Ax = A^*b$. The least equation, known as the normal equation of $Ax = b$, has a solution x if and only if x is a least-squares solution to the equation $Ax = b$ (i.e. $\|Ax - b\| \leq \|Az - b\|$, for all z). The least-squares solution of $Ax = b$ is an often used approximate solution in statistical applications [3].

Theorem 4.1 implies the following result for $b \in R(AA^\diamond)$.

Corollary 4.2. *Let $A \in \mathcal{B}(H)^{d,-}$ and $A^- \in A\{1\}$ be arbitrary but fixed. The general solution form of equation*

$$Ax = b, \quad b \in R(AA^\diamond), \tag{4.3}$$

is

$$x = A^-b + (I - A^-A)z,$$

for arbitrary $z \in H$.

Proof. The hypothesis $b \in R(AA^\diamond)$ yields $b = AA^\diamond b$. We finish this proof applying Theorem 4.1. □

We also consider when the solution of equation (4.1) is unique.

Theorem 4.3. *Let $b \in H$, $A \in \mathcal{B}(H)^{d,-}$ and $A^- \in A\{1\}$ be arbitrary but fixed. Then $A^{-,\diamond}b$ is uniquely determined solution in $R(A^-A(A^{ⓓ}A)^*)$ of the equation* (4.1).

Proof. By Theorem 4.1, we conclude that the equation (4.1) has a solution $A^{-,\diamond}b \in \mathcal{R}(A^{-,\diamond}) = \mathcal{R}(A^{-}A(A^{\textcircled{d}}A)^*)$.

Let $x = A^{-,\diamond}b$ and y be two solutions in $\mathcal{R}(A^{-}A(A^{\textcircled{d}}A)^*) = \mathcal{R}(A^{-,\diamond})$ of (4.1). Then

$$x - y \in N(A) \cap \mathcal{R}(A^{-,\diamond}) = N(A^{-,\diamond}A) \cap \mathcal{R}(A^{-,\diamond}A) = \{0\}$$

yields $x = y$. Hence, the equation (4.1) has the unique solution $A^{-,\diamond}b$ in $\mathcal{R}(A^{-}A(A^{\textcircled{d}}A)^*)$. □

We can verify the solvability of the next equation and determined its unique solution as Theorem 4.1 and Theorem 4.3.

Theorem 4.4. *Let $b \in H$, $A \in \mathcal{B}(H)^{d,-}$ and $A^{-} \in A\{1\}$ be arbitrary but fixed. The general solution form of equation*

$$A^{\textcircled{d}}Ax = A^{\textcircled{d}}b \tag{4.4}$$

is

$$x = A^{-,\diamond}b + (I - A^{-,\diamond}A)z,$$

for arbitrary $z \in H$.

Theorem 4.5. *Let $b \in H$, $A \in \mathcal{B}(H)^{d,-}$ and $A^{-} \in A\{1\}$ be arbitrary but fixed. Then $A^{-,\diamond}b$ is uniquely determined solution in $\mathcal{R}(A^{-}A(A^{\textcircled{d}}A)^*)$ of the equation* (4.4).

In a similar manner, we solve the following equation based on the gMP-inner inverse.

Theorem 4.6. *Let $b \in H$, $A \in \mathcal{B}(H)^{d,-}$ and $A^{-} \in A\{1\}$ be arbitrary but fixed. The general solution form of equation*

$$A^{\textcircled{d}}Ax = A^{\textcircled{d}}AA^{-}b \tag{4.5}$$

is

$$x = A^{\diamond,-}b + (I - A^{\diamond}A)z,$$

for arbitrary $z \in H$.

Theorem 4.7. *Let $b \in H$, $A \in \mathcal{B}(H)^{d,-}$ and $A^{-} \in A\{1\}$ be arbitrary but fixed. Then $A^{\diamond,-}b$ is uniquely determined solution in $\mathcal{R}((A^{\textcircled{d}}A)^*)$ of the equation* (4.5).

To illustrate Theorem 4.1 and Theorem 4.6, we give the next example.

Example 4.8. Suppose that A is represented as in Example 2.4, $z = [z_1\ z_2\ z_3]^*$ and $b = [2\ 1\ 0]^*$. Firstly, we observe that

$$\begin{aligned} x &= A^{-,\diamond}b + (I - A^{-}A)z \\ &= \begin{bmatrix} 2a + (1-3a)z_1 - az_2 - dz_3 \\ 2(1-3a) - (3-9a)z_1 + 3az_2 + 3dz_3 \\ 0 \end{bmatrix} \end{aligned}$$

satisfies

$$Ax = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = AA^{\diamond}b.$$

Theorem 4.3 implies that $A^{-,\diamond}b = [2a\ 2(1-3a)\ 0]^*$ is uniquely determined solution of the equation (4.1) in

$$\mathcal{R}(A^{-}A(A^{\textcircled{d}}A)^*) = \{[ay_1\ \ (1-3a)y_1\ \ 0]^* \ :\ y_1 \in \mathbb{C}\}.$$

Also, for

$$\begin{aligned} x &= A^{\diamond,-}b + (I - A^{\diamond}A)z \\ &= \begin{bmatrix} \frac{3}{5} + \frac{1}{10}z_1 - \frac{3}{10}z_2 \\ \frac{1}{5} - \frac{3}{10}z_1 + \frac{9}{10}z_2 \\ 0 \end{bmatrix}, \end{aligned}$$

note that

$$A^{\circledcirc} A x = \begin{bmatrix} \frac{2}{3} \\ 0 \\ 0 \end{bmatrix} = A^{\circledcirc} A A^{-} b.$$

According to Theorem 4.7, we have that $A^{\diamond,-}b = [\frac{3}{5}\ \frac{1}{5}\ 0]^*$ is the unique solution of (4.5) in

$$R((A^{\circledcirc} A)^*) = \left\{ \begin{bmatrix} y_1 & \frac{1}{3} y_1 & 0 \end{bmatrix}^* \ : \ y_1 \in \mathbb{C} \right\}.$$

5. Applications of inner-*gMP and *gMP-inner inverses

Similarly, we verify the following results for solving several linear equations using inner-*gMP and *gMP-inner inverses.

Theorem 5.1. *Let $b \in H$, $A \in \mathcal{B}(H)^{d,-}$ and $A^- \in A\{1\}$ be arbitrary but fixed. The general solution form of equation*

$$Ax = AA_{\diamond} b \tag{5.1}$$

is

$$x = A_{-,\diamond} b + (I - A^- A)z,$$

for arbitrary $z \in H$.

If $b \in R(AA_{\circledcirc})$ in Theorem 5.1, we obtain the next consequence.

Corollary 5.2. *Let $A \in \mathcal{B}(H)^{d,-}$ and $A^- \in A\{1\}$ be arbitrary but fixed. The general solution form of equation*

$$Ax = b, \quad b \in R(AA_{\circledcirc}), \tag{5.2}$$

is

$$x = A^- b + (I - A^- A)z,$$

for arbitrary $z \in H$.

The uniqueness of the solution to equation (5.1) can be obtained.

Theorem 5.3. *Let $b \in H$, $A \in \mathcal{B}(H)^{d,-}$ and $A^- \in A\{1\}$ be arbitrary but fixed. Then $A_{-,\diamond} b$ is uniquely determined solution in $R(A^- A)$ of the equation* (5.1)*.*

One more equation can be solved using inner-*gMP inverse.

Theorem 5.4. *Let $b \in H$, $A \in \mathcal{B}(H)^{d,-}$ and $A^- \in A\{1\}$ be arbitrary but fixed. The general solution form of equation*

$$(AA_{\circledcirc})^* A x = (AA_{\circledcirc})^* b \tag{5.3}$$

is

$$x = A_{-,\diamond} b + (I - A_{-,\diamond} A)z,$$

for arbitrary $z \in H$.

Theorem 5.5. *Let $b \in H$, $A \in \mathcal{B}(H)^{d,-}$ and $A^- \in A\{1\}$ be arbitrary but fixed. Then $A_{-,\diamond} b$ is uniquely determined solution in $R(A^- AA_{\circledcirc})$ of the equation* (5.3)*.*

We now consider solvability of equations by *gMP-inner inverse.

Theorem 5.6. *Let $b \in H$, $A \in \mathcal{B}(H)^{d,-}$ and $A^- \in A\{1\}$ be arbitrary but fixed. The general solution form of equation*

$$A_{\circledcirc} A x = A_{\diamond,-} b \tag{5.4}$$

is

$$x = A_{\diamond,-} b + (I - A_{\circledcirc} A)z,$$

for arbitrary $z \in H$.

Corollary 5.7. *Let $A \in \mathcal{B}(H)^{d,-}$ and $A^- \in A\{1\}$ be arbitrary but fixed. The general solution form of equation*

$$A_{\textcircled{d}}Ax = A_{\diamond}b, \quad b \in R(A),$$

is

$$x = A_{\diamond}b + (I - A_{\textcircled{d}}A)z,$$

for arbitrary $z \in H$.

Theorem 5.8. *Let $b \in H$, $A \in \mathcal{B}(H)^{d,-}$ and $A^- \in A\{1\}$ be arbitrary but fixed. Then $A_{\diamond,-}b$ is uniquely determined solution in $R(A_{\textcircled{d}})$ of the equation* (5.4).

Theorem 5.9. *Let $b \in H$, $A \in \mathcal{B}(H)^{d,-}$ and $A^- \in A\{1\}$ be arbitrary but fixed. The general solution form of equation*

$$(AA_{\textcircled{d}})^*Ax = (AA_{\textcircled{d}})^*AA^-b \tag{5.5}$$

is

$$x = A_{\diamond,-}b + (I - A_{\diamond}A)z,$$

for arbitrary $z \in H$.

Corollary 5.10. *Let $A \in \mathcal{B}(H)^{d,-}$ and $A^- \in A\{1\}$ be arbitrary but fixed. The general solution form of equation*

$$(AA_{\textcircled{d}})^*Ax = (AA_{\textcircled{d}})^*b, \quad b \in R(A),$$

is

$$x = A_{\diamond}b + (I - A_{\diamond}A)z,$$

for arbitrary $z \in H$.

Theorem 5.11. *Let $b \in H$, $A \in \mathcal{B}(H)^{d,-}$ and $A^- \in A\{1\}$ be arbitrary but fixed. Then $A_{\diamond,-}b$ is uniquely determined solution in $R(A_{\textcircled{d}})$ of the equation* (5.5).

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