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Research Article

Directional Tube Surface in Euclidean 4-Space

Başak YAĞBASAN [1], Cumali EKİCİ [2], Hatice TOZAK [3]

Abstract
The aim of this paper is to study characterization of tube surfaces (called directional tube surfaces) with respect to the q-frame in Euclidean 4-space $\mathbb{E}^4$. First, a parametrization of these directional tube surfaces in $\mathbb{E}^4$ is established. Then, the normals of the directional tube surfaces, denoted as $\mathbf{U}_1$ and $\mathbf{U}_2$, are determined respectively. Furthermore, the Gaussian curvature K and the mean curvature H of the directional tube surfaces are investigated. Subsequently, an example of a directional tube surface is given in $\mathbb{E}^4$, together with visual representations of this tube surfaces in projection space.

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[1,2] *Department of Mathematics and Computer Science, Eskişehir Osmangazi University, 26040, Eskişehir, Türkiye*
[3] *Department of Strategy, MEB Denizli Directorate of National Education, 20030, Denizli, Türkiye*
[1] ✉bskyagbasan@gmail.com, [2] ✉cekici@ogu.edu.tr, [3] ✉hatice.tozak@gmail.com
Corresponding author: Cumali EKİCİ
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1. Introduction

The canal surfaces, a special type of surfaces, have been originally introduced by Monge in 1850. Tube surfaces, a special case of canal surfaces, arise as the envelope of a moving sphere with a constant radius function, denoted $r(t)$ [1,2]. The tube surfaces which is also called as a pipe were parameterized using the Frenet frame [3]. Blaga (2005) introduced a method to parameterize tube surfaces in which one of the parameters is traced along the generating curve, and a point on the surface is denoted by the position vector ψ [4]. Relatively simple analytical and functional descriptions of tube surfaces have generated considerable interest in various fields, including medicine and computer-aided design etc., due to their potential applications. This interest arises from their potential applications in different fields. Many researchers have been interested in relations between the curvatures and characterizations of these surfaces in different spaces. In the field of differential geometry, the Frenet frame has been a very important tool in the analysis of curves and surfaces. However, there are other alternative frame to the Frenet frame, such as the Darboux frame and the Bishop frame etc. [5]. Bishop showed that we can define more frame along a space curve [6]. In addition, singularities of parallel surface and directional tubes are investigated in [7–9]. Several geometers have explored tubes in both Euclidean 3-space and Minkowski 3-space, by exploring equations in terms of Gaussian curvature, mean curvature, and second Gaussian curvature [10–12]. In addition to these frames Coquillart introduced an alternative approach known as the q-frame, obtained by using the quasi-normal vector in 1987 [13]. Dede et al. have introduced the q-frame adapted along a spatial curve and have established its connection with the Frenet frame [14] . This q-frame has been employed for parametric representations of directional tube surfaces, referred to as directional tubular surfaces, in various spaces [10, 15, 16]. Gezer and Ekici (2023) have introduced the q-frame, q-frame formulas, and the relation between the q-frame and Frenet frame in Euclidean 4-space [17]. Gluck (1966) has presented a simple algorithm, by using a single formula for all curvature, based on the Gram-Schmidt orthonormalization process for determining curvatures of the curve in Euclidean n-space and has provided an example to illustrate these concepts in Euclidean 4-space [18]. Focusing on Euclidean 4-space $\mathbb{E}^4$, researchers have investigated Frenet elements and derivative equations for space curves with unit speed [19–26]. The canal surface can be parametrized using Frenet frame and alternative frame in both 3-dimensional Euclidean space $\mathbb{E}^3$ and 4-dimensional Euclidean space $\mathbb{E}^4$ [5, 27–32].

Also in Euclidean 4-space, the ruled surfaces with quasi-vectors are studied by Coşkun and Akça [33], and Weingarten map of the hypersurface is given by Yüce [34]. It is worth remarking that one of the main challenges with surfaces is to visualize their images in Euclidean 4-space $\mathbb{E}^4$, and researchers such as Mello (2009) have investigated the properties and conditions of such surfaces [35]. Besides, Kişi (2018) has studied canal surfaces in $\mathbb{E}^4$ under parallel transport frame vectors, exploring conditions for flatness, minimality, linear Weingarten conditions, and providing normal vectors [27]. The research done by Bulca (2012) offers characterizations of surfaces in $\mathbb{E}^4$ based on the coefficients of the first and second fundamental forms [28]. Additionally, Yağbasan et al. (2023) have determined as some algebraic invariants of the parametrization of the tube surfaces according to the Frenet frame in $\mathbb{E}^4$ [36, 37].

In this study, firstly, basic definitions and theorems about q-frame and tube surface are mentioned. Then, the parametrization of directional tube surfaces is given in Euclidean 4-space. Then, the normals of the directional tube surfaces are established respectively. Furthermore, Gaussian curvature K and mean curvature H of the directional tube surfaces are given. Subsequently, this study has been made more understandable with an example of directional tube surfaces in $\mathbb{E}^4$. The figures of these directional tube surfaces are presented in projection spaces.

2. Preliminaries

A canal surface is defined as an envelope of a non-parameter set of spheres, centered at a spine curve $\alpha(t)$ with radius $r(t)$. The canal surface is parametrized by

$$C(t,v) = \alpha(t) + r(t)\left(\sqrt{(1-r'(t))^2}\cos v\mathbf{N}(t) + \sqrt{(1-r'(t))^2}\sin v\mathbf{B}(t) - r'(t)\mathbf{T}(t)\right),$$

where $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is the Frenet frame of the spine curve $\alpha(t)$ [1]. The tube surface, a special case taken at a constant distance r from the canal surface, with Frenet frame can be parametrized as

$$\psi(t,v) = \alpha(t) + r\cos v\mathbf{N}(t) + r\sin v\mathbf{B}(t).$$

A tube surface can often be parameterized by using the Frenet frame of a space curve [15]. However, various alternative methods such as by using the Bishop frame, the Darboux frame, the q-frame etc. have been proposed for computing the tube surfaces Dede et al. [14] introduced the directional q-frame $\{\mathbf{T},\mathbf{N}_q,\mathbf{B}_q,\mathbf{k}\}$ of a regular curve $\alpha(t)$ as follows

$$\mathbf{T} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{N}_q = \frac{\mathbf{T}\wedge\mathbf{k}}{\|\mathbf{T}\wedge\mathbf{k}\|}, \mathbf{B}_q = \mathbf{T}\wedge\mathbf{N}_q,$$

where $\mathbf{k}$ is the projection vector. In Euclidean 3-space; the q-frame and Frenet frame vectors are shown in Fig. 1.

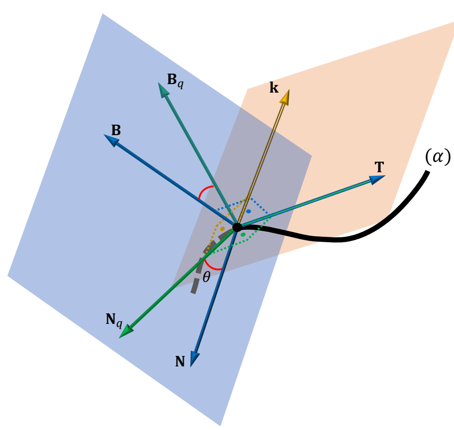

Fig. 1. The quasi frame and Frenet frame

The variation equations of the directional q-frame is given by

$$\begin{bmatrix}\mathbf{T}'\\ \mathbf{N}_q'\\ \mathbf{B}_q'\end{bmatrix} = \|\alpha'\|\begin{bmatrix}0 & k_1 & k_2\\ -k_1 & 0 & k_3\\ -k_2 & -k_3 & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}_q\\ \mathbf{B}_q\end{bmatrix},$$

where the q-curvatures are expressed as follows

$$k_1 = \frac{\langle \mathbf{T}', \mathbf{N}_q \rangle}{\|\alpha'\|}, k_2 = \frac{\langle \mathbf{T}', \mathbf{B}_q \rangle}{\|\alpha'\|}, k_3 = -\frac{\langle \mathbf{N}_q, \mathbf{B}_q' \rangle}{\|\alpha'\|}.$$

Let $\mathbf{X} = (x_1, x_2, x_3, x_4)$, $\mathbf{Y} = (y_1, y_2, y_3, y_4)$ and $\mathbf{Z} = (z_1, z_2, z_3, z_4)$ be three vectors in $\mathbb{E}^4$. The inner product of the two vectors $\mathbf{X}$ and $\mathbf{Y}$ is defined as $< \mathbf{X}, \mathbf{Y} >= x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$. The norm of the vector $\mathbf{X} \in \mathbb{E}^4$ is given by $||\mathbf{X}|| = \sqrt{< \mathbf{X}, \mathbf{X} >}$. The vector product of the three vectors $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ is given by the determinant as follows

$$\mathbf{X} \times \mathbf{Y} \times \mathbf{Z} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix},$$

where $\mathbf{e}_1 \times \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_4$, $\mathbf{e}_2 \times \mathbf{e}_3 \times \mathbf{e}_4 = \mathbf{e}_1$, $\mathbf{e}_3 \times \mathbf{e}_4 \times \mathbf{e}_1 = \mathbf{e}_2$ and $\mathbf{e}_3 \times \mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{e}_4$ [19,20] .

Let $\alpha(t) = \alpha : I \subset \mathbb{R} \to \mathbb{E}^4$ be any space curve in $\mathbb{E}^4$. The curve is said to be parameterized by arc length s if $< \alpha', \alpha' >= 1$. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ be a Frenet frame where $\mathbf{T}, \mathbf{N}, \mathbf{B}_1$ and $\mathbf{B}_2$ are called the tangent, normal, first and second binormal vector fields, respectively. The Frenet formulas of a unit speed curve $\alpha(t)$ is written as

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}_1' \\ \mathbf{B}_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \eta \\ 0 & 0 & -\eta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix},$$

where the functions

$$\kappa =< \mathbf{T}', \mathbf{N} >, \tau =< \mathbf{N}', \mathbf{B}_1 >, \eta =< \mathbf{B}_1', \mathbf{B}_2 >$$

are called the first, second and third curvatures, respectively [18]. The tube surface with respect to the Frenet frame in $\mathbb{E}^4$ is parametrized as follows [28]:

$$\psi(t, v) = \alpha(t) + r(\mathbf{B}_1(t) \cos v + \mathbf{B}_2(t) \sin v).$$

Let $\alpha = \alpha(s)$ be a space curve with according to the quasi-frame $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q, \mathbf{C}_q\}$ is introduced by Gezer and Ekici (2023) where $\mathbf{T}$ is the unit tangent vector field, $\mathbf{N}_q$ is the quasi-normal vector field, $\mathbf{B}_q$ and $\mathbf{C}_q$ are the first and second quasi-binormal vector fields respectively in $\mathbb{E}^4$ [17]. The q-frame is given by

$$\begin{cases} \mathbf{T} = \dfrac{\alpha'(s)}{\|\alpha'(s)\|}, & \mathbf{N}_q = \dfrac{\mathbf{T} \wedge \mathbf{k}_x \wedge \mathbf{k}_y}{\|\mathbf{T} \wedge \mathbf{k}_x \wedge \mathbf{k}_y\|}, \\ \mathbf{B}_q = \mathbf{C}_q \wedge \mathbf{T} \wedge \mathbf{N}_q, & \mathbf{C}_q = \dfrac{\alpha'(s) \wedge \mathbf{N}_q\,(\mathbf{s}) \wedge \alpha'''(s)}{\|\alpha'(s) \wedge \mathbf{N}_q\,(\mathbf{s}) \wedge \alpha'''(s)\|}, \end{cases}$$

where $\mathbf{k}_x = (1, 0, 0, 0)$ and $\mathbf{k}_y = (0, 1, 0, 0)$ are the projection vectors [17,20].

Let $\alpha(s)$ be a curve that is parameterized by arc length s. The variation equations of the quasi-frame are given as [17]

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}_q' \\ \mathbf{B}_q' \\ \mathbf{C}_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & 0 \\ -k_1 & 0 & k_3 & 0 \\ -k_2 & -k_3 & 0 & k_4 \\ 0 & 0 & -k_4 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N}_q \\ \mathbf{B}_q \\ \mathbf{C}_q \end{bmatrix},$$

where q-curvatures are

$$\begin{cases} k_1 = \dfrac{\langle \mathbf{T}', \mathbf{N}_q \rangle}{\|\alpha'\|}, & k_2 = \dfrac{\langle \mathbf{T}', \mathbf{B}_q \rangle}{\|\alpha'\|}, \\ k_3 = \dfrac{\langle \mathbf{N}_q', \mathbf{B}_q \rangle}{\|\alpha'\|}, & k_4 = \dfrac{\langle \mathbf{B}_q', \mathbf{C}_q \rangle}{\|\alpha'\|}. \end{cases}$$

A point lies on a circle of radius r normal to the generating curve at a point s, with its center at point $\alpha(s)$ from the curve. Denote by $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q, \mathbf{k}\}$ the q-frame moving along the unit velocity curve $\alpha(s)$. Suppose a vector ρ and an angle θ are denoted respectively by the vector connecting the point from the curve to the point from the surface and the vector $\mathbf{N}_q(s)$ to the vector ρ lying in the plane normal to the surface. A parameterization of the tube surface

$$\psi(s,\theta)=\alpha(s)+r\mathbf{N}_q(s)\cos\theta+r\mathbf{B}_q(s)\sin\theta \tag{1}$$

is obtained by [15].

Let M be a regular surface given with the parameterization $\psi(u,v)$ in $\mathbb{E}^4$ such that where $\psi:U\subset\mathbb{E}^2\to\mathbb{E}^4$. The tangent space of M at an arbitrary point is spanned by the vectors ψ_u and ψ_v. The first fundamental form I of the surface M are, respectively, given by

$$I=Edu^2+2Fdudv+Gdv^2,$$

where the equations

$$E=<\psi_u,\psi_u>,\ F=<\psi_u,\psi_v>\ \text{and}\ G=<\psi_v,\psi_v> \tag{2}$$

are the coefficients of the first fundamental form, $<,>$ is the inner product and $\wedge$ is the vector product [23, 35].

Let $\psi_{uu},\psi_{uv},\psi_{vv}$ be the second order partial derivatives and let $\mathbf{U}_1,\mathbf{U}_2,...,\mathbf{U}_{n-2}$ be the normal vector fields of M such that the second fundamental form coefficients of M are

$$L_k=<\psi_{uu},\mathbf{U}_k>,\ M_k=<\psi_{uv},\mathbf{U}_k>\ \text{and}\ N_k=<\psi_{vv},\mathbf{U}_k>,\quad 1\le k\le n-2. \tag{3}$$

Let $M\subset\mathbb{E}^n$ be a surface defined by the regular patch $\psi(u,v):(u,v)\in D\subset\mathbb{R}^2$. Then, Gaussian curvature function of the surface M is defined by

$$K=\frac{1}{W^2}\sum_{k=1}^{n-2}(L_kN_k-(M_k)^2), \tag{4}$$

where $W^2=EG-F^2$ [26, 28, 35, 38].

Let $M\subset\mathbb{E}^n$ be a surface defined by the regular patch $\psi(u,v):(u,v)\in D\subset\mathbb{R}^2$. In this case, for every $\{X_1,X_2\}\in\chi(M)$ and orthonormal bases $\{\mathbf{U}_1,\ \mathbf{U}_2,...,\mathbf{U}_{n-2}\}$, the mean curvature vector field of the surface M is given by

$$\overrightarrow{H}=\frac{1}{W^2}\sum_{i=1}^{n-2}H_i\mathbf{U}_i,$$

where $W^2=EG-F^2$ and function

$$H_i=\frac{1}{W^2}\sum_{k=1}^{n-2}(GL_k-2FM_k+EN_k) \tag{5}$$

is the i-th mean curvature function of M. Additionally, the mean curvature function of M is $H=||\overrightarrow{H}||$ [26, 28, 35, 38].

Let $M\subset\mathbb{E}^n$ be a surface defined by the regular patch $\psi:U\subset\mathbb{R}^2\to\mathbb{E}^n$. Thus the vectors X_1 and X_2 being an orthonormal base of $T_p(M)$ are defined by

$$\begin{cases} X_1 &= \dfrac{\psi_u}{||\psi_u||},\\ X_2 &= \dfrac{\sqrt{E}}{W}\left(\psi_u-<\psi_v,\psi_u>\dfrac{\psi_u}{||\psi_u||^2}\right),\end{cases}$$

where $W^2=EG-F^2$ [28].

3. Directional tube surfaces in $\mathbb{E}^4$

Let $M\subset\mathbb{E}^4$ be tube surface with respect to the q-frame $\{\mathbf{T},\mathbf{N}_q,\mathbf{B}_q,\mathbf{C}_q\}$ and $r\in\mathbb{R}$ moving along the unit velocity curve $\alpha(t)$ in $\mathbb{E}^4$. Let the center curve $\alpha(t)$ be on the directional tube surface ψ and let the angle between the vector $\mathbf{B}_q(s)$ and the vector ρ that lies in the normal plane be denoted by v. Then, we have

$$\rho=r(\cos v\mathbf{B}_q+\sin v\mathbf{C}_q). \tag{6}$$

From (6), we see that a parameterization of directional tube surface at a distance r with the q-frame can be introduced as

$$\psi(t,v)=\alpha(t)+r(\cos v\mathbf{B}_q+\sin v\mathbf{C}_q).$$

Theorem 1. *For the directional tube surface at a distance r from the spin curve $\alpha(t)$ with respect to the q-frame $\{\mathbf{T},\mathbf{N}_q,\mathbf{B}_q,\mathbf{C}_q\}$ and parametrization $\psi(t,v)$, the first and second unit normal vectors $\mathbf{U}_1$ and $\mathbf{U}_2$ are given as, respectively,*

$$\mathbf{U}_1=\frac{rk_3\cos v\mathbf{T}+(1-rk_2\cos v)\mathbf{N}_q+\cos v\mathbf{B}_q+\sin v\mathbf{C}_q}{\sqrt{(1-rk_2\cos v)^2+1+r^2k_3^2\cos^2 v}}$$

and

$$\mathbf{U}_2=\frac{\sqrt{(1-rk_2\cos v)^2+r^2k_3^2\cos^2 v}}{\sqrt{(1-rk_2\cos v)^2+r^2k_3^2\cos^2 v+1}}\Bigg[\frac{rk_3\cos v\mathbf{T}}{(1-rk_2\cos v)^2+r^2k_3^2\cos^2 v}+\frac{(1-rk_2\cos v)\mathbf{N}_q}{(1-rk_2\cos v)^2+r^2k_3^2\cos^2 v}$$
$$-\cos v\mathbf{B}_q-\sin v\mathbf{C}_q\Bigg].$$

Proof. Let the parametrization of the general equation of the directional tube surface be given as the following equation

$$\psi(t,v)=\alpha(t)+r(\cos v\mathbf{B}_q+\sin v\mathbf{C}_q). \tag{7}$$

The first partial derivatives of the equation (7), with respect to s and v, are determined respectively as

$$\psi_t=(1-rk_2\cos v)\mathbf{T}-rk_3\cos v\mathbf{N}_q-rk_4\sin v\mathbf{B}_q+rk_4\cos v\mathbf{C}_q \tag{8}$$

and

$$\psi_v=r(-\sin v\mathbf{B}_q+\cos v\mathbf{C}_q). \tag{9}$$

For the first and second unit normal vectors $\mathbf{U}_1$ and $\mathbf{U}_2$ of the directional tube surface, the following equations must be satisfied

$$\begin{cases} <\psi_t,\mathbf{U}_1>=0, & <\psi_t,\mathbf{U}_2>=0,\\ <\psi_v,\mathbf{U}_1>=0, & <\psi_v,\mathbf{U}_2>=0,\\ <\mathbf{U}_1,\mathbf{U}_1>=1, & <\mathbf{U}_2,\mathbf{U}_2>=1, \quad <\mathbf{U}_1,\mathbf{U}_2>=0. \end{cases} \tag{10}$$

Assuming that the equation of the vector $\mathbf{U}_1$ is given as

$$\mathbf{U}_1=\frac{a_1\mathbf{T}+a_2\mathbf{N}_q+a_3\mathbf{B}_q+a_4\mathbf{C}_q}{\sqrt{a_1^2+a_2^2+a_3^2+a_4^2}}. \tag{11}$$

If equations (8), (9), (10) and (11) are used with the coefficients $a_3=\cos v$ and $a_1=rk_3\cos v$, the coefficients $a_4=\sin v$ and $a_2=1-rk_2\cos v$ are obtained. Therefore, the first normal vector $\mathbf{U}_1$ is seen to be as

$$\mathbf{U}_1=\frac{rk_3\cos v\mathbf{T}+(1-rk_2\cos v)\mathbf{N}_q+\cos v\mathbf{B}_q+\sin v\mathbf{C}_q}{\sqrt{(1-rk_2\cos v)^2+1+r^2k_3^2\cos^2 v}}.$$

Hence, it is easy to clear that the first unit normal vector $\mathbf{U}_1$ satisfies the equations $<\psi_t,\mathbf{U}_1>=0$, $<\psi_v,\mathbf{U}_1>=0$ and $<\mathbf{U}_1,\mathbf{U}_1>=1$.

For the second unit normal vector, let the vector $\mathbf{Y}$ be taken as

$$\mathbf{Y}=\frac{b_1\mathbf{T}+b_2\mathbf{N}_q+b_3\mathbf{B}_q+b_4\mathbf{C}_q}{\sqrt{b_1^2+b_2^2+b_3^2+b_4^2}}. \tag{12}$$

By considering the coefficients $b_3=0$ and $b_1=rk_3\cos v$, we derive the coefficients $b_4=0$ and $b_2=(1-rk_2\cos v)$. Then, expressed in (12), the coefficients give the vector

$$\mathbf{Y}=\frac{rk_3\cos v\mathbf{T}+(1-rk_2\cos v)\mathbf{N}_q}{\sqrt{(1-rk_2\cos v)^2+r^2k_3^2\cos^2 v}}.$$

So, it is easily seen that the first normal vector $\mathbf{U}_1$ satisfies the equations $<\psi_t, \mathbf{Y}>=0$, $<\psi_v, \mathbf{Y}>=0$ and $<\mathbf{U}_1, \mathbf{Y}>\neq 1$. As the vectors $\mathbf{U}_1$ and $\mathbf{Y}$ are linearly independent, a second unit normal vector $\mathbf{U}_2$ orthogonal to the vector $\mathbf{U}_1$ is represented using the Gram-Schmidt method as follows

$$\mathbf{U}_2 = \frac{\sqrt{(1-rk_2\cos v)^2 + r^2k_3^2\cos^2 v}}{\sqrt{(1-rk_2\cos v)^2 + r^2k_3^2\cos^2 v + 1}}\Bigg[\frac{rk_3\cos v\mathbf{T}}{(1-rk_2\cos v)^2 + r^2k_3^2\cos^2 v} + \frac{(1-rk_2\cos v)\mathbf{N}_q}{(1-rk_2\cos v)^2 + r^2k_3^2\cos^2 v}$$
$$-\cos v\mathbf{B}_q - \sin v\mathbf{C}_q\Bigg].$$

Consequently, it is clearly seen that the first and second normal vectors $\mathbf{U}_1$ and $\mathbf{U}_2$ satisfy equation (10). ■

Theorem 2. *For the directional tube surface at a distance r from the spin curve $\alpha(t)$ with respect to the q-frame $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q, \mathbf{C}_q\}$ and parametrization $\psi(t,v)$ in $\mathbb{E}^4$,*

i) Gaussian curvatures according to the first and second normal vectors $\mathbf{U}_1$ and $\mathbf{U}_2$ are respectively as follows:

$$K_1 = \frac{1}{2r - 6r^2k_2\cos v + r^3\cos^2 v(7k_2^2 + 3k_3^2) - r^4(k_2^2 + k_3^2)\cos^3 v(4k_2 - r\cos v(k_2^2 + k_3^2))}[-k_1 + rk_3k_4\sin v$$
$$+r\cos^2 v(rk_2'k_3 - rk_2k_3' - rk_1k_3^2 + 2k_3^2 - rk_1k_2^2) - rk_3^2 + \cos v(2rk_1k_2 + rk_2^2 - k_2 + rk_3')] \tag{13}$$

and

$$K_2 = \frac{(r^2(1-rk_2\cos v)^2 + r^4\cos^2 vk_3^2)^{-1}}{2 - 6rk_2\cos v + r^2\cos^2 v(7k_2^2 + 3k_3^2) - 4r^3k_2\cos^3 v(k_2^2 + k_3^2) + r^2(k_2^2\cos^2 v + k_3^2\cos^2 v)^2}[r^2k_2k_3k_4\sin 2v$$
$$+r^3\cos^3 v(2k_2k_2'k_3 - 4k_1k_2k_3^2 - k_3^2k_3' - 4k_1k_3^3 - 3k_2^2k_3') + r^5\cos^6 v(k_2^6 + k_3^6 + 3k_2^4k_3^2 + 3k_2^2k_3^4) - rk_3^2$$
$$-r^3k_3k_4\cos^2 v\sin v(k_2^2 + k_3^2) - \cos v(k_2 + rk_3' + 4rk_1k_2) - r^2k_2\cos^3 v(10k_2^2 - 6k_3^2) - rk_3k_4\sin v + k_1$$
$$+r^3\cos^4 v(10k_2^4 + 12k_2^2k_3^2 + 2k_3^4) + r^2\cos^2 v(6k_1k_2^2 + 3k_2k_3' - k_2'k_3 + 2k_1k_3^2) + r\cos^2 v(5k_2^2 + 2k_3^2)$$
$$-5r^4\cos^5 v(k_2^5 - 2k_2^3k_3^2 - k_2k_3^4) + r^4\cos^4 v(k_2^3k_3' - k_2^2k_2'k_3 + 2k_1k_2^2k_3^2 + k_2k_3^2k_3' - k_2'k_3^3 + k_1k_3^4 + k_1k_2^4)]. \tag{14}$$

ii) The mean curvatures according to the first and second normal vectors $\mathbf{U}_1$ and $\mathbf{U}_2$ are respectively as follows:

$$H_1 = \frac{((1-r\cos v)^2 + r^2k_3^2\cos^2 v)^{-1}}{\sqrt{1 + r^2k_3^2\cos^2 v + (1-rk_2\cos v)^2}}\left[2r^2k_1k_2 - r^2k_3k_4\sin v + rk_1 - r^2k_3'\cos v - 2r^2\cos^2 v(k_2^2 + k_3^2)\right.$$
$$\left.-1 + 3rk_2\cos v + r^3\cos^2 v(k_1k_2^2 - k_2'k_3 + k_1k_3^2 + k_2k_3')\right] \tag{15}$$

and

$$H_2 = \frac{r((1-rk_2\cos v)^2 + r^2k_3^2\cos^2 v)^{3/2}}{\sqrt{(1-rk_2\cos v)^2 + r^2k_3^2\cos^2 v + 1}}\left[1 + rk_1 - 5rk_2\cos v + 2r^4(k_2^2\cos^2 v + k_3^2\cos^2 v)^2 - 2r^2k_1k_2\cos v\right.$$
$$\left.-r^2k_3'\cos v + r^2\cos^2 v(3k_3^2 + 9k_2^2 - rk_2'k_3 + rk_1k_3^2 + rk_1k_2^2 + rk_2k_3') - 7r^3k_2\cos^3 v(k_2^2 + k_3^2) - r^2k_3k_4\sin v\right]. \tag{16}$$

Proof. Let $\psi(t,v)$ be parametrized the directional tube surface in Euclidean 4-space. Substituting equations (8) and (9) into equation (2), the components of the first fundamental form of directional tube surface are get as

$$\begin{cases} E = & (1-rk_2\cos v)^2 + r^2k_3^2\cos^2 v + r^2k_4^2, \\ F = & r^2k_4, \\ G = & r^2, \end{cases} \tag{17}$$

where $W^2 = EG - F^2 = r^2(1-rk_2\cos v)^2 + r^4k_3^2\cos^2 v$.

The second partial derivatives of the equation (7), with respect to s and v, are obtained as respectively as

$$\begin{cases} \psi_{tt} = & (-rk_2'\cos v + rk_2k_4\sin v + rk_1k_3\cos v)\mathbf{T} + (k_1 - rk_1k_2\cos v - \mathbf{r}k_3' + rk_3k_4\sin v)\mathbf{N}_q \\ & +(k_2 - rk_2^2 - rk_3^2\cos v - rk_4^2)\mathbf{B}_q + (rk_4'\cos v - rk_4^2\sin v)\mathbf{C}_q, \\ \psi_{tv} = & rk_2\sin v\mathbf{T} + rk_3\sin v\mathbf{N}_q - rk_4\cos v, \mathbf{B}_q - rk_4\sin v\mathbf{C}_q, \\ \psi_{vv} = & r(-\cos v\mathbf{B}_q - \sin v\mathbf{C}_q). \end{cases} \tag{18}$$

From equation (3) and (18), the components of the second fundamental form according to the normal vectors $\mathbf{U}_1$ and $\mathbf{U}_2$ are respectively determined as

$$\begin{cases} L_1 = & \dfrac{1}{\sqrt{1 + r^2k_3^2\cos^2 v + (1 - rk_2\cos v)^2}}[rk_3k_4\sin v + k_1 - rk_4^2 + \cos v(k_2 - 2rk_1k_2 - rk_3') - r^2k_2'k_3) \\ & +\cos^2 v(r^2k_1k_3^2 - rk_2^2 - rk_3^2 + r^2k_1k_2^2 + r^2k_2k_3')], \\ M_1 = & \dfrac{rk_3\sin v - rk_4}{\sqrt{1 + r^2k_3^2\cos^2 v + (1 - rk_2\cos v)^2}}, \\ N_1 = & -\dfrac{r}{\sqrt{1 + r^2k_3^2\cos^2 v + (1 - rk_2\cos v)^2}} \end{cases} \tag{19}$$

and

$$\begin{cases} L_2 = & \dfrac{((1 - rk_2\cos v)^2 + r^2k_3^2\cos^2 v)^{-1/2}}{\sqrt{(1 - rk_2\cos v)^2 + r^2k_3^2\cos^2 v + 1}}[rk_4^2 + rk_3k_4\sin v + k_1 - k_2\cos v + r^3\cos^4 v(k_2^4 + k_3^4 + 2k_2^2k_3^2) \\ & +r\cos^2 v(3k_2^2 + k_3^2 + r(k_1k_3^2 + k_1k_2^2 + k_2^2(\dfrac{k_3}{k_2})') + r^2k_4^2(k_2^2 + k_3^2)) - r\cos v(2k_1k_2 + k_3' + 2rk_2k_4^2) \\ & -3r^2k_2\cos^3 v(k_2^2 + k_3^2)], \\ M_2 = & \dfrac{rk_3\sin v + rk_4 - 2r^2k_2k_4\cos v + r^3k_4\cos^2 v(k_2^2 + k_3^2)}{\sqrt{(1 - rk_2\cos v)^2 + r^2k_3^2\cos^2 v}\sqrt{(1 - rk_2\cos v)^2 + r^2k_3^2\cos^2 v + 1}}, \\ N_2 = & \dfrac{r\sqrt{(1 - rk_2\cos v)^2 + r^2k_3^2\cos^2 v}}{\sqrt{(1 - rk_2\cos v)^2 + r^2k_3^2\cos^2 v + 1}}, \end{cases} \tag{20}$$

respectively. In addition, using equations (17), (19), (20) and (4), the Gaussian curvatures of the directional tube surfaces with respect to the normal vectors $\mathbf{U}_1$ and $\mathbf{U}_2$ are determined as presented in equations (13) and (14), respectively. Similarly, using the same steps by substituting equations (17), (19) and (20) in equation (5), the mean curvatures with respect to the first and second normal vectors $\mathbf{U}_1$ and $\mathbf{U}_2$ are obtained as equations (15) and (16), respectively. ■

Theorem 3. *For the directional tube surface at a distance r from the spin curve $\alpha(t)$ with respect to the q-frame $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q, \mathbf{C}_q\}$ and parametrization $\psi(t,v)$ in $\mathbb{E}^4$, the Gaussian and mean curvatures are presented as*

$$K = \frac{r\cos^2 v(2k_3^2 + 3k_2^2) + r^3\cos^4 v(k_2^2 + k_3^2)^2 - rk_3^2 - 3r^2k_2\cos^3 v(k_2^2 + k_3^2) - k_2\cos v}{r - 4r^2k_2\cos v(1 + \cos^2 v(k_2^2 + k_3^2)) + 2r^3\cos^2 v(k_3^2 + 3k_2^2) + r^5\cos^4 v(k_2^2 + k_3^2)^2} \tag{21}$$

and

$$\begin{aligned} H &= \frac{1}{r^2(1-rk_2\cos v)^2+r^4k_3^2\cos^2 v}\Bigg[\frac{r^2}{\sqrt{(1-r\cos v)^2+r^2k_3^2\cos^2 v+1}}\big[rk_3k_4\sin v+k_1-rk_4^2 \\ &-r\cos^2 v(k_2^2+k_3^2)+r^2\cos^2 v(k_1(k_3^2+k_2^2)-k_2'k_3+k_2k_3')+\cos v(k_2-rk_3'-2rk_1k_2)\big] \\ &-\frac{((1-r\cos v)^2+r^2k_3^2\cos^2 v)^{-1/2}}{\sqrt{(1-r\cos v)^2+r^2k_3^2\cos^2 v+1}}[2r^3k_3k_4\sin v+2r^3k_4^2-4r^4k_2k_4^2\cos v+2r^5k_4^2\cos^2 v(k_2^2+k_3^2)] \\ &+\frac{((1-r\cos v)^2+r^2k_3^2\cos^2 v)^{-1/2}}{\sqrt{(1-r\cos v)^2+r^2k_3^2\cos^2 v+1}}[r^2k_1+r^3k_4^2-3r^3k_2\cos^3 v(k_3^2+k_2^2)+r^4(k_2^2\cos^2 v+k_3^2\cos^2 v)^2 \\ &+r^4k_4^2\cos^2 v(k_2^2+k_3^2)-r^3\cos v(2k_2k_4^2+2k_1k_2+k_3')+r^3\cos^2 v(3k_2^2+k_3^2+k_1(k_3^2+k_2^2)+k_2k_3' \\ &-k_2'k_3)+r^3k_3k_4\sin v-r^2k_2\cos v] \\ &+\frac{r\sqrt{(1-r\cos v)^2+r^2k_3^2\cos^2 v}((1-rk_2\cos v)^2+r^2\cos^2 vk_3^2+r^2k_4^2))}{\sqrt{(1-r\cos v)^2+r^2k_3^2\cos^2 v+1}} \\ &+\frac{r^3k_4^2-2r^3k_4k_3\sin v-r^3k_3^2\cos^2 v-r(1-rk_2\cos v)^2}{\sqrt{(1-r\cos v)^2+r^2k_3^2\cos^2 v+1}}\Bigg], \end{aligned} \tag{22}$$

respectively.

Proof. Substituting equations (13), (14), (15) and (16) into equation (4) and (5), it can be easily seen that the Gaussian and mean curvatures of the directional tube surface in $\mathbb{E}^4$ are as in equations (21) and (22), respectively. ∎

4. Example

In this section, we give an example of directional tube surface in Euclidean 4-space.

Example 4. *Let us consider a curve parameterized by*

$$\alpha(s)=\left(\cos\frac{s}{\sqrt5}-2,1+\sin\frac{s}{\sqrt5},\cos\frac{2s}{\sqrt5},\sin\frac{2s}{\sqrt5}-3\right)$$

with respect to the Frenet vectors

$$\begin{cases} \mathbf{T} &= \left(-\frac{1}{\sqrt5}\sin\frac{s}{\sqrt5},\frac{1}{\sqrt5}\cos\frac{s}{\sqrt5},-\frac{2}{\sqrt5}\sin\frac{2s}{\sqrt5},\frac{2}{\sqrt5}\cos\frac{2s}{\sqrt5}\right), \\ \mathbf{N} &= \left(-\frac{1}{\sqrt{17}}\cos\frac{s}{\sqrt5},-\frac{1}{\sqrt{17}}\sin\frac{s}{\sqrt5},-\frac{4}{\sqrt{17}}\cos\frac{2s}{\sqrt5},-\frac{4}{\sqrt{17}}\sin\frac{2s}{\sqrt5}\right), \\ \mathbf{B}_2 &= \left(-\frac{4}{\sqrt{17}}\cos\frac{s}{\sqrt5},-\frac{4}{\sqrt{17}}\sin\frac{s}{\sqrt5},\frac{1}{\sqrt{17}}\cos\frac{2s}{\sqrt5},\frac{1}{\sqrt{17}}\sin\frac{2s}{\sqrt5}\right), \\ \mathbf{B}_1 &= \left(\frac{2}{\sqrt5}\sin\frac{s}{\sqrt5},-\frac{2}{\sqrt5}\cos\frac{s}{\sqrt5},-\frac{1}{\sqrt5}\sin\frac{2s}{\sqrt5},\frac{1}{\sqrt5}\cos\frac{2s}{\sqrt5}\right), \end{cases}$$

where the curvatures are $\kappa(s)=\frac{17}{5\sqrt{17}}$, $\tau(s)=-\frac{6}{5\sqrt{17}}$ *and* $\eta(s)=-\frac{2}{\sqrt{17}}$. *From equation (1), for* $r=2$ *the tube surface in* $\mathbb{E}^4$ *is parametrized as*

$$\begin{aligned}\psi(s,v) &= \Big(\cos\frac{s}{\sqrt5}-2+\frac{4}{\sqrt5}\cos v\cos\frac{s}{\sqrt5}-\frac{8}{\sqrt{17}}\sin v\cos\frac{s}{\sqrt5},1+\sin\frac{s}{\sqrt5}-\frac{4}{\sqrt5}\cos v\cos\frac{s}{\sqrt5}-\frac{8}{\sqrt{17}}\sin v\sin\frac{s}{\sqrt5}, \\ &\cos\frac{2s}{\sqrt5}-\frac{2}{\sqrt5}\cos v\sin\frac{2s}{\sqrt5}+\frac{2}{\sqrt{17}}\sin v\cos\frac{2s}{\sqrt5},\sin\frac{2s}{\sqrt5}-3+\frac{2}{\sqrt5}\cos v\sin\frac{2s}{\sqrt5}+\frac{2}{\sqrt{17}}\sin v\sin\frac{2s}{\sqrt5}\Big).\end{aligned}$$

Finally for $r = 2$, the tube surfaces with respect to the Frenet frame are shown in Fig. 2.(a) and Fig. 2.(b) in projection spaces xyz and xyt, respectively.

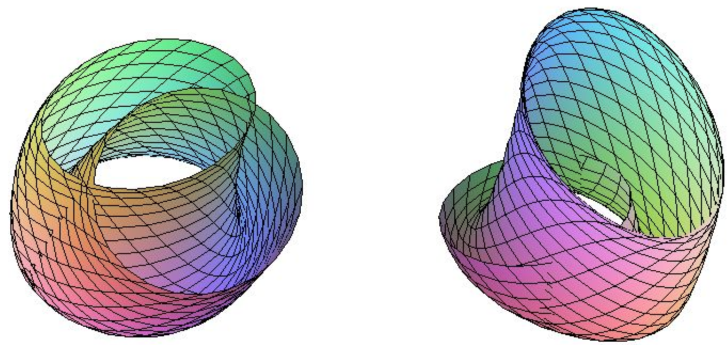

Fig. 2. (a) Tube surface in space xyz (b) Tube surface in space xyt

Similarly, in Fig. 3.(a) and Fig. 3.(b), these tube surfaces are shown in projection space xzt and yzt, respectively.

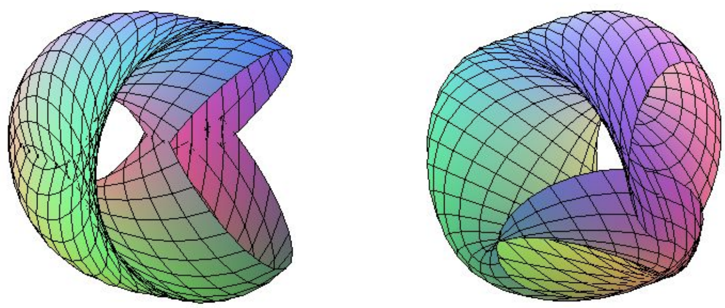

Fig. 3. (a) Tube surface in space xzt (b) Tube surface in space yzt

Instead, when considering calculations for directional tube surfaces, the q-frame vectors can be determined as

$$\begin{cases} \mathbf{T} &= \left(-\frac{1}{\sqrt{5}}\sin\frac{s}{\sqrt{5}}, \frac{1}{\sqrt{5}}\cos\frac{s}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\sin\frac{2s}{\sqrt{5}}, \frac{2}{\sqrt{5}}\cos\frac{2s}{\sqrt{5}}\right), \\ \mathbf{N}_q &= \left(0, 0, -\cos\frac{2s}{\sqrt{5}}, -\sin\frac{2s}{\sqrt{5}}\right), \\ \mathbf{B}_q &= \left(\cos\frac{s}{\sqrt{5}}, \sin\frac{s}{\sqrt{5}}, 0, 0\right), \\ \mathbf{C}_q &= \left(\frac{2}{\sqrt{5}}\sin\frac{s}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\cos\frac{s}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\sin\frac{2s}{\sqrt{5}}, \frac{1}{\sqrt{5}}\cos\frac{2s}{\sqrt{5}}\right), \end{cases}$$

where q-curvatures are $k_1(s) = \frac{4}{5}$, $k_2(s) = -\frac{1}{5}$, $k_3(s) = 0$ and $k_4(s) = -\frac{2}{5}$. If we calculate for the q-frame, the directional tube surface in $\mathbb{E}^4$ from equation (7) is parametrized as follows

$$\begin{aligned} \psi(s,v) \quad = \quad & \left(\cos\frac{s}{\sqrt{5}} - 2 + r(\cos v\cos\frac{s}{\sqrt{5}} + \frac{2}{5}\sin v\sin\frac{s}{\sqrt{5}}), 1 + \sin\frac{s}{\sqrt{5}} + r(\cos v\sin\frac{s}{\sqrt{5}} - \frac{2}{5}\sin v\cos\frac{s}{\sqrt{5}}),\right. \\ & \left.\cos\frac{2s}{\sqrt{5}} - \frac{1}{5}r\sin v\sin\frac{2s}{\sqrt{5}}, \sin\frac{2s}{\sqrt{5}} - 3 + \frac{1}{5}r\sin v\cos\frac{2s}{\sqrt{5}}\right). \end{aligned}$$

Here, for $r=2$ it is clearly seen as

$$\begin{aligned}\psi(s,v) \quad = \quad & \Big(\cos\frac{s}{\sqrt{5}}-2+2\cos v\cos\frac{s}{\sqrt{5}}+\frac{4}{5}\sin v\sin\frac{s}{\sqrt{5}},1+\sin\frac{s}{\sqrt{5}}+2\cos v\sin\frac{s}{\sqrt{5}}-\frac{4}{5}\sin v\cos\frac{s}{\sqrt{5}},\\ & \cos\frac{2s}{\sqrt{5}}-\frac{2}{5}\sin v\sin\frac{2s}{\sqrt{5}},\sin\frac{2s}{\sqrt{5}}-3+\frac{2}{5}\sin v\cos\frac{2s}{\sqrt{5}}\Big).\end{aligned}$$

Then for $r=2$, the first and second normal vectors $\mathbf{U}_1$ and $\mathbf{U}_2$ of the directional tube surface are respectively given as

$$\mathbf{U}_1=\frac{1}{\sqrt{1+(1+\frac{2}{5}\cos v)^2}}\Big(0,1+\frac{2}{5}\cos v,\cos v,\sin v\Big)$$

and

$$\mathbf{U}_2=\frac{1}{\sqrt{1+(1+\frac{2}{5}\cos v)^2}}\Big(0,1,-\cos v(1+\frac{2}{5}\cos v),-\sin v(1+\frac{2}{5}\cos v)\Big).$$

In addition, for $r=2$, the Gaussian and mean curvatures with the first and second normal vectors $\mathbf{U}_1$ and $\mathbf{U}_2$ of the directional tube surface are respectively computed as

$$K_1=-\frac{5(3\cos v+20)}{4(4\cos^3 v+30\cos^2 v+100\cos v+125)},\qquad H_1=-\frac{4\cos v-15}{2(2\cos v+5)\sqrt{50+20\cos v+4\cos^2 v}}$$

and

$$K_2=\frac{2\cos^2 v+5\cos v+20}{4(25+10\cos v+2\cos^2 v)},\qquad H_2=\frac{8\cos^2 v+30\cos v+65}{2(2\cos v+5)\sqrt{50+20\cos v+4\cos^2 v}}.$$

From (4) and (5), for $r=2$ the Gaussian and mean curvatures of the directional tube surface are respectively determined as

$$K=\frac{\cos v}{2(5+2\cos v)},\qquad H=\frac{25}{8}\frac{\sqrt{16\cos^2 v+40\cos v+89}}{(2\cos v+5)(25+20\cos v+4\cos^2 v)}.$$

Finally, the directional tube surfaces shown in Fig. 4. (a) and Fig. 4. (b) are parametrized as

$$\psi_1(s,v)=\Big(\cos\frac{s}{\sqrt{5}}-2+2\cos v\cos\frac{s}{\sqrt{5}}+\frac{4}{5}\sin v\sin\frac{s}{\sqrt{5}},1+\sin\frac{s}{\sqrt{5}}+2\cos v\sin\frac{s}{\sqrt{5}}-\frac{4}{5}\sin v\cos\frac{s}{\sqrt{5}},\cos\frac{2s}{\sqrt{5}}-\frac{2}{5}\sin v\sin\frac{2s}{\sqrt{5}}\Big),$$

$$\psi_2(s,v)=\Big(\cos\frac{s}{\sqrt{5}}-2+2\cos v\cos\frac{s}{\sqrt{5}}+\frac{4}{5}\sin v\sin\frac{s}{\sqrt{5}},1+\sin\frac{s}{\sqrt{5}}+2\cos v\sin\frac{s}{\sqrt{5}}-\frac{4}{5}\sin v\cos\frac{s}{\sqrt{5}},\sin\frac{2s}{\sqrt{5}}-3+\frac{2}{5}\sin v\cos\frac{2s}{\sqrt{5}}\Big)$$

for $r=2$ in projection spaces xyz and xyt, respectively.

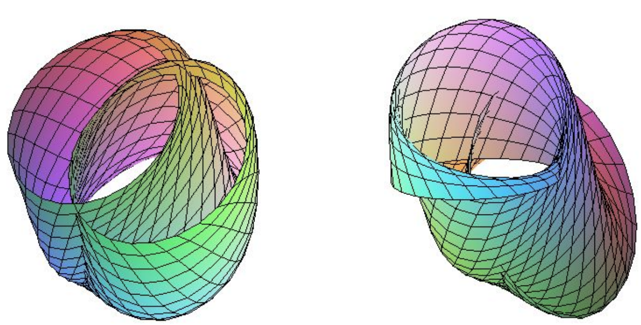

Fig. 4. (a) Directional tube surface in space *xyz* (b) Directional tube surface in space *xyt*

Likewise, the directional tube surfaces shown in Fig. 5. (a) and Fig. 5. (b) are parametrized as

$$\psi_3(s,v) = \left(\cos\frac{s}{\sqrt{5}} - 2 + 2\cos v\cos\frac{s}{\sqrt{5}} + \frac{4}{5}\sin v\sin\frac{s}{\sqrt{5}}, \cos\frac{2s}{\sqrt{5}} - \frac{2}{5}\sin v\sin\frac{2s}{\sqrt{5}}, \sin\frac{2s}{\sqrt{5}} - 3 + \frac{2}{5}\sin v\cos\frac{2s}{\sqrt{5}}\right),$$

$$\psi_4(s,v) = \left(1 + \sin\frac{s}{\sqrt{5}} + 2\cos v\sin\frac{s}{\sqrt{5}} - \frac{4}{5}\sin v\cos\frac{s}{\sqrt{5}}, \cos\frac{2s}{\sqrt{5}} - \frac{2}{5}\sin v\sin\frac{2s}{\sqrt{5}}, \sin\frac{2s}{\sqrt{5}} - 3 + \frac{2}{5}\sin v\cos\frac{2s}{\sqrt{5}}\right)$$

for $r = 2$ in projection spaces xzt and yzt, respectively.

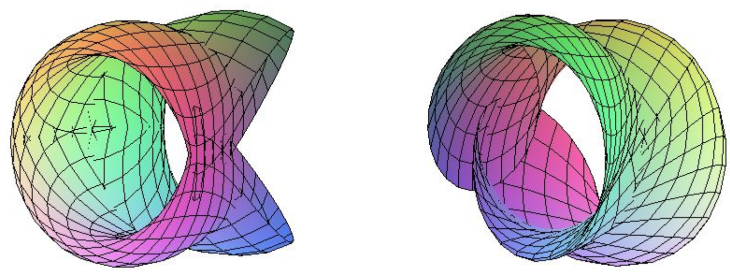

Fig. 5. (a) Directional tube surface in space *xzt* (b) Directional tube surface in space *yzt*

All the figures in this study were created by using maple programme.

5. Conclusions

In this study, the tube surface was parametrized using the q-frame. Additionally, tube surfaces parametrized with the q-frame are called directional tube surfaces. Parametrizing with the q-frame offers advantages over the Frenet frame in terms of avoiding singularity and unnecessary torsion. A directional tube surface is formed by joining an infinite number of circles drawn on planes generated by the q-frame vector fields of the centre curve, where the radius remains constant. The centres of these circles are oriented along the centre curve of the directional tube surface.

After introducing directional tube surfaces in Euclidean 4-space, unit normal vectors, fundamental form coefficients, Gaussian, and mean curvatures are computed. Finally, an example is provided by taking a center curve in Euclidean 4-space. First, a general equation for tube surface is formulated and drawn using the Frenet frame. Subsequently, the directional tube surface is parametrized with respect to the q-frame, and the normal vectors, Gaussian and mean curvatures of the directional tube surface are calculated. Finally, the Gaussian and mean curvatures are determined using coefficients of the fundamental form and the directional tube surface is represented in projection spaces.

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