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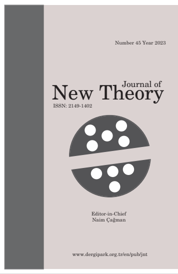

On Non-Archimedean $\mathcal{L}$-Fuzzy Vector Metric Spaces

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Abstract — This paper contributes to the broader studies of fuzzy vector metric spaces and fuzzy metric spaces based on order structures beyond the unit interval. It defines the notions of the left (right) order convergence and continuity in non-Arcimedean $\mathcal{L}$-fuzzy vector metric spaces. The notation $\mathcal{M}_E(a, b, s)$ means the nearness between a and b according to any positive vector s. This study exemplifies definitions and reaches some well-known results. Moreover, it proposes the concept of $\mathcal{L}$-fuzzy vector metric diameter and studies some of its basic properties. Further, the present paper proves the Cantor intersection theorem and the Baire category theorem via these concepts. Finally, this study discusses the need for further research.

Keywords *Non-Archimedean $\mathcal{L}$-fuzzy vector metrics, left and right order convergence, $\mathcal{L}$-fuzzy vector diameter, Riesz spaces*

Mathematics Subject Classification (2020) 46A40, 47H10

1. Introduction

In the field of engineering design, it is often the case that there is no clear solution or design, which often leads to fuzziness, and Zadeh [1] proposed a rule to address such issues in engineering and design. Goguen [2] expanded Zadeh's study with a fresh viewpoint, considering the ordered structures beyond the unit interval. It is typically necessary for a partially ordered set (poset) to be at least a complete lattice with distributive law to query what the maximum and minimum values of a fuzzy set are. A detailed study about these concepts can be found in [2, 3].

Moreover, Menger [4] presented probabilistic metric spaces and associated ideas. The notion was then greatly improved by Schweizer and Sklar [5, 6]. Subsequently, Kramosil and Michálek in [7] provided an equivalent definition for the term probabilistic metric in the form of fuzzy metric spaces, which George and Veeramani [8] later adapted to provide a Hausdorff topology. The degree of nearness between two elements a and b of a set X concerning the real number s is the subject of the notion of fuzzy metric. The reality of X having a vector space structure is a common occurrence (for more details, see [9–11]). Alternatively, the distance in a Riesz space can be defined as a vector; more details can be found in [12–15].

In this study, we consider the parameter s as a vector based on $\mathcal{L}$-fuzzy sets given by Goguen and the fuzzy metric space provided by Kramosil and Michálek. In this case, the order structure must be added to the concept of left-hand continuity. Thus, we define left (right) order continuity to construct

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$\mathcal{L}$-fuzzy vector metric spaces and non-Archimedean $\mathcal{L}$-fuzzy vector metric spaces. Then, we obtain some new results and provide Cantor's intersection theorem and Baire's theorem in non-Archimedean $\mathcal{L}$-fuzzy vector metric spaces.

2. Preliminaries

This section provides some basic notions to be needed in the next section. The concept of an $\mathcal{L}$-fuzzy set was introduced by Goguen [2], who generalized the notion of a fuzzy set nicely introduced by Zadeh. Goguen defined an $\mathcal{L}$-fuzzy set as a function that maps elements of a universe of discourse to elements of a complete lattice $\mathcal{L}$, where each lattice element represents the degree of membership of the corresponding universe element in the fuzzy set. He defined $\mathcal{L}$-fuzzy set in the following manner.

Definition 2.1. [2] Let $X \neq \emptyset$ and $\mathcal{L} = (L, \leqslant_L)$ be a complete lattice with distributive law. Then, an $\mathcal{L}$-fuzzy set $\mathcal{A}$ is a function such that $\mathcal{A} : X \to L$ and $\mathcal{A}(a)$, for each $a \in X$, means the degree of a in L.

Definition 2.2. [10] Let $X \neq \emptyset$. Then, an intuitionistic $\mathcal{L}$-fuzzy set $\mathcal{A}_{\xi,\vartheta}$ is an object on X such that $\mathcal{A}_{\xi,\vartheta} = \{(\xi_{\mathcal{A}}(a), \vartheta_{\mathcal{A}}(a)) : a \in X\}$, where the notations $\xi_{\mathcal{A}}(a)$ and $\vartheta_{\mathcal{A}}(a)$ represent the membership and non-membership degrees of a, respectively, and satisfy the condition $\xi_{\mathcal{A}}(a) + \vartheta_{\mathcal{A}}(a) \leqslant_L 1_{\mathcal{L}}$.

Goguen [2] and Sadati et al. [10] provided the definitions of t-norm, decreasing negation function, and involutive negation as follows:

Definition 2.3. [2, 10] A t-norm on $\mathcal{L}$ is a function $\mathcal{T} : L^2 \to L$ holding following properties, for all $k, l, m, n \in L$, where $\inf L = 0_{\mathcal{L}}$ and $\sup L = 1_{\mathcal{L}}$.

i. $\mathcal{T}(k, 1_{\mathcal{L}}) = k$ (boundary condition)

ii. $\mathcal{T}(k, l) = \mathcal{T}(l, k)$ (commutativity)

iii. $\mathcal{T}(k, \mathcal{T}(l, m)) = \mathcal{T}(\mathcal{T}(k, l), m)$ (associativity)

iv. $k \leqslant_L m$ and $l \leqslant_L n \Rightarrow \mathcal{T}(k, l) \leqslant_L \mathcal{T}(m, n)$ (monotonicity)

Definition 2.4. [2, 10] Let $\mathcal{L} = (L, \leqslant_L)$ be a complete lattice. Then, $\mathcal{N} : L \to L$ is a decreasing negation function on $\mathcal{L}$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$. Furthermore, $\mathcal{N}$ is called an involutive negation if $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L$.

Aliprantis, in his books Infinite Dimensional Analysis [12] and Positive Operators [13], discussed the concept of ordered vector space in the following fashion.

Definition 2.5. [12,13] Let E be a real vector space. If E has an order relation $\leq$, which is compatible with the algebraic structure of E in terms of the following two axioms:

i. if $s \leq u$, then $s + w \leq u + w$, for all $w \in E$

ii. if $s \leq u$, then $\gamma s \leq \gamma u$, for all $\gamma \in \mathbb{R}^+$

then E is called an ordered vector space.

For any two vectors $s, u \in E$, the notation $s \leq u$ can be represented by $u \geq s$ in another way. If $\theta \leq s$ where θ represents the zero vector of E, then the vector s is called positive. The set of all the positive vectors of E is denoted by $E_+ := \{s \in E : \theta \leq s\}$.

Aliprantis et al. [12, 13] also proposed the concept of Riesz spaces and some related concepts in the following form.

Definition 2.6. [12, 13] Let E be an ordered vector space. For all $s, u \in E$, if E has the supremum and the infimum of the set $\{s, u\}$, then E is called a Riesz space or a vector lattice. The notations used for $\sup\{s,u\}$ and $\inf\{s,u\}$ are as follows:

$$s \vee u = \sup\{s,u\} \quad \text{and} \quad s \wedge u = \inf\{s,u\}$$

An example of a Riesz space is the space of real-valued continuous functions on a set X, considering the pointwise ordering, defined as follows: $f_1 \leq f_2$ in E if and only if $f_1(a) \leq f_2(a)$, for all $a \in X$. The lattice operation in any function space E can be defined as

$$[f_1 \vee f_2](a) = \max\{f_1(a), f_2(a)\} \quad \text{and} \quad [f_1 \wedge f_2](a) = \min\{f_1(a), f_2(a)\}$$

for each pair $f_1, f_2 \in E$ and for all $a \in X$.

We will denote Riesz spaces with the letter E in the rest of this study.

Theorem 2.7. [12,13] For all $s,u,w \in E$, the following properties hold:

i. $s \vee u = -[(-s) \wedge (-u)]$ and $s \wedge u = -[(-s) \vee (-u)]$

ii. $s + u = (s \wedge u) + (s \vee u)$

iii. $s + (u \vee w) = (s+u) \vee (s+w)$ and $s + (u \wedge w) = (s+u) \wedge (s+w)$

iv. $\gamma(s \vee u) = (\gamma s) \vee (\gamma u)$ and $\gamma(s \wedge u) = (\gamma s) \wedge (\gamma u)$, for all $\gamma \geqslant 0$

For any vector $s \in E$, the positive part, negative part, and absolute value of s are denoted by s^+, s^-, and $|s|$, respectively, and defined as follows:

$$s^+ := s \vee \theta, \quad s^- := (-s) \vee \theta, \quad \text{and} \quad |s| = s \vee (-s)$$

Theorem 2.8. [12,13] For any vector $s \in E$, the following properties hold:

i. $s = s^+ - s^-$

ii. $|s| = s^+ + s^-$

iii. $s^+ \wedge s^- = \theta$

A sequence $(s_n) \subseteq E$ is decreasing, denoted by $s_n \downarrow$, if and only if $n \geqslant m$ implies $s_n \leq s_m$. In addition the notation $s_n \downarrow s$ means $s_n \downarrow$ and $\inf\{s_n\} = s$. Similarly, a sequence $(s_n) \subseteq E$ is increasing, represented by $s_n \uparrow$, if and only if $n \leqslant m$ implies $s_n \leq s_m$. In addition the notation $s_n \uparrow s$ means $s_n \uparrow$ and $\sup\{s_n\} = s$.

Aliprantis et al. [12,13] set forth the concepts of ordered convergence and lattice norm in the following way.

Definition 2.9. [12,13] Let $(s_n) \subseteq E$ be a sequence and $s \in E$ be a vector. Then, (s_n) is called order convergent to s, denoted by $s_n \xrightarrow{o} s$, if there exists another sequence (u_n) satisfying $|s_n - s| \leq u_n \downarrow \theta$.

Definition 2.10. [3,13] Let s and u be some vectors of E and $\|\cdot\|$ be a defined norm on E. If $|s| \leq |u|$ implies $\|s\| \leq \|u\|$, then $\|\cdot\|$ is called a lattice norm. In addition, a Riesz space equipped with this norm is called a normed Riesz space.

The notion of vector metric spaces, where the distance function takes values in Riesz spaces, was first mentioned in [14].

Definition 2.11. [14] Let $X \neq \emptyset$, E be a Riesz space, and $d_E : X \times X \to E$ be a function. Then, (X, d_E) is called a vector metric space if the function d_E satisfies the following properties, for all $a,b,c \in X$:

i. $\theta \leq d_E(a,b)$

ii. $d_E(a,b) = \theta$ if and only if $a = b$

iii. $d_E(a,b) = d_E(b,a)$

iv. $d_E(a,c) \leq d_E(a,b) + d_E(b,c)$

Since the set of real numbers $\mathbb{R}$ is a Riesz space with the usual ordering, it is obvious that every metric space is a vector metric space.

Example 2.12. [14] Every Riesz space E is a vector metric space with the function $d_E : E \times E \to E$ defined by $d_E(a,b) = |a-b|$. This vector metric is called the absolute valued vector metric on E.

To set up the definition of non-Archimedean $\mathcal{L}$-fuzzy vector metric spaces, we benefit from the definition of fuzzy metric space suggested by Kramosil and Michálek [7].

Definition 2.13. [7] Let $X \neq \emptyset$, M be a fuzzy set on $X \times X \times [0,\infty)$, and $\mathcal{T}$ be a continuous t-norm. Then, the triple $(X, M, *)$ is a fuzzy metric space as Kramosil and Michálek describe, if for all $a,b,c \in X$ and $0 < s,u$, the following properties hold:

i. $M(a,b,0) = 0$

ii. $M(a,b,s) = 1$ if and only if $a = b$

iii. $M(a,b,s) = M(b,a,s)$

iv. $\mathcal{T}(M(a,b,s), M(b,c,u)) \leqslant M(a,c,s+u)$

v. $M(a,b,.) : [0,\infty) \to [0,1]$ is left-continuous

Here, the notation $M(a,b,s)$ denotes the nearness degree between a and b according to s.

3. Main Results

We define the concepts of left and right-order convergence and continuity. Thanks to these concepts, new ideas on $\mathcal{L}$-fuzzy vector metric space will be built.

Definition 3.1. Let $(s_n) \subseteq E$ be a sequence and $s \in E$ be a vector. Then,

i. (s_n) is called left-order convergent to some vector s, denoted by $s_n \overset{o^-}{\to} s$, if there exists another sequence (u_n) satisfying $(s_n - s)^- \leq u_n \downarrow \theta$.

ii. (s_n) is called right-order convergent to some vector s, denoted by $s_n \overset{o^+}{\to} s$, if there exists another sequence (u_n) satisfying $(s_n - s)^+ \leq u_n \downarrow \theta$.

Definition 3.2. Let $X \neq \emptyset$, $\mathcal{M}_E$ be an $\mathcal{L}$-fuzzy set on $X \times X \times E^+$, and $\mathcal{T}$ be a continuous t-norm on $\mathcal{L}$. Then, the triple $(X, \mathcal{M}_E, \mathcal{T})$ is an $\mathcal{L}$-fuzzy vector metric space if for all $a,b,c \in X$ and $s,u \in E_+$, the following properties hold:

i. $\mathcal{M}_E(a,b,\theta) = 0_{\mathcal{L}}$

ii. $\mathcal{M}_E(a,b,s) = 1_{\mathcal{L}}$ if and only if $a = b$

iii. $\mathcal{M}_E(a,b,s) = \mathcal{M}_E(b,a,s)$

iv. $\mathcal{T}(\mathcal{M}_E(a,b,s), \mathcal{M}_E(b,c,u)) \leqslant_L \mathcal{M}_E(a,c,s+u)$

v. $\mathcal{M}_E(a,b,.) : E_+ \to L$ is left-order-continuous

If the condition *vi* below is used instead of the condition *iv*, then the triple $(X, \mathcal{M}_E, \mathcal{T})$ is said to be a non-Archimedean $\mathcal{L}$-fuzzy vector metric space.

vi. $\mathcal{T}(\mathcal{M}_E(a,b,s), \mathcal{M}_E(b,c,u)) \leqslant_L \mathcal{M}_E(a,c,s \vee u)$

It can be observed that every non-Archimedean $\mathcal{L}$-fuzzy vector metric space is an $\mathcal{L}$-fuzzy vector metric space because the triangular inequality *vi* implies *iv*. Moreover, if $s \wedge u = 0$, then every $\mathcal{L}$-fuzzy vector metric space becomes a non-Archimedean $\mathcal{L}$-fuzzy vector metric space.

Lemma 3.3. In a non-Archimedean $\mathcal{L}$-fuzzy vector metric space, the function $\mathcal{M}_E(a, b, .)$ is non-decreasing, for all $a, b \in X$.

Lemma 3.4. In a non-Archimedean $\mathcal{L}$-fuzzy vector metric space, the following statements hold:

i. If $s_n \overset{o}{\to} s$ and $s_n \overset{o}{\to} u$, then $\mathcal{M}_E(a, b, s) = \mathcal{M}_E(a, b, u)$

ii. If $s_n \overset{o}{\to} s$ and $u \leqslant s_n$ hold for $n \in \mathbb{N}$, then $\mathcal{M}_E(a, b, u) \leqslant_L \mathcal{M}_E(a, b, s)$

iii. If $s_n \downarrow$ and $s_n \overset{o}{\to} s$, which means both $s_n \overset{o^+}{\to} s$ and $s_n \downarrow s$, then for all $n \in \mathbb{N}$, $\mathcal{M}_E(x, y, s) \leqslant_L \mathcal{M}_E(x, y, s_n)$

iv. If $s_n \uparrow$ and $s_n \overset{o}{\to} s$, which means both $s_n \overset{o^-}{\to} s$ and $s_n \uparrow s$, then for all $n \in \mathbb{N}$, $\mathcal{M}_E(a, b, s_n) \leqslant_L \mathcal{M}_E(a, b, s)$

v. If $s_n \overset{o}{\to} s$ and $u_n \overset{o}{\to} u$, then $\lim_{n\to\infty} \mathcal{M}_E(a, b, ks_n + ru_n) = \mathcal{M}_E(a, b, ks + ru)$, for all $n \in \mathbb{N}$ and $k, r \in \mathbb{R}$

Corollary 3.5. By the definition $s^+ := s \vee \theta$, if $s, u \in E_+$, then $\mathcal{M}_E(a, c, s \vee u) = \mathcal{M}_E(a, c, s^+ \vee u^+)$.

Theorem 3.6. Let $\emptyset \neq A \subseteq E_+$ and $s \in E_+$. If $\inf A$ exists, then the infimum of the set $(s \vee A)$ exists and

$$\mathcal{T}(\mathcal{M}_E(a, b, s), \mathcal{M}_E(b, c, \inf A)) \leqslant_L \mathcal{M}_E(a, c, s \vee \inf A) = \mathcal{M}_E(a, c, \inf(s \vee A))$$

PROOF.
Assume that $\inf A$ exists. Let $u = \inf A$, then $s \vee u \leq s \vee w$, for all $w \in A$ and $s \in E$, which means that $s \vee u$ is a lower bound of the set $s \vee A$ and $\mathcal{M}_E(a, b, s \vee u) \leqslant_L \mathcal{M}_E(a, b, s \vee w)$ holds. Let r be another lower bound. To show that $s \vee u$ is the greatest lower bound of $s \vee A$, we must show $r \leq s \vee u$. Besides, $w + s = (s \wedge w) + (s \vee w)$, for all $w \in E$. From the properties in Theorem 2.7,

$$w = (s \wedge w) + (s \vee w) - s \geq (s \wedge w) + r - s \geq (s \wedge u) + r - s$$

Because $\inf A = u$, it follows that $u \geq (s \wedge u) + r - s$. This implies $u \geq (u + s) - (s \vee u) + r - s$. Thus, $s \vee u \geq r$ is obtained. It means that $s \vee u$ is the greatest lower bound. Then, $\inf(s \vee A)$ exists and $\inf(s \vee A) = s \vee \inf A$. Consequently,

$$\mathcal{T}(\mathcal{M}_E(a, b, s), \mathcal{M}_E(b, c, \inf A)) \leqslant_L \mathcal{M}_E(a, c, s \vee \inf A) = \mathcal{M}_E(a, c, \inf(s \vee A))$$

□

Example 3.7. Let $(X, \mathcal{M}_E, \mathcal{T})$ be an $\mathcal{L}$-fuzzy vector space with (s_n) and (u_n) in $C[0, 1] = \{h \mid h : [0, 1] \to \mathbb{R}$ is a continuous function$\}$ define as follows:

$$s_n = \begin{cases} 0 & , \quad x \in [0, \frac{1}{n+1}] \\ \frac{(n+1)x-1}{n} & , \quad x \in (\frac{1}{n+1}, 1] \end{cases}$$

$$u_n = \begin{cases} -(n+1)x + 1 & , \quad x \in [0, \frac{1}{n+1}] \\ 0 & , \quad x \in (\frac{1}{n+1}, 1] \end{cases}$$

Since $s_n \uparrow 1_{\mathcal{L}} = \mathbb{1}$ and $u_n \downarrow 0_{\mathcal{L}} = \theta$, then $s_n \wedge u_n = \theta$ holds, where $\mathbb{1}(x) = 1$ and $\theta(x) = 0$ are constant functions in $C[0, 1]$. Hence, $(X, \mathcal{M}_E, \mathcal{T})$ becomes a non-Archimedean $\mathcal{L}$-fuzzy vector metric space.

Example 3.8. Let the pair (X, d_E) be a bounded vector metric space such that $d_E(a, b) < k$, for all $a, b \in X$ and $k \in E$. In addition, let $g : E_+ \to (\|k\|, +\infty)$ be an increasing continuous function.

Define $\mathcal{T}(l,t) = \sup\{l + t - 1_{\mathcal{L}}, 0_{\mathcal{L}}\}$ and the function $\mathcal{M}_E$ by

$$\mathcal{M}_E(a,b,s) = 1_{\mathcal{L}} - \frac{d_E(a,b)}{g(s)}$$

In this case, $(X, \mathcal{M}_E, \mathcal{T})$ becomes a non-Archimedean $\mathcal{L}$-fuzzy vector metric space.

Example 3.9. For $\mathcal{T}(k,l) = \inf\{k,l\}$, define the function $\mathcal{M}_E$ by

$$\mathcal{M}_E(a,b,s) = \begin{cases} 1, & a = b \\ \varphi(s), & a \neq b \end{cases}$$

where $\varphi : E_+ \to [0_{\mathcal{L}}, 1_{\mathcal{L}})$ is an increasing continuous function. In this case, $(X, \mathcal{M}_E, \mathcal{T})$ becomes a non-Archimedean $\mathcal{L}$-fuzzy vector metric space.

Example 3.10. Let the pair (X, d_E) be a vector metric space and E be a normed Riesz space. For all $a, b \in X$ and $s \in E_+$ and for $\mathcal{T}(k,l) = \inf\{k,l\}$, define the function $\mathcal{M}_E$ by

$$\mathcal{M}_E(a,b,s) = \frac{\|s\|}{\|s\| + \|d_E(a,b)\|}$$

Particularly, $\mathcal{M}_E$ is called the standard $\mathcal{L}$-fuzzy vector metric induced by the vector metric d_E. Then, $(X, \mathcal{M}_E, \mathcal{T})$ becomes a non-Archimedean $\mathcal{L}$-fuzzy vector metric space.

Moreover, this example is used successfully in color image processing in [9,11] as a real-life application. For this, let F_i and F_j be two image pixels. In this case, the spatial closeness between F_i and F_j is calculated with

$$\mathcal{S}(F_i, F_j, s) = \frac{s}{s + \|d_E(F_i, F_j)\|}$$

where $s \in \mathbb{R}^+$ is a parameter adjusting the sensitivity of S.

Definition 3.11. Let $(X, \mathcal{M}_E, \mathcal{T})$ be a non-Archimedean $\mathcal{L}$-fuzzy vector metric space. In this case, $\mathcal{B}_E(a,r,s)$ and $\mathcal{B}_E[a,r,s]$, for $s \in E_+$, with center $a \in X$ and radius $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ are defined as follows:

$$\mathcal{B}_E(a,r,s) = \{b \in X : \mathcal{M}_E(a,b,s) >_L \mathcal{N}(r)\}$$

and

$$\mathcal{B}_E[a,r,s] = \{b \in X : \mathcal{M}_E(a,b,s) \geqslant_L \mathcal{N}(r)\}$$

Corollary 3.12. A subset $\Omega \subseteq X$ is said to be open if for $a \in \Omega$, there exist an $s \in E_+$ and a radius $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $\mathcal{B}_E(a,r,s) \subset \Omega$. Then, every open ball is an open set. Futhermore, $\tau_{\mathcal{M}_E} = \{\Omega \subseteq X : \Omega \text{ is open}\}$ is a topology on X.

Definition 3.13. Let $(X, \mathcal{M}_E, \mathcal{T})$ be a non-Archimedean $\mathcal{L}$-fuzzy vector metric space.

i. Let $\emptyset \neq \Omega \subseteq X$. For every $a, b \in \Omega$ and $s \in E_+$, if there exists an $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $\mathcal{M}_E(a,b,s) \geqslant_L \mathcal{N}(r)$, then Ω is bounded. Moreover, for all $n \in \mathbb{N}$, $(a_n) \subseteq X$ is called bounded if there exists an $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $(a_n) \subseteq \mathcal{B}_E[a,r,s]$.

ii. For every $\varepsilon \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and $s \in E_+$, $(a_n) \subseteq X$ is convergent to $a \in X$ if there exists $n_0 \in \mathbb{N}$ such that $\mathcal{M}_E(a_n, a, s) >_L \mathcal{N}(\varepsilon)$, for all $n \geqslant n_0$ and denoted by

$$\lim_{n\to\infty} \mathcal{M}_E(a_n, a, s) = 1_{\mathcal{L}} \text{ or } a_n \overset{\mathcal{M}_E}{\to} a$$

iii. For each $\varepsilon \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and $s \in E_+$, $(a_n) \subseteq X$ is a Cauchy sequence in X if there exists $n_0 \in \mathbb{N}$ such that $\mathcal{M}_E(a_n, a_m, s) >_L \mathcal{N}(\varepsilon)$, for all $n, m \geqslant n_0$.

iv. $(X, \mathcal{M}_E, \mathcal{T})$ is complete if and only if every Cauchy sequence in X is convergent.

v. Let $\Omega \subseteq X$. Then, Ω is said to be closed if $(a_n) \subseteq \Omega$ and $a_n \overset{\mathcal{M}_E}{\to} a$ imply $a \in \Omega$.

In the following example, we provide a nonconvergent sequence in a non-Archimedean $\mathcal{L}$-fuzzy vector metric space.

Example 3.14. Let $X = (a_n) \cup \{1\}$ for $(a_n) \subseteq \mathbb{R}^+$ with $a_n \uparrow 1$. Define $\mathcal{M}_E(a_n, a_n, s) = 1_\mathcal{L}$, $\mathcal{M}_E(1, 1, s) = 1_\mathcal{L}$, and

$$\mathcal{M}_E(a_n, 1, s) = \begin{cases} \inf\{a_n, s\} & , \quad \theta < s < \mathbb{1} \\ a_n & , \quad s > \mathbb{1} \end{cases}$$

for all n and $s \in E^+$. Then, $(X, \mathcal{M}_E, \mathcal{T})$ is a non-Archimedean $\mathcal{L}$-fuzzy vector metric space where $\mathcal{T}(k, l) = \inf\{k, l\}$. Since $\lim_{n\to\infty} \mathcal{M}_E(a_n, 1, \frac{\mathbb{1}}{3}) = \frac{\mathbb{1}}{3}$, (a_n) is not a convergent sequence in this space.

Proposition 3.15. Let $(X, \mathcal{M}_{E_1}, \mathcal{T})$ and $(Y, \mathcal{M}_{E_2}, \mathcal{T})$ be two non-Archimedean $\mathcal{L}$-fuzzy vector metric spaces. If

$$\mathcal{M}_E((a_1, b_1), (a_2, b_2), s) = \mathcal{T}(\mathcal{M}_{E_1}(a_1, a_2, s), \mathcal{M}_{E_2}(b_1, b_2, s))$$

for $(a_1, b_1), (a_2, b_2) \in X \times Y$ and for all $s \in E_+$, then $\mathcal{M}_E$ is a non-Archimedean $\mathcal{L}$-fuzzy vector metric on $X \times Y$.

Note 3.16. For the rest of this study, $\mathcal{T}$ stands for a continuous t-norm on L such that for any $s \in E_+$ and $\varepsilon \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$, there exists an element $r \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ satisfying the condition $\mathcal{T}(\mathcal{N}(r), \mathcal{N}(r)) \geqslant_L \mathcal{N}(\varepsilon)$.

Theorem 3.17. Let $\mathcal{M}_E$ be defined as in Proposition 3.15 and $(a_n) \subseteq X$ and $(b_n) \subseteq Y$ be two sequences. If $a_n \overset{\mathcal{M}_{E_1}}{\to} a$ in X and $b_n \overset{\mathcal{M}_{E_2}}{\to} b$ in Y, then $(a_n, b_n) \overset{\mathcal{M}_E}{\to} (a, b)$ in $X \times Y$.

Proof.
Let $a_n \overset{\mathcal{M}_{E_1}}{\to} a$ in X and $b_n \overset{\mathcal{M}_{E_2}}{\to} b$ in Y. Then, according to Definition 3.13 (*ii*) there exist $n_1 \in \mathbb{N}$ and $n_2 \in \mathbb{N}$ such that $\mathcal{M}_{E_1}(a_n, a, s) >_L \mathcal{N}(r)$, for all $n \geqslant n_1$ and $\mathcal{M}_{E_2}(b_n, b, s) >_L \mathcal{N}(r)$, for all $n \geqslant n_2$. If $n_0 = \max\{n_1, n_2\}$, then

$$\begin{aligned} \mathcal{M}_E((a_n, b_n), (a, b), s) &= \mathcal{T}(\mathcal{M}_{E_1}(a_n, a, s), \mathcal{M}_{E_2}(b_n, b, s)) \\ &>_L \mathcal{T}(\mathcal{N}(r), \mathcal{N}(r)) \\ &\geqslant_L \mathcal{N}(\varepsilon) \end{aligned}$$

is obtained. Thus, the proof is completed. □

Theorem 3.18. Suppose $(X, \mathcal{M}_E, \mathcal{T})$ be a non-Arcimedean $\mathcal{L}$-fuzzy vector metric space and $(a_n) \subseteq X$ be a convergent sequence. Then, the following properties hold:

i. (a_n) is bounded and its limit is unique.

ii. (a_n) is a Cauchy sequence.

iii. Any subsequence (a_{n_k}) of (a_n) converges to the same limit.

Proof.
Suppose $(X, \mathcal{M}_E, \mathcal{T})$ be a non-Arcimedean $\mathcal{L}$-fuzzy vector metric space and $(a_n) \subseteq X$ be a convergent sequence.

i. Let $a_n \overset{\mathcal{M}_E}{\to} a$. Then, for each $\varepsilon, \eta \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ and $s \in E_+$, there exists $n_1 \in \mathbb{N}$ such that $\mathcal{M}_E\left(a_n, a, s/2\right) \geqslant_L \mathcal{N}(\varepsilon)$, for all $n \geqslant n_1$ and $a_0 \in X$ such that $\mathcal{M}_E\left(a_0, a, s/2\right) \geqslant_L \mathcal{N}(\eta)$. For some $\lambda \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$, suppose

$$\min\left\{\mathcal{M}_E\left(a_n, a, s/2\right) : n_1 > n\right\} = \mathcal{N}(\lambda)$$

Then, an $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ can be found such that

$$\min\{\mathcal{T}(\mathcal{N}(\eta), \mathcal{N}(\lambda)), \mathcal{T}(\mathcal{N}(\eta), \mathcal{N}(\varepsilon))\} = \mathcal{N}(r)$$

Thereby, for all $n \in \mathbb{N}^+$

$$\mathcal{M}_E(a_0, a_n, s) \geqslant_L \mathcal{T}(\mathcal{M}_E(a_0, a, s/2), \mathcal{M}_E(a_n, a, s/2)) \geqslant_L \mathcal{N}(r)$$

is obtained. As a result, $(a_n) \subseteq \mathcal{B}_E[a_0, r, s]$, which means (a_n) is bounded. To illustrate the uniqueness of the limit, suppose the sequence (a_n) has two different limits a and b. Let $\varepsilon = \mathcal{N}(\mathcal{M}_E(a, b, s))$, for any $s \in E_+$. Since (a_n) is convergent, then there exist $n_1, n_2 \in \mathbb{N}$ such that $\mathcal{M}_E(a_n, a, s/2) \geqslant_L \mathcal{N}(\lambda)$ and $\mathcal{M}_E(a_n, b, s/2) \geqslant_L \mathcal{N}(\lambda)$, for all $n \geqslant n_1, n_2$. Let $n_0 = \max\{n_1, n_2\}$. Then, for $n \geqslant n_0$,

$$\begin{aligned} \mathcal{M}_E(a, b, s) &\geqslant_L \mathcal{T}(\mathcal{M}_E(a_n, a, s/2), \mathcal{M}_E(a_n, b, s/2)) \\ &>_L \mathcal{T}(\mathcal{N}(\lambda), \mathcal{N}(\lambda)) \\ &\geqslant_L \mathcal{N}(\varepsilon) \end{aligned}$$

which means a contraction. Hence, the limit of the convergent sequence is unique.

ii. Let $s \in E_+$ and $\varepsilon \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$. Because of the convergent of the sequence (a_n), there exists $n_0 \in \mathbb{N}$ such that $\mathcal{M}_E(a_n, a, s/2) > \mathcal{N}(\lambda)$, for all $n \geqslant n_0$. Then, for all $m \geqslant n_0$,

$$\begin{aligned} \mathcal{M}_E(a_n, a_m, s) &\geqslant_L \mathcal{T}(\mathcal{M}_E(a_n, a, s/2), \mathcal{M}_E(a, a_m, s/2)) \\ &>_L \mathcal{T}(\mathcal{N}(r), \mathcal{N}(r)) \\ &\geqslant_L \mathcal{N}(\varepsilon) \end{aligned}$$

Thus, every convergent sequence is a Cauchy sequence.

iii. Let $a_n \overset{\mathcal{M}_E}{\to} a$ and $(a_{n_i}) \subseteq (a_n)$. Thus, for all $\varepsilon \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and $s \in E_+$, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{M}_E(a_n, a, s/2) > \mathcal{N}(\varepsilon)$, for all $n \geqslant n_0$. If $i \geqslant n_0$, then $n_0 \leqslant i \leqslant n_i$ and thus $\mathcal{M}_E(a_{n_i}, a, s) > \mathcal{N}(\varepsilon)$.

□

Definition 3.19. Let $(X, \mathcal{M}_E, \mathcal{T})$ be a non-Archimedean $\mathcal{L}$-fuzzy vector metric space and $\Omega \subseteq X$. Then, the $\mathcal{L}$-fuzzy vector metric diameter $\mathcal{D}_E(\Omega)$ is defined as follows:

$$\mathcal{D}_E(\Omega) = \sup_{s \in E_+} \{\inf \mathcal{M}_E(a, b, s) : a, b \in \Omega\}$$

If $\mathcal{D}_E(\Omega) = 1_L$, then Ω is said to be bounded.

Remark 3.20. If Ω is a singleton set, then $\mathcal{D}_E(\Omega) = 1_L$. However, unlike crisp sets, the converse may not always be true. For example, for the standard non-Archimedean $\mathcal{L}$-fuzzy vector metric defined in Example 3.10 as follows

$$\mathcal{M}_E(a, b, s) = \frac{\|s\|}{\|s\| + \|d_E(a, b)\|}$$

and for $\Omega = \{a_0, b_0\} \subset X$,

$$\mathcal{D}_E(\Omega) = \sup_{s \in E_+} \frac{\|s\|}{\|s\| + \|d_E(a_0, b_0)\|} = 1_L$$

is obtained.

Theorem 3.21. For $\mathcal{D}_E(\Omega)$, the following statements hold:

i. Let $\Omega \subseteq \Psi$. Then, $\mathcal{D}_E(\Psi) \leqslant_L \mathcal{D}_E(\Omega)$

ii. $\mathcal{D}_E(\Omega) \leqslant_L \mathcal{M}_E(a, b, s)$, for any $a, b \in \Omega$

iii. Let $\Omega = \{a, b\}$. Then, $\mathcal{D}_E(\Omega) = \mathcal{M}_E(a, b, s)$

iv. Let $\Omega \cap \Psi \neq \emptyset$. Then, $\mathcal{T}(\mathcal{D}_E(\Omega), \mathcal{D}_E(\Psi)) \leqslant_L \mathcal{D}_E(\Omega \cup \Psi)$

Definition 3.22. Let $(X, \mathcal{M}_E, \mathcal{T})$ be a non-Archimedean $\mathcal{L}$-fuzzy vector metric space. For $\emptyset \neq (\Omega_n) \subseteq X$ if

$$\lim_{n\to\infty} \mathcal{D}_E(\Omega_n) = 1_L$$

then it is said to be Ω has appearing $\mathcal{L}$-fuzzy vector metric diameter. Moreover, for all $r \in L\setminus\{0_\mathcal{L}, 1_\mathcal{L}\}$ and $s \in E_+$, a number $n_0 \in \mathbb{N}$ can be found such that $\mathcal{M}_E(a,b,s) >_L \mathcal{N}(r)$, for all $a, b \in \Omega_{n_0}$.

Theorem 3.23 (Theorem of Cantor Intersection)**.** Let $(X, \mathcal{M}_E, \mathcal{T})$ be a non-Archimedean $\mathcal{L}$-fuzzy vector metric space. Let $\emptyset \neq \Omega_n$ be closed and decreasing sequence of subsets of X. Suppose that $\lim_{n\to\infty} \mathcal{D}_E(\Omega_n) = 1_L$. Then, X is complete if and only if the intersection of the sequence is a singleton.

Proof.
Let X be complete. For each $n \in \mathbb{N}$ by considering a point $a_n \in \Omega_n$, a sequence (a_n) can be formed. If $m \geqslant n$ is chosen, $\Omega_m \subseteq \Omega_n$ is obtained such that all the points $\{a_m : m \geqslant n\}$ of the sequence belong to the set Ω_n. According to Theorem 3.21, $\mathcal{D}_E(\Omega_n) \leqslant_L \mathcal{M}_E(a_m, a_n, s)$, for $s \in E_+$ and for all $m \geqslant n$. Since the sequence (Ω_n) has an appearing $\mathcal{L}$-fuzzy vector diameter, $\lim_{n,m\to\infty} \mathcal{M}_E(a_m, a_n, s) = 1_L$. Thus, (a_n) is a Cauchy sequence. Since X is complete, there is a point $a \in X$ such that $\lim_{n\to\infty} \mathcal{M}_E(a_n, a, s) = 1_L$. If a set Ω_{n_0} is taken and formed the sequence $(a_n) \subset \Omega_{n_0}$, for $n \geqslant n_0$, then $\lim_{n\to\infty} \mathcal{M}_E(a_n, a, s) = 1_L$. Moreover, $a \in \Omega_{n_0}$ because Ω_{n_0} is closed. As a result, it follows that a belongs to all the members of the sequence (Ω_n). Hence, $a \in \bigcap_{n=1}^{\infty} \Omega_n$ is obtained. Considering another point $a' \in \bigcap_{n=1}^{\infty} \Omega_n$, $\mathcal{D}_E(\Omega_n) \leqslant_L \mathcal{M}_E(a, a', s)$, for all $s \in E^+$. Since the sequence (Ω_n) has an appearing $\mathcal{L}$-fuzzy vector diameter, $\mathcal{M}_E(a, a', s) = 1_L$. As a result, it follows that $\bigcap_{n=1}^{\infty} \Omega_n = \{a\}$ because of $a = a'$.

Conversely, considering a Cauchy sequence $(a_n) \subseteq X$ and closed nonempty subset $\Omega_n = \overline{\{a_m : m \geqslant n\}}$ of X, then $\lim_{n\to\infty} \mathcal{D}_E(\Omega_n) = 1_L$ because the sequence (Ω_n) is decreasing and (a_n) is a Cauchy sequence. According to the assumption of the theorem, there is only a single point a such that $\bigcap_{n=1}^{\infty} \Omega_n = \{a\}$. Then, because of the definition of $\mathcal{L}$-fuzzy vector diameter there is a natural number n_0 such that $\mathcal{D}_E(\Omega_{n_0}) >_L \mathcal{N}(\varepsilon)$, for each $\varepsilon \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$. Moreover, since $a \in \Omega_{n_0}$, $\mathcal{M}(a_n, a, s) >_L \mathcal{N}(\varepsilon)$, for all $n \geqslant n_0$. It means that $a_n \overset{\mathcal{M}_E}{\to} a$. Consequently, the non-Archimedean $\mathcal{L}$-fuzzy vector metric space X is a complete space. □

Theorem 3.24 (Baire Category Theorem)**.** Let $(X, \mathcal{M}_E, \mathcal{T})$ be a non-Archimedean $\mathcal{L}$-fuzzy vector metric space and let $(\Omega_n) \subset X$ be a countable collection of open and dense subsets. Then, the intersection of (Ω_n) is also dense in X.

Proof.
For proof, it is necessary that

$$\mathcal{B}_E(a, r, s) \cap \left(\bigcap_{n=1}^{\infty} \Omega_n \right) \neq \emptyset$$

is satisfied for all $a \in X$, $r \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ and $s \in E_+$. For Ω_1, $\mathcal{B}_E(a, r, s) \cap \Omega_1$ is open and nonempty because $\Omega_1 \subset X$ is dense. Considering the element $a_1 \in \mathcal{B}_E(a, r, s) \cap \Omega_1$, then there exist $r_1 \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ and $s_1 \in E_+$ such that $\mathcal{B}_E[a_1, r_1, s_1] \subset \mathcal{B}_E(a, r, s) \cap \Omega_1$. Let $\mathcal{B}_{E_1} = \mathcal{B}_E(a_1, r_1, s_1)$. $\mathcal{B}_{E_1} \cap \Omega_2$ is open and nonempty because $\Omega_2 \subset X$ is dense. Considering the element $a_2 \in \mathcal{B}_{E_1} \cap \Omega_2$, then there exist $r_2 \in (0_L, 1_L/2)$ and $s_2 \in E_+$ such that $\mathcal{B}_E[a_2, r_2, s_2] \subset \mathcal{B}_{E_1} \cap \Omega_2$. Let $\mathcal{B}_{E_2} = \mathcal{B}_E(a_2, r_2, s_2)$.

If continued inductively, two sequences $(a_n) \subseteq X$ and $(r_n) \subseteq \mathbb{R}$ are obtained such that

$$\mathcal{B}_E[a_{n+1}, r_{n+1}, s_{n+1}] \subset \mathcal{B}_{E_n} \cap \Omega_{n+1} \subset \mathcal{B}_E[a_n, r_n, s_n] \quad \text{and} \quad r_n \in (0_L, 1_L/n)$$

for all $n \in \mathbb{N}$. According to Theorem 3.23, $\bigcap_{n=1}^{\infty} \mathcal{B}_E[a_n, r_n, s_n]$ has only one element. As a result, from

$$\bigcap_{n=1}^{\infty} \mathcal{B}_E[a_n, r_n, s_n] \subset \mathcal{B}_E(a, r, s) \cap \left(\bigcap_{n=1}^{\infty} \Omega_n \right)$$

we reach the conclusion $\mathcal{B}_E(a, r, s) \cap \left(\bigcap_{n=1}^{\infty} \Omega_n \right) \neq \emptyset$. This completes the proof. □

4. Conclusion

In conclusion, this article contributes to the field of fuzzy metric spaces by defining left and right-order convergence and continuity within the framework of non-Archimedean $\mathcal{L}$-fuzzy vector metric spaces. Left and right-order continuity concepts are used to construct $\mathcal{L}$-fuzzy vector metric spaces and non-Archimedean $\mathcal{L}$-fuzzy vector metric spaces. Furthermore, some non-trivial examples are built, and as an implication, the findings are used to prove Cantor's intersection theorem and Baire's theorem. In the next stages, as a continuation of this study, examples of these spaces can be multiplied, and fixed point theorems can be studied.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

References

[1] L. A. Zadeh, *Fuzzy Sets*, Information and Control 8 (3) (1965) 338–353.

[2] J. Goguen, *$\mathcal{L}$-Fuzzy Sets*, Journal of Mathematical Analysis and Applications 18 (1) (1967) 145–174.

[3] G. D. Birkhoff, Lattice Theory, 3rd Edition, American Mathematical Society, New York, 1973.

[4] K. Menger, *Statistical Metrics*, Proceedings of the National Academy of Sciences 28 (12) (1942) 535–537.

[5] B. Schweizer, A. Sklar, *Statistical Metric Spaces*, Pacific Journal of Mathematics 10 (1) (1960) 313–334.

[6] B. Schweizer, A. Sklar, Probabilistic Metric Spaces, Dover Publications, New York, 2011.

[7] I. Kramosil, J. Michálek, *Fuzzy Metrics and Statistical Metric Spaces*, Kybernetica 11 (5) (1975) 336–344.

[8] A. George, P. Veeramani, *On Some Results in Fuzzy Metric Spaces*, Fuzzy Sets and Systems 64 (3) (1994) 395–399.

[9] V. Gregori, S. Morillas, A. Sapena, *Examples of Fuzzy Metrics and Applications*, Fuzzy Sets and Systems 170 (1) (2011) 95–111.

[10] R. Saadati, A. Razani, H. Adibi, *A Common Fixed Point Theorem in $\mathcal{L}$-Fuzzy Metric Spaces*, Chaos, Solitons & Fractals 33 (2) (2007) 358–363.

[11] S. Morillas, V. Gregori, G. Peris-Fajarnés, P. Latorre, *A Fast Impulsive Noise Color Image Filter Using Fuzzy Metrics*, Real-Time Imaging 11 (5-6) (2005) 417–428.

[12] C. D. Aliprantis, K. C. Border, Infinite Dimensional Analysis, Springer-Verlag Berlin, Heidelberg, 1999.

[13] C. D. Aliprantis, O. Burkinshaw, Positive Operators, Springer, Dordrecht, 2006.

[14] C. Çevik, I. Altun, *Vector Metric Spaces and Some Properties*, Topological Methods in Nonlinear Analysis 34 (2) (2009) 375–382.

[15] Ş. Eminoğlu, C. Çevik, *Fuzzy Vector Metric Spaces and Some Results*, Journal of Nonlinear Sciences and Applications 10 (2017) 3429–3436.