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Elliptic Inversions in Taxicab Geometry

Taksi Geometride Eliptik İnversiyonlar

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Abstract

The goal of this research is to introduce inversion with respect to an ellipse which is a generalization of the classical circular inversion in taxicab plane and to investigate general properties and basic concepts of this transformation in taxicab geometry. The cross ratio is preserved under the elliptic inversion in taxicab plane though this transformation is not an isometry. Thus some properties such as cross ratio and harmonic conjugates of the elliptic inversions in $\mathbb{R}^2_T$ are also studied.

Anahtar Kelimeler Inversion; Elliptic inversion; Taxicab metric; Cross ratio; Harmonic conjugates

Öz

Bu çalışmanın amacı klasik çembersel inversiyonların bir genelleştirmesi olan elipse göre inversiyonları taksi düzleminde çalışmak ve bu dönüşümün genel özelliklerini ve temel yapılarını taksi geometride araştırmaktır. Bu dönüşüm bir izometri olmasa da çapraz oran taksi düzleminde eliptik inversiyonlar altında korunur. Bu yüzden bu araştırmada eliptik inversiyonların çapraz oran ve harmonik eşlenik gibi bazı özellikleri üzerine de $\mathbb{R}^2_T$ de çalışılmıştır.

Keywords *İnversiyon; Eliptik inversiyon; Taksi metriği; Çapraz oran; Harmonik eşlenik*

1. Introduction

A family of metrics; l_p-metric (also known as the Minkowski distance) was published in (Minkowski 1967) by Minkowski. This family of metrics includes taxicab (also known as l_1 or Manhattan), and Euclidean (also known as l_2) metrics as special cases. Later, Menger introduced the taxicab plane geometry in (Menger 1952). Krause subsequently developed the taxicab geometry in (Krause 1975) and this geometry has been studied by many authors, for some of the studies on taxicab geometry see (Djvak 2000, Akça and Kaya 1997, Schattschneider 1984, Chen 1992, So 2002, Ho and Lıu 1996, Laatsch 1982, Reynolds 1982, Tian *et al.* 1997, Kaya 2004, Özcan and Kaya 2002). Taxicab plane geometry is derived simply by substituting the metric

$$d_T(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2| \tag{1}$$

with the well known Euclidean metric

$$d_E(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{2}$$

for the distance between any two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in the analytical plane, thus taxicab plane is denoted by $\mathbb{R}^2_T$.

Taxicab plane geometry is a Minkowski geometry. Minkowski geometry is a non- Euclidean geometry in a finite number of dimensions that is different from elliptic and hyperbolic geometry (and from the Minkowskian geometry of space-time). In this geometry the linear structure is the same as the Euclidean one but the distance is not "uniform" in all directions. Instead of the usual sphere in Euclidean space, the unit ball is a general symmetric convex set (Thompson 1996). That is, Euclidean and taxicab planes have the same points, lines and the way of measuring the angles. Since the only difference of the taxicab geometry from the Euclidean geometry is the distance function, it is interesting to study the taxicab analogues of issues that include the distance concept in Euclidean geometry. One of these concepts is inversion. Since it reveals difficult questions and many challenging problems in geometry and many problems become much manageable when it is applied, inversion is one of the most gripping transformation in the plane. As it has been stated in (Patterson 1933) this transformation was probably first introduced by Apollonious of Perga in his last book *Plane Loci* and systematically investigated by Jakob Steiner in the 1820s. This transformation would be used to study on several theorems and problems in geometry as Ptolemy's theorem, Steiner porism, the problem of Apollonius, the Pappus chain, etc.

When an inversion is considered the first thing that comes to mind is an inversion with respect to a circle, but Childress introduced inversions with respect to the

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central conics in real Euclidean plane in (Childress 1965) and authors studied inversions with respect to an ellipse in real Euclidean plane in (Ramirez 2014) and (Ramirez and Rubiano 2014). In (Bayar and Ekmekçi 2014) and (Nickel 1995) inversions with respect to taxicab circles and in (Gelişgen and Ermiş 2019) inversions with respect to circles in alpha plane (alpha plane includes taxicab plane as a special case) are investigated.

In this paper first, the inversion in an ellipse in taxicab plane ($\mathbb{R}_T^2$) is introduced. Then general properties and basic concepts are investigated and some illustrations of taxicab elliptic inversion for particular conditions via GeoGebra are given as examples. Furthermore some properties related with this inversion such as cross ratio and harmonic conjugates are studied.

2. Preliminaries

2.1 Some Basics of Taxicab Plane

In this section some properties of taxicab plane and some relations between Euclidean and taxicab planes are given without proofs which are briefly taken from (Gelişgen 2007).

Proposition 2.1.1 Taxicab distance function $d_T: \mathbb{R} \times \mathbb{R} \to (0, \infty]$ is defined as

$d_T(P,Q) = |x_1 - x_2| + |y_1 - y_2|$ for any points points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in $\mathbb{R}_T^2$ and d_T is a metric.

Proposition 2.1.2 Each of Euclidean translations is an isometry of $\mathbb{R}_T^2$.

Proposition 2.1.3 Let P_1 and P_2 be two points on a line l with the slope m in the analytical plane and d_E denotes the Euclidean distance function, then

$$d_E(P_1, P_2) = \frac{\sqrt{1+m^2}}{1+|m|} d_T(P_1, P_2). \tag{3}$$

Corollary 2.1.4 For any three collinear points P_1, P_2 and X in $\mathbb{R}^2$, $d_E(P_1, X) = d_E(P_2, X)$ if and only if $d_T(P_1, X) = d_T(P_2, X)$.

Corollary 2.1.5 For any three collinear points P_1, P_2 and X in $\mathbb{R}^2$, $d_E(P_1, X)/d_E(P_2, X)$ if and only if $d_T(P_1, X)/ d_T(P_2, X)$.

2.2 Preliminaries about Inversions

In this section definition of inversion in Euclidean plane as have been stated in (Blair 200) is given and basic properties of this mapping are mentioned.

Definition 2.2.1 Let C be a circle with the center O and radius r. If P is any point other than O, then inverse of P with respect to C is the point P' on the ray $\overrightarrow{OP}$ such that the product of the distances of P and P' from O is equal to r^2; that is,

$$d_E(O, P). d_E(O, P) = r^2. \tag{4}$$

The inversion mapping defined above is a conformal mapping. Obviously P and P' are the inverses of each other. Also by a little observation the mapping shifts the interior and exterior of C and points on C are fixed. This shifting excepts O, since O has no image, and no point of the plane is mapped to O. Note also that points close to O are mapped to the points far from O, and vice versa. Thus to include O in the domain and range of an inversion, one "ideal point", or "point at infinity" would be adjoined to the Euclidean plane.

3. Taxicab Elliptic Inversion

In this section, taxicab elliptic inversions are examined. First, inversion with respect to an ellipse in taxicab plane is introduced and basic properties of this inversion are given. Then inversions of lines and ellipses in $\mathbb{R}_T^2$ are investigated. In addition, properties of taxicab elliptic inversion, such as cross ratio and harmonic conjugates are given.

Definition 3.1 Let F_1 and F_2 be two points in $\mathbb{R}_T^2$. The taxicab ellipse $\mathcal{E}$ with the center $O = (a, b)$, the constant k and foci F_1 and F_2 in $\mathbb{R}_T^2$ is the set of points

$$\{P = (x, y): d_T(P, F_1) + d_T(P, F_2) = k, P \in \mathbb{R}_T^2, \mathrm{k} \geq d_T(F_1, F_2)\}.$$

There are two types of taxicab ellipses according to the slope of the line through foci of the ellipse. If the slope of the line is 0 or ∞ then the ellipse is a hexagon otherwise the ellipse is an octagon.

Definition 3.2 Let $\mathcal{E}$ be an ellipse centered at the point $O = (a, b)$ with foci F_1 and F_2 and constant k in $\mathbb{R}_T^2$ and let the ideal point enclosed to the taxicab plane is P_∞. In $\mathbb{R}_T^2$ the taxicab elliptic inversion with respect to $\mathcal{E}$ is the mapping

$$I_{\mathcal{E}}(O, k): \mathbb{R}_T^2 \cup \{P_\infty\} \to \mathbb{R}_T^2 \cup \{P_\infty\}$$

defined by $I_{\mathcal{E}}(O, k)(O) = P_\infty$, $I_{\mathcal{E}}(O, k)(P_\infty) = O$ and $I_{\mathcal{E}}(O, k)\ (P) = P'$, where P' lies on the ray $\overrightarrow{OP}$ for $P \neq O$ and

$$d_T(O, P). d_T(O, P') = [d_T(O, Q)]^2, \tag{5}$$

where Q is the intersection of the ray $\overrightarrow{OP}$ with the ellipse $\mathcal{E}$. O is named by the center of the inversion, $\mathcal{E}$ is named by the ellipse of the inversion, the point P' is named by the inverse of the point P with respect to the ellipse $\mathcal{E}$, and the positive real number $\mathcal{E} := d_T(O,Q)$ is said to be the radius of the inversion, see Figure 1.

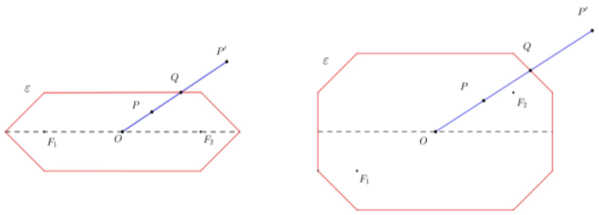

Figure 1. Taxicab elliptic inverse of a point

Unlike the circular inversions in taxicab plane (and in Euclidean plane), the radii of taxicab elliptic inversions are not constant. Also it is obvious that the only invariant points under $I_{\mathcal{E}}(O,k)$ are the points on the ellipse $\mathcal{E}$ since $I_{\mathcal{E}}(O,k)$ is an involution like reflections.

Lemma 3.3 Let $I_{\mathcal{E}}(O,k)$ be a taxicab elliptic inversion in the ellipse $\mathcal{E}$ with the center (O,O) and constant k. $I_{\mathcal{E}}(O,k)$ shifts the interior and exterior of $\mathcal{E}$.

Proof. Let P is a point in the interior of $\mathcal{E}$, then

$$d_T(O,P) < d_T(O,Q). \tag{6}$$

Since $P' = I_{\mathcal{E}}(O,k)(P)$, by using (5) and (6) we get

$$\begin{aligned}[d_T(O,Q)]^2 &= d_T(O,P).d_T(O,P') \\ &< d_T(O,Q).d_T(O,P')\end{aligned} \tag{7}$$

and thus

$$d_T(O,P') > d_T(O,Q). \tag{8}$$

So P' is in the exterior of $\mathcal{E}$. The proof for the case that P is in the exterior of $\mathcal{E}$ is similar.

Proposition 3.4 Let $I_{\mathcal{E}}(O,k)$ be a taxicab elliptic inversion in the ellipse $\mathcal{E}$ with foci $F_1 = (x_1,y_1)$ and $F_2 = (x_2,y_2)$, the center $O = (0,0)$ and the constant k. If $P = (x,y)$ and $P' = (x',y')$ are inverse points according to $I_{\mathcal{E}}(O,k)$,then

$$(x',y') = \begin{cases} \left(\frac{k^2x}{4(x+y)^2}, \frac{k^2y}{4(x+y)^2}\right), & if \left|\frac{2y_1}{k-2y_1}\right| < m < \left|\frac{k-2x_1}{2x_1}\right| \\ \left(\frac{(k-2x_1)^2x}{4y^2}, \frac{(k-2x_1)^2}{4y}\right), & if \left|\frac{k-2x_1}{2x_1}\right| < |m| \\ \left(\frac{k^2x}{4(x-y)^2}, \frac{k^2y}{4(x-y)^2}\right), & if -\left|\frac{k-2x_1}{2x_1}\right| < m < -\left|\frac{2y_1}{k-2y_1}\right| \\ \left(\frac{(k-2y_1)^2}{4x}, \frac{(k-2y_1)^2y}{4x^2}\right), & if\ m < \left|\frac{2y_1}{k-2y_1}\right| \end{cases} \tag{9}$$

where $x_i \neq 0, y_i \neq 0, x_1 > y_1$ and m is the slope of the $\overrightarrow{OP}$. Note that if $x_i \neq 0, y_i = 0$ and $x_1 > y_1$, then

$$(x',y') = \begin{cases} \left(\frac{k^2x}{4(x+y)^2}, \frac{k^2y}{4(x+y)^2}\right), & if\ 0 < m < \left|\frac{k-2x_1}{2x_1}\right| \\ \left(\frac{(k-2x_1)^2x}{4y^2}, \frac{(k-2x_1)^2}{4y}\right), & if \left|\frac{k-2x_1}{2x_1}\right| < |m| \\ \left(\frac{k^2x}{4(x-y)^2}, \frac{k^2y}{4(x-y)^2}\right), & if\ -\left|\frac{k-2x_1}{2x_1}\right| < m < 0 \end{cases} \tag{10}$$

and if $x_i = 0, y_i \neq 0,\ and\ x_1 > y_1$, then

$$(x',y') = \begin{cases} \left(\frac{k^2x}{4(x+y)^2}, \frac{k^2y}{4(x+y)^2}\right), & if \left|\frac{2y_1}{k-2y_1}\right| < m \\ \left(\frac{k^2x}{4(x-y)^2}, \frac{k^2y}{4(x-y)^2}\right), & if\ m < -\left|\frac{2y_1}{k-2y_1}\right| \\ \left(\frac{(k-2y_1)^2}{4x}, \frac{(k-2y_1)^2y}{4x^2}\right), & if\ |m| < \left|\frac{2y_1}{k-2y_1}\right| \end{cases} \tag{11}$$

Proof. The central ellipse $\mathcal{E}$ with foci $F_1 = (x_1,y_1)$, $F_2 = (x_2,y_2)$ and the constant k is the set

$$\{Q \in \mathbb{R}_T^2 : | x - x_1 | + | y - y_1 | + | x - x_2 | + | y - y_2 | = k, Q = (x,y), \mathrm{k} \geq d_T(F_1,F_2)\ \}. \tag{12}$$

Assume that $P = (x,y)$ and $P' = (x',y')$ are inverse pair of points under $I_{\mathcal{E}}(O,k)$, thus $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ have the same direction since P, P' and O are collinear. Then $\overrightarrow{OP'} = t\overrightarrow{OP}$ for $t \in \mathbb{R}^+$. Since P and P' are inverse points and by (5) we have

$$t = \frac{[d_T(O,Q)]^2}{(|x|+|y|)^2} \tag{13}$$

and by substituting the value of t in $(x',y') = (tx,ty)$ required results are obtained. For instance if $\frac{k-2x_1}{2x_1} <\mid m \mid$, then $d_T(O,Q) = (1+\mid m \mid)(\frac{k-2x_1}{2m})$. Thus $(x',y') = (\frac{(k-2x_1)^2x}{4y^2}, \frac{(k-2x_1)^2}{4y^2})$.

Corollary 3.5 Let $I_{\mathcal{E}}(O,k)$ be a taxicab elliptic inversion in the ellipse $\mathcal{E}$ with foci $F_1(x_1,y_1)$ and $F_2 = (x_2,y_2)$, the center $O = (a,b)$, and the constant k in $\mathbb{R}_T^2$. If $P'(x',y')$ and $P = (x,y)$ are inverse pair of points according to $I_{\mathcal{E}}(O,k)$, then

$$(x',y') = \begin{cases} \left(a+\frac{k^2(x-a)}{4(x-a+y-b)^2}, b+\frac{k^2(y-b)}{4(x-a+y-b)^2}\right), & if \left|\frac{2(y_1-b)}{k-2(y_1-b)}\right| < m < \left|\frac{k-2(x_1-a)}{2(x_1-a)}\right| \\ \left(a+\frac{(k-2x_1-2a)^2(x-a)}{4(y-b)^2}, b+\frac{(k-2x_1-2a)^2}{4(y-b)}\right), & if \left|\frac{k-2(x_1-a)}{2(x_1-a)}\right| < |m| \\ \left(a+\frac{k^2(x-a)}{4(x-a-y+b)^2}, b+\frac{k^2(y-b)}{4(x-a-y+b)^2}\right), & if -\left|\frac{k-2(x_1-a)}{2(x_1-a)}\right| < m < -\left|\frac{2(y_1-b)}{k-2(y_1-b)}\right| \\ \left(a+\frac{(k-2y_1-2b)^2}{4(x-a)}, b+\frac{(k-2y_1-2b)^2(y-b)}{4(x-a)^2}\right), & if\ m < \left|\frac{2(y_1-b)}{k-2(y_1-b)}\right| \end{cases} \quad (14)$$

where $x_i \neq 0, y_i \neq 0,\ x_1 > y_1$ and m is the slope of the ray $\overrightarrow{OP}$.

Proof. The result is obvious by the Proposition 2.1.2.

The statements of the following theorem would be proven by the definition of $I_{\mathcal{E}}(O,k)$, so it is given without proof but some conditions are illustrated.

Theorem 3.6

i. If a line passes through the center of inversion O, then it is invariant under $I_{\mathcal{E}}(O,k)$.

ii. If a line doesn't pass through the center of inversion O, then it is not mapped onto a taxicab ellipse centered at O under $I_{\mathcal{E}}(O,k)$, see Figure 2.

iii. Inverses of taxicab ellipses centered at O which are the same type of ellipse of the inversion under $I_{\mathcal{E}}(O,k)$ are same type of taxicab ellipses with the center O.

iv. Inverses of taxicab ellipses centered at O which are not the same type of ellipse of the inversion under $I_{\mathcal{E}}(O,k)$ are not taxicab ellipses with the center O, see Figure 3.

v. Inverses of taxicab ellipses not passing through O under $I_{\mathcal{E}}(O,k)$ are not any taxicab ellipses.

vi. Inverses of taxicab ellipses passing through the center of inversion O under $I_{\mathcal{E}}(O,k)$ are not lines not containing the center O.

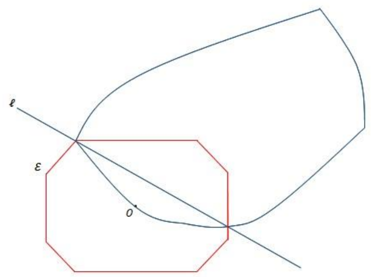

Figure 2. Inverse of a line l not passing through O under $I_{\mathcal{E}}(O,k)$

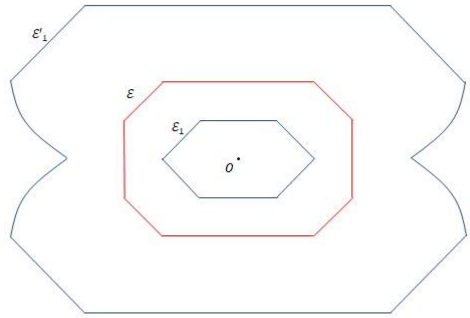

Figure 3. Inverse of a taxicab ellipse $\mathcal{E}_1$ centered at O which is not the same type of the ellipse of the inversion $I_{\mathcal{E}}(O,k)$

$I_{\mathcal{E}}(O,k)$ is not an isometry in $\mathbb{R}^2_T$, thus distance is not invariant under $I_{\mathcal{E}}(O,k)$. However the cross ratio is a concept that includes distance and it is preserved under inversion under some conditions. Thus next the cross-ratio and harmonic conjugates under $I_{\mathcal{E}}(O,k)$ are investigated.

Proposition 3.7 Let $\mathcal{E}$ be a taxicab ellipse of inversion with the center $O=(a,b)$, foci $F_1=(x_1,y_1)$ and $F_2=(x_2,y_2)$, and the constant k. If P_1, P_2 are two points collinear with O in $\mathbb{R}^2_T$ and if $\{P_1,P_1'\}$ and $\{P_2,P_2'\}$ are inverse pairs with respect to $I_{\mathcal{E}}(O,k)$, then

$$d_T(P_1{}', P_2{}') = \frac{[d_T(O,Q)]^2 d_T(P_1,P_2)}{d_T(O,P_1)d_T(O,P_2)}, \quad (15)$$

where Q is the intersection point of the ray $\overrightarrow{OP_1}$ and the ellipse $\mathcal{E}$, m is the slope of the ray $\overrightarrow{OP_1}$ and

$$d_T(O,Q) = \begin{cases} \left|\frac{k}{2(1+m)}\right|(1+|m|), & if \left|\frac{2(y_1-b)}{k-2(y_1-b)}\right| < m < \left|\frac{k-2(x_1-a)}{2(x_1-a)}\right| \\ \left|\frac{k-2(x_1-a)}{2m}\right|(1+|m|), & if \left|\frac{k-2(x_1-a)}{2(x_1-a)}\right| < |m| \\ \left|\frac{k}{2(m-1)}\right|(1+|m|), & if -\left|\frac{k-2(x_1-a)}{2(x_1-a)}\right| < m < -\left|\frac{2(y_1-b)}{k-2(y_1-b)}\right| \\ \left|\frac{k-2(y_1-b)}{2}\right|(1+|m|), & if\ m < \left|\frac{2(y_1-b)}{k-2(y_1-b)}\right| \end{cases}$$

(16)

Proof. Suppose that O, P_1, P_2 are collinear and P_1', P_2' are inverses of P_1 and P_2 respectively. By the Definition 3.2

$$d_T(O,P_1).d_T(O,P_1') = [d_T(O,Q)]^2 = d_T(O,P_2).d_T(O,\ P_2'). \quad (17)$$

By using Corollary 2.1.5

$$\begin{aligned} d_T(P_1{}', P_2{}') &= |d_T(O,P_1{}') - d_T(O, P_2{}')| \\ &= \left|\frac{[d_T(O.Q)]^2}{d_T(O,P_1)} - \frac{[d_T(O.Q)]^2}{d_T(O,P_2)}\right| \\ &= \frac{[d_T(O.Q)]^2 d_T(P_1,P_2)}{d_T(O,P_1)d_T(O,P_2)}. \end{aligned} \tag{18}$$

Clearly the equality (15) holds if the points P_1, P_2 and O are collinear, that is the equality isn't valid for every points in $\mathbb{R}_T^2$. The next proposition suggests under which other conditions an analogue of (15) is satisfied.

Proposition 3.8 Let $\{P_1\ , P_2, O\}$ is a set of any distinct, non-collinear points in $\mathbb{R}_T^2$ and $\{P_1, P_1'\}, \{P_2 P_2'\}$ are inverse pairs of points under $I_{\mathcal{E}}(O,k)$. Let $m_{\overrightarrow{OP_1}}$ and $m_{\overrightarrow{OP_2}}$ denote the slopes of $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ respectively. If $m_{\overrightarrow{OP_1}}, m_{\overrightarrow{OP_2}} \in \left\{m: \left|\frac{k-2(x_1-a)}{2(x_1-a)}\right| < |m|\right\}$ and P_1 and P_2 lie on the line with slope 0 or, if $m_{\overrightarrow{OP_1}}, m_{\overrightarrow{OP_2}} \in \left\{m: m < \left|\frac{2(y_1-b)}{k-2(y_1-b)}\right|\right\}$ and P_1 and P_2 lie on the line with slope ∞, then

$$d_T(P_1{}', P_2{}') = \frac{d_T(O,Q_1)\, d_T(O,Q_2) d_T(P_1,P_2)}{d_T(O,P_1)d_T(O,P_2)}, \tag{19}$$

where Q_1 is the intersection point of the $\overrightarrow{OP_1}$ and the ellipse $\mathcal{E}$, and Q_2 is the intersection point of the $\overrightarrow{OP_2}$ and the ellipse $\mathcal{E}$.

Proof. Note that if $m_{\overrightarrow{OP_1}}, m_{\overrightarrow{OP_2}} \in \left\{m: \left|\frac{k-2(x_1-a)}{2(x_1-a)}\right| < |m|\right\}$, then $P_1 = (x_3, y_0)$ and $P_2 = (x_4, y_0)$ are mapped to

$P_1{}' = \left(a + \frac{(k-2x_1-2a)^2(x_3-a)}{4(y_0-b)^2}, b + \frac{(k-2x_1-2a)^2}{4(y_0-b)}\right)$ and

$P_2{}' = \left(a + \frac{(k-2x_1-2a)^2(x_4-a)}{4(y_0-b)^2}, b + \frac{(k-2x_1-2a)^2}{4(y_0-b)}\right)$,

respectively. If $m_{\overrightarrow{OP_1}}, m_{\overrightarrow{OP_2}} \in \left\{m: m < \left|\frac{2(y_1-b)}{k-2(y_1-b)}\right|\right\}$, then $P_1 = (x_0, y_3)$ and $P_2 = (x_0, y_4)$ are mapped to

$P_1{}' = \left(a + \frac{(k-2y_1+2b)^2}{4(x_0-a)}, b + \frac{(k-2y_1+2b)^2(y_3-b)}{4(x_0-a)^2}\right)$ and

$P_2{}' = \left(a + \frac{(k-2y_1+2b)^2}{4(x_0-a)}, b + \frac{(k-2y_1-2b)^2(y_4-b)}{4(x_0-a)^2}\right)$,

respectively. Thus for both conditions it can easily be shown that (19) holds.

$d_T[P_1, P_2]$ is used to indicate the taxicab directed distance from P_1 to P_2 in the taxicab plane. If P_1 is the initial point of the ray and the side P_2 is contained has the positive direction of orientation, then $d_T[P_1, P_2] = d_T(P_1, P_2)$, and $d_T[P_1, P_2] = -\, d_T(P_1, P_2)$ when the ray has the opposite direction.

Let the four distinct points on an oriented line in the taxicab plane be P_1, P_2, P_3 and P_4, then their taxicab cross ratio $(P_1P_2, P_3P_4)_T$ is defined by

$$(P_1P_2, P_3P_4)_T = \frac{d_T[P_1,P_3] d_T[P_2,P_4]}{d_T[P_1,P_4] d_T[P_2,P_3]}. \tag{20}$$

If $P_3, P_4 \in [P_1P_2]$ or $P_3, P_4 \notin [P_1P_2]$, then the taxicab cross ratio is positive and if pairs $\{P_1, P_2\}$ and $\{P_3, P_4\}$ seperate each other, then the taxicab cross ratio is negative. Also a taxicab elliptic inversion with respect to $\mathcal{E}$ with a center which is different from P_1, P_2, P_3 and P_4, and collinear with these four points, preserves the taxicab croos ratio.

Theorem 3.9 Taxicab elliptic inversion preserves the taxicab croos ratio.

Proof. Let the four points P_1, P_2, P_3 and P_4 are collinear in the taxicab plane and $P_1{}'$, $P_2{}'$, $P_3{}'$ and $P_4{}'$ be inverses of P_1, P_2, P_3 and P_4, respectively according to taxicab elliptic inversion $I_{\mathcal{E}}(O,k)$. Observe that the seperation or non-seperation of the pairs $\{P_1, P_2\}$ and $\{P_3, P_4\}$ is invariant under taxicab elliptic inversion and also taxicab elliptic inversion reverses the taxicab directed distance from the point P_1 to the point P_2 along a line l to taxicab-directed distance from point P_2' to the point P_1'. By considering these observations and Proposition 3.7 we have

$$\begin{aligned} (P_1{}'P_2{}', P_3{}'P_4{}')_T &= \frac{d_T[P_1{}', P_3{}'].\, d_T[P_2{}', P_4{}']}{d_T[P_1{}', P_4{}'].\, d_T[P_2{}', P_3{}']} \\ &= \frac{\frac{[d_T(O,Q)]^2 d_T(P_1,P_3)}{d_T(O,P_1)d_T(O,P_3)} \cdot \frac{[d_T(O,Q)]^2 d_T(P_2,P_4)}{d_T(O,P_2)d_T(O,P_4)}}{\frac{[d_T(O,Q)]^2 d_T(P_1,P_4)}{d_T(O,P_1)d_T(O,P_4)} \cdot \frac{[d_T(O,Q)]^2 d_T(P_2,P_3)}{d_T(O,P_2)d_T(O,P_3)}} \\ &= \frac{d_T(P_1,P_3)d_T(P_2,P_4)}{d_T(P_1,P_4)d_T(P_2,P_3)} \\ &= (P_1P_2, P_3P_4)_T. \end{aligned} \tag{21}$$

Suppose that P_1, P_2, P_3 and P_4 are four points on a line l in $\mathbb{R}_T^2$. If $(P_1, P_2, P_3, P_4)_T = -1$, then P_1, P_2, P_3 and P_4 form a harmonic set and it is denoted by $H(P_1P_2, P_3P_4)_T$. That is, any pair P_3 and P_4 on l is said to divide P_1 and P_2 harmonically if

$$\frac{d_T[P_1,P_3] d_T[P_2,P_4]}{d_T[P_1,P_4] d_T[P_2,P_3]} = -1. \tag{22}$$

Then the points P_3 and P_4 are called taxicab harmonic conjugates with respect to P_1 and P_2.

Theorem 3.10 Let $\mathcal{E}$ be a taxicab ellipse with the center O, the constant k and P_1, $P_2 \in \mathcal{E}$ be any two points collinear with O in $\mathbb{R}_T^2$. Let P_3, P_4 be a pair of distinct points of the ray $\overrightarrow{OP_1}$, which seperates $\{P_1, P_2\}$. Thus, P_3 and P_4 are taxicab harmonic conjugates with respect to P_1 and P_2 if and only if P_3 and P_4 is a pair of inverse points under $I_{\mathcal{E}}(O,k)$.

Proof. Let P_3 and P_4 are taxicab harmonic conjugates with respect to P_1 and P_2. So

$$(P_1P_2, P_3P_4)_T = -1 \tag{23}$$

or

$$\frac{d_T[P_1,P_3]d_T[P_2,P_4]}{d_T[P_1,P_4]d_T[P_2,P_3]} = -1. \tag{24}$$

Since P_3 is between $\{P_1, P_2\}$ and P_3 is on the ray $\overrightarrow{OP_2}$ and $d_T(O,Q) = d_T(O,P_1) = d_T(O,P_2)$,

$$d_T(P_3,P_2) = d_T(O,Q) - d_T(O,P_3) \tag{25}$$

and

$$d_T(P_1,P_3) = d_T(O,Q) + d_T(O,P_3). \tag{26}$$

Since P_4 is not between $\{ P_1, P_2\}$ and P_4 is on the ray $\overrightarrow{OP_2}$ and $d_T(O,Q) = d_T(O,P_1) = d_T(O,P_2)$,

$$d_T(P_1,P_4) = d_T(O,Q) + d_T(O,P_4) \tag{27}$$

and

$$d_T(P_2,P_4) = d_T(O,Q) - d_T(O,P_3). \tag{28}$$

Thus,

$$\frac{\big(d_T(O,Q)+d_T(O,P_3)\big)\big(d_T(O,Q)-d_T(O,P_4)\big)}{\big(d_T(O,Q)+d_T(O,P_4)\big)\big(d_T(O,Q)-d_T(O,P_3)\big)} = -1. \tag{29}$$

By rearranging (29)

$$\big(d_T(O,Q) + d_T(O,P_3)\big)\big(d_T(O,Q) - d_T(O,P_4)\big) = \big(d_T(O,Q) + d_T(O,P_4)\big)\big(d_T(O,P_3) - d_T(O,Q)\big) \tag{30}$$

and by simplifying (30)

$$d_T(O,P_3).\,d_T(O,P_4) = (d_T(O,Q))^2 \tag{31}$$

is obtained. So, P_3 and P_4 are the taxicab inverse points with respect to $I_{\mathcal{E}}(O,k)$. The same conclusion would be obtained by similar calculations for the conditions P_3 and P_4 are on the ray $\overrightarrow{OP_1}$. For the converse statement of the theorem the proof is similar.

4. Discussion and Conclusion

This study deals with a generalization of the classical circular inversion in Euclidean geometry. Inversion is a very important, popular and useful transformation of the analytical plane since not only it reveals challenging problems but also it makes many problems in geometry much manageable when it is applied.

In this work inversions with respect to taxicab ellipses are introduced and some properties of these inversions are investigated. By this work we think that this generalization would provoke further studies by interested readers.

Declaration of Ethical Standards

The authors declare that they comply with all ethical standards.

Declaration of Competing Interest

The authors have no conflicts of interest to declare regarding the content of this article.

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