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Fixed Point Theorems on Partial Metric Spaces Involving Rational Type Expressions with C-Class Functions

Isa Yildirim[a], Arslan H. Ansari[b,*], Mohammad S. Khan[c], Brian Fisher[d]

[a]Department of Mathematics, Faculty of Science, Ataturk University, 25240 Erzurum, Turkey.
[b]Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.
[c]Department of Mathematics and Statistics, Sultan Qaboos University, P. O.Box 36, Al-Khoud 123, Muscat, Sultanate of Oman.
[d]Department of Mathematics, The University of Leicester, LEI 7RH, England.

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Abstract. In this paper, we present some fixed point theorems for contraction of rational type by using a class of pairs of functions satisfying certain assumptions with C-class functions in a complete partial metric space. Also, an example are given to support our main result. Our result extends and generalizes some well known results of [7] and [8] in metric spaces.

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1. Introduction

Banach's contraction principle is one of the cornerstones in the development of fixed point theory. Its significance lies in its vast applicability to a great number of branches of mathematics and other sciences, for example, theory of existence of solutions for nonlinear differential, integral and functional equations, variational inequalities and optimization and approximation theory. There are many generalization of this principle. These generalizations are made either by using contractive conditions or by imposing some additional conditions on the ambient spaces.

On the other hand, a number of generalizations of metric spaces have been done and one of such generalization is partial metric space initiated by Matthews ([15, 16]) in 1992. In partial metric spaces, the distance of a point in the self may not be zero. After the definition of partial metric space, Matthews proved the partial metric version of Banach fixed point theorem. Then, many authors gave some generalizations of the result of Matthews and proved some fixed point theorems on this space (see, e.g. [1–3, 10–13, 17, 19]).

Das and Gupta [6] proved the following fixed point theorem using contractive conditions involving rational expressions.

*Corresponding Author
Email addresses: isayildirim@atauni.edu.tr (I. Yildirim), analsisamirmath2@gmail.com (A. H. Ansari), mohammad@squ.edu.om (M. S. Khan), fbr@leicester.ac.uk (B. Fisher)

Theorem 1.1 ([6]). *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping such that there exists $\alpha, \beta > 0$ with $\alpha + \beta < 1$ satisfying*

$$d(Tx,Ty) \le \alpha d(x,y) + \beta \frac{d(y,Ty)[1 + d(x,Tx)]}{1 + d(x,y)}, \tag{1.1}$$

for all $x, y \in X$. Then T has a unique fixed point.

Recently, Karapinar et al. [11] proved some fixed point theorems for mappings satisfying rational type contractive condition using auxiliary functions.

The main purpose of this paper is to give fixed point theorems for contractions of rational type by using C-class functions in partial metric spaces.

2. Preliminaries

First, we recall some definitions of partial metric spaces and some properties of theirs.

A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p_1) $x = y \Leftrightarrow p(x,x) = p(y,y) = p(x,y)$,
(p_2) $p(x,x) \le p(x,y)$,
(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. It is clear that if $p(x,y) = 0$, then $x = y$. But if $x = y$ then $p(x,y)$ may need not be zero. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open $p-$balls $\left\{B_p(x,\varepsilon) : x \in X,\ \varepsilon > 0\right\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ is given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y), \tag{2.1}$$

is a metric on X.

Let (X,p) be a partial metric space, then we have the following.

(i) A sequence $\{x_n\}$ in a partial metric space (X,p) converges to $x \in X$, if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

(ii) A sequence $\{x_n\}$ in a partial metric space (X,p) is called a Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n,x_m)$ exist and finite.

(iii) A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\} \in X$ converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

(iv) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$, if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0,\delta)) \subset B(f(x_0),\varepsilon)$.

In September 2014, the concept of C-class functions (see Definition 2.1) was introduced by A.H. Ansari in [4] that is pivotal result in fixed point theory, for example see numbers (1), (2), (9) and (15) from Example 2.2. Also see [5] and [9] .

Definition 2.1 ([4]). A mapping $F : [0,\infty)^2 \to \mathbb{R}$ is called *C-class* function if it is continuous and satisfies following axioms:

(1) $F(s,t) \le s$;
(2) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$; for all $s, t \in [0,\infty)$.

Note for some F we have that $F(0,0) = 0$.

We denote C-class functions as C.

Example 2.2 ([4]). The following functions $F : [0,\infty)^2 \to \mathbb{R}$ are elements of C, for all $s, t \in [0,\infty)$:

(1) $F(s,t) = s - t$, $F(s,t) = s \Rightarrow t = 0$;
(2) $F(s,t) = ms$, $0<m<1$, $F(s,t) = s \Rightarrow s = 0$;
(3) $F(s,t) = \frac{s}{(1+t)^r}$; $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$;
(4) $F(s,t) = \log(t + a^s)/(1 + t)$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$;
(5) $F(s,t) = \ln(1 + a^s)/2$, $a > e$, $F(s,1) = s \Rightarrow s = 0$;

(6) $F(s,t) = (s+l)^{(1/(1+t)^r)} - l$, $l > 1$, $r \in (0,\infty)$, $F(s,t) = s \Rightarrow t = 0$;
(7) $F(s,t) = s\log_{t+a} a$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$;
(8) $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$, $F(s,t) = s \Rightarrow t = 0$;
(9) $F(s,t) = s\beta(s)$, $\beta : [0,\infty) \to [0,1)$, and is continuous, $F(s,t) = s \Rightarrow s = 0$;
(10) $F(s,t) = s - \frac{t}{k+t}$, $F(s,t) = s \Rightarrow t = 0$;
(11) $F(s,t) = s - \varphi(s)$, $F(s,t) = s \Rightarrow s = 0$, here $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;
(12) $F(s,t) = sh(s,t)$, $F(s,t) = s \Rightarrow s = 0$, here $h : [0,\infty)\times[0,\infty) \to [0,\infty)$ is a continuous function such that $h(t,s) < 1$ for all $t, s > 0$;
(13) $F(s,t) = s - (\frac{2+t}{1+t})t$, $F(s,t) = s \Rightarrow t = 0$.

(14) $F(s,t) = \sqrt[n]{\ln(1+s^n)}$, $F(s,t) = s \Rightarrow s = 0$.
(15) $F(s,t) = \phi(s)$, $F(s,t) = s \Rightarrow s = 0$, here $\phi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$,
(16) $F(s,t) = \frac{s}{(1+s)^r}$; $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$;

Definition 2.3 ([14]). A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:
(i) ψ is non-decreasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.4 ([4]). An ultra altering distance function is a continuous, nondecreasing mapping $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0$, $t > 0$ and $\varphi(0) \geq 0$.

Remark 2.5. We denote Φ_u set ultra altering distance functions.

We will use the following lemmas of [11] and [16] in order to prove our main results.

Lemma 2.6. *1) A sequence $\{x_n\}$ is Cauchy in a partial metric space (X,p) if and only if $\{x_n\}$ is Cauchy in a metric space (X,p^s).*
2) A partial metric space (X,p) is said to complete if a metric space (X,p^s) is complete. That is, $\lim\limits_{n\to\infty} p^s(x,x_n) = 0 \Leftrightarrow p(x,x) = \lim\limits_{n\to\infty} p(x,x_n) = \lim\limits_{n,m\to\infty} p(x_n,x_m)$.

Lemma 2.7. *Let $x_n \to z$ as $n \to \infty$ in a partial metric space (X,p), where $p(z,z) = 0$. Then $\lim\limits_{n\to\infty} p(x_n,y) = p(z,y)$ for every $y \in X$.*

Lemma 2.8 ([18]). *If $\{x_n\}$ with $\lim_{n\to\infty} p^s(x_{n+1},x_n) = 0$ is not a Cauchy sequence in (X,p), then for each $\varepsilon > 0$ there are two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$, that the following four sequences*

$$p(x_{m(k)},x_{n(k)+1}),\, p(x_{m(k)},x_{n(k)}),\, p(x_{m(k)-1},x_{n(k)+1}),\, p(x_{m(k)-1},x_{n(k)})$$

tend to $\varepsilon > 0$, when $k \to \infty$.

3. Main Results

We start with the following our main result.

Theorem 3.1. *Let (X,p) be a complete partial metric space and $T : X \to X$ be a self map such that there exists a pair of functions $\varphi \in \Psi$, $\phi \in \Phi_u$, $F \in C$ such that*

$$\varphi(p(Tx,Ty)) \leq \max\left\{ \begin{array}{c} F(\varphi(p(x,y)), \phi(p(x,y))), \\ F(\varphi\left(p(y,Ty)\,\frac{1+p(x,Tx)}{1+p(x,y)}\right), \phi\left(p(y,Ty)\,\frac{1+p(x,Tx)}{1+p(x,y)}\right)) \end{array} \right\}$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Consider the sequence $\{x_n\}$ in X defined by

$$x_{n+1} = Tx_n \quad \text{for} \quad n \geq 0. \tag{3.1}$$

From contractive condition, we have

$$\begin{aligned}\varphi(p(x_{n+1},x_n)) &= \varphi(p(Tx_n,Tx_{n-1})) \\ &\le \max\Big\{F(\varphi(p(x_n,x_{n-1})),\phi(p(x_n,x_{n-1}))), \\ &F(\varphi\Big(p(x_{n-1},Tx_{n-1})\frac{1+p(x_n,Tx_n)}{1+p(x_n,x_{n-1})}\Big),\phi\Big(p(x_{n-1},Tx_{n-1})\frac{1+p(x_n,Tx_n)}{1+p(x_n,x_{n-1})}\Big))\Big\} \\ &\le \max\Big\{F(\varphi(p(x_n,x_{n-1})),\phi(p(x_n,x_{n-1})), \\ &F(\varphi\Big(p(x_{n-1},x_n)\frac{1+p(x_n,x_{n+1})}{1+p(x_n,x_{n-1})}\Big),\phi\Big(p(x_{n-1},x_n)\frac{1+p(x_n,x_{n+1})}{1+p(x_n,x_{n-1})}\Big))\Big\}.\end{aligned} \tag{3.2}$$

Now, let us assume that there exists $n_0 \in \mathbb{N}$ such that $p(x_{n_0+1},x_{n_0}) = 0$. In this case, $x_{n_0+1} = x_{n_0}$ and consequently, by (3.1)

$$Tx_{n_0} = x_{n_0+1} = x_{n_0},$$

i.e. x_{n_0} would be the fixed point of T .
Consequently, we can suppose that $p(x_{n+1},x_n) \neq 0$ for all $n \in \mathbb{N}$.
Now we consider the following cases.
Case 1. If

$$\begin{aligned}&\max\Big\{F(\varphi(p(x_n,x_{n-1})),\phi(p(x_n,x_{n-1})), \\ &F(\varphi\Big(p(x_{n-1},x_n)\frac{1+p(x_n,x_{n+1})}{1+p(x_n,x_{n-1})}\Big),\phi\Big(p(x_{n-1},x_n)\frac{1+p(x_n,x_{n+1})}{1+p(x_n,x_{n-1})}\Big))\Big\} \\ &= F(\varphi(p(x_n,x_{n-1})),\phi(p(x_n,x_{n-1})),\end{aligned}$$

From (3.2), we have

$$\varphi(p(x_{n+1},x_n)) \le F(\varphi(p(x_n,x_{n-1})),\phi(p(x_n,x_{n-1})) \le \varphi(p(x_n,x_{n-1})), \tag{3.3}$$

and since $\varphi \in \Psi$, we deduce that $p(x_{n+1},x_n) \le p(x_n,x_{n-1})$.
Case 2. If

$$\begin{aligned}&\max\Big\{F(\varphi(p(x_n,x_{n-1})),\phi(p(x_n,x_{n-1})), \\ &F(\varphi\Big(p(x_{n-1},x_n)\frac{1+p(x_n,x_{n+1})}{1+p(x_n,x_{n-1})}\Big),\phi\Big(p(x_{n-1},x_n)\frac{1+p(x_n,x_{n+1})}{1+p(x_n,x_{n-1})}\Big))\Big\} \\ &= F(\varphi\Big(p(x_{n-1},x_n)\frac{1+p(x_n,x_{n+1})}{1+p(x_n,x_{n-1})}\Big),\phi\Big(p(x_{n-1},x_n)\frac{1+p(x_n,x_{n+1})}{1+p(x_n,x_{n-1})}\Big))\end{aligned}$$

then from (3.2), we get

$$\begin{aligned}\varphi(p(x_{n+1},x_n)) &\le F(\varphi\Big(p(x_{n-1},x_n)\frac{1+p(x_n,x_{n+1})}{1+p(x_n,x_{n-1})}\Big), \\ &\phi\Big(p(x_{n-1},x_n)\frac{1+p(x_n,x_{n+1})}{1+p(x_n,x_{n-1})}\Big)) \\ &\le \varphi\Big(p(x_{n-1},x_n)\frac{1+p(x_n,x_{n+1})}{1+p(x_n,x_{n-1})}\Big)\end{aligned} \tag{3.4}$$

and since $\varphi \in \Psi$, we deduce that

$$p(x_{n+1},x_n) \le p(x_{n-1},x_n)\frac{1+p(x_n,x_{n+1})}{1+p(x_n,x_{n-1}},$$

which implies that $p(x_{n+1},x_n) \le p(x_n,x_{n-1})$.

In any case, we see that $\{p(x_{n+1},x_n)\}$ is a decreasing sequence.

Put $r = \lim_{n\to\infty} p(x_{n+1},x_n)$, where $r \ge 0$, and denote

$$\begin{aligned}A &= \{n \in \mathbb{N} : n \ \text{ satisfies } (3.3)\}, \\ B &= \{n \in \mathbb{N} : n \ \text{ satisfies } (3.4)\}.\end{aligned}$$

Now, we make the following remarks.

(1) If Card $A = \infty$, then from (3.2), we can find infinitely many natural numbers satisfying (3.3) and since

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = \lim_{n\to\infty} p(x_n, x_{n-1}) = r \qquad \text{and} \qquad \varphi(r) \le F(\varphi(r), \phi(r)),$$

so, $\varphi(r) = 0$ or $\phi(r) = 0$, we deduce that $r = 0$.
(2) If Card $B = \infty$, then from (3.2), we can find infinitely many natural numbers satisfying (3.4) and using the same argument one used in Case 2, we obtain

$$\varphi(r) \le F(\varphi(r), \phi(r)),$$

so, $\varphi(r) = 0$ or $\phi(r) = 0$, we deduce that $r = 0$.

Therefore,

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = 0. \tag{3.5}$$

From (2.1), we have

$$p^s(x_{n+1}, x_n) \le 2p(x_{n+1}, x_n).$$

Thus

$$\lim_{n\to\infty} p^s(x_{n+1}, x_n) = 0.$$

Now, we showe that $\{x_n\}$ is a Cauchy sequence in X, i.e. we prove that

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

In the contrary we suppose that the sequence $\{x_n\}$ is not a Cauchy sequence in (X, p), then sequences in Lemma 2.8 tend to $\varepsilon > 0$, when $k \to \infty$.

Now applying the contractive condition, we have

$$\begin{aligned} \varphi(p(x_{m(k)}, x_{n(k)})) &= \varphi(p(Tx_{m(k)-1}, Tx_{n(k)-1})) \\ &\le \max\Big\{F(\varphi(p(x_{m(k)-1}, x_{n(k)-1})), \phi(p(x_{m(k)-1}, x_{n(k)-1}))), \\ &\quad F(\varphi\Big(p(x_{n(k)-1}, Tx_{n(k)-1})\frac{1 + p(x_{m(k)-1}, Tx_{m(k)-1})}{1 + p(x_{m(k)-1}, x_{n(k)-1})}\Big), \\ &\quad \phi\Big(p(x_{n(k)-1}, Tx_{n(k)-1})\frac{1 + p(x_{m(k)-1}, Tx_{m(k)-1})}{1 + p(x_{m(k)-1}, x_{n(k)-1})}\Big))\Big\} \\ &= \max\Big\{F(\varphi(p(x_{m(k)-1}, x_{n(k)-1})), \phi(p(x_{m(k)-1}, x_{n(k)-1}))), \\ &\quad F(\varphi\Big(p(x_{n(k)-1}, x_{n(k)})\frac{1 + p(x_{m(k)-1}, x_{m(k)})}{1 + p(x_{m(k)-1}, x_{n(k)-1})}\Big), \\ &\quad \phi\Big(p(x_{n(k)-1}, x_{n(k)})\frac{1 + p(x_{m(k)-1}, x_{m(k)})}{1 + p(x_{m(k)-1}, x_{n(k)-1})}\Big))\Big\}, \quad \text{for all} \quad k \in \mathbb{N}. \end{aligned} \tag{3.6}$$

Let us put

$$C = \{k \in \mathbb{N} : \varphi(p(x_{m(k)}, x_{n(k)})) \le \phi(p(x_{m(k)-1}, x_{n(k)-1}))\},$$

$$D = \Big\{k \in \mathbb{N} : \varphi(p(x_{m(k)}, x_{n(k)})) \le \phi\Big(p(x_{n(k)-1}, x_{n(k)})\frac{1 + p(x_{m(k)-1}, x_{m(k)})}{1 + p(x_{m(k)-1}, x_{n(k)-1})}\Big)\Big\}.$$

By (3.6), we have Card $C = \infty$ or Card $D = \infty$.

Let us suppose that Card $C = \infty$. Then there exists infinitely many $k \in \mathbb{N}$ such that

$$\varphi(p(x_{m(k)}, x_{n(k)})) \le F(\varphi(p(x_{m(k)-1}, x_{n(k)-1})), \phi(p(x_{m(k)-1}, x_{n(k)-1}))).$$

Taking the limit as $k \to \infty$ in the above inequality, we get

$$\varphi(\varepsilon) \le F(\varphi(\varepsilon), \phi(\varepsilon)),$$

so, $\varphi(\varepsilon) = 0$ or $\phi(\varepsilon) = 0$, which implies that $\varepsilon = 0$, a contradiction, $\varepsilon > 0$.

Let us suppose that Card $D = \infty$. In this case, we can find infinitely many $k \in \mathbb{N}$ such that

$$\begin{aligned}\varphi(p(x_{m(k)}, x_{n(k)})) \quad &\leq \quad F(\varphi\Big(p(x_{n(k)-1}, x_{n(k)})\frac{1 + p(x_{m(k)-1}, x_{m(k)})}{1 + p(x_{m(k)-1}, x_{n(k)-1})}\Big), \\ &\phi\Big(p(x_{n(k)-1}, x_{n(k)})\frac{1 + p(x_{m(k)-1}, x_{m(k)})}{1 + p(x_{m(k)-1}, x_{n(k)-1})}\Big)).\end{aligned}$$

Taking limit as $k \to \infty$ and in view of (3.5), it follows that

$$\varphi(\varepsilon) \leq F(\varphi(0), \phi(0)) \leq \varphi(0) = 0,$$

so, $\varepsilon = 0$ and we get a contradiction.

Therefore, in both possibilities, we obtain a contradiction and so we have $\lim_{n,m\to\infty} p(x_n, x_m) = 0$.

Since $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and finite, we have $\{x_n\}$ is a Cauchy sequence in (X, p).

From (2.1), we know $p^s(x_n, x_m) \leq 2p(x_n, x_m)$. Therefore

$$\lim_{n,m\to\infty} p^s(x_n, x_m) = 0.$$

Thus, by Lemma 2.6, $\{x_n\}$ is a Cauchy sequence in (X, p^s) and (X, p).

Since (X, p) is a complete partial metric space, there exists $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = p(x, x)$. Since $\lim_{n,m\to\infty} p(x_n, x_m) = 0$, then again by Lemma 2.6, we have $p(x, x) = 0$.

Next, we will show that x is a fixed point of T. Suppose that $Tx \neq x$.

From contractive condition and Lemma 2.7, we have

$$\begin{aligned}\varphi(p(Tx, Tx_n)) \quad &\leq \quad \max\Big\{F(\varphi(p(x, x_n)), \phi(p(x, x_n))), \\ &F(\varphi\Big(p(x_n, Tx_n)\frac{1 + p(x, Tx)}{1 + p(x, x_n)}\Big), \phi\Big(p(x_n, Tx_n)\frac{1 + p(x, Tx)}{1 + p(x, x_n)}\Big))\Big\}.\end{aligned}$$

We can distinguish two cases again.

Case **(i)** There exist infinitely many $n \in \mathbb{N}$ such that

$$\varphi(p(Tx, Tx_n)) \leq F(\varphi(p(x, x_n)), \phi(p(x, x_n))).$$

Letting $n \to \infty$, we get

$$\varphi(\lim_{n\to\infty} p(Tx, Tx_n)) \leq F(\varphi(0), \phi(0)) \leq \varphi(0).$$

so,

$$\lim_{n\to\infty} Tx_n = Tx. \tag{3.7}$$

where to simplify our consideration, we will denote the subsequence by the same symbol $\{Tx_n\}$. By (3.1)

$$Tx = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1}. \tag{3.8}$$

Since $x_n \to x$ in X, this means that $\limsup p(x_n, x) \to 0$ and consequently, $\lim_{n\to\infty} x_{n+1} = x$. From this last result and (3.8), we deduce that $Tx = x$, which means that x is a fixed point of T.

Case **(ii)** There exist infinitely many $n \in \mathbb{N}$ such that

$$\varphi(p(Tx, Tx_n)) \leq F(\varphi\Big(p(x_n, Tx_n)\frac{1 + p(x, Tx)}{1 + p(x, x_n)}\Big), \phi\Big(p(x_n, Tx_n)\frac{1 + p(x, Tx)}{1 + p(x, x_n)}\Big)).$$

We will denote the subsequence by the same symbol $\{Tx_n\}$ in order to simplify our consideration. Since $F \in C$ and $Tx_n = x_{n+1}$, we have

$$p(Tx, Tx_n) \leq p(x_n, Tx_n)\frac{1 + p(x, Tx)}{1 + p(x, x_n)} \quad \text{for any} \quad n \in \mathbb{N}.$$

If we take the limit as $n \to \infty$ and by (3.5), $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$. That is, we get (3.7). From the above Case (i), we obtain that x is a fixed point of T.

Therefore, in both the cases we have shown that x is a fixed point of T.

Finally, we will prove the uniqueness of the fixed point. Suppose that y is another fixed point of T such that $x \neq y$.

Now using contractive condition, we get

$$\begin{aligned}\varphi(p(x,y)) &= \varphi(p(Tx,Ty)) \\ &\leq \max\Big\{F(\varphi(p(x,y)),\phi(p(x,y))), \\ &\quad F\Big(\varphi\Big(p(y,Ty)\frac{1+p(x,Tx)}{1+p(x,y)}\Big),\phi\Big(p(y,Ty)\frac{1+p(x,Tx)}{1+p(x,y)}\Big)\Big)\Big\} \\ &\leq \max\{F(\varphi(p(x,y)),\phi(p(x,y))),F(\varphi(0),\phi(0))\}.\end{aligned} \tag{3.9}$$

Now there are two cases.

1. Consider $\max\{F(\varphi(p(x,y)),\phi(p(x,y))),F(\varphi(0),\phi(0))\} = F(\varphi(p(x,y)),\phi(p(x,y)))$. From (3.9), we get

$$\varphi(p(x,y)) \leq F(\varphi(p(x,y)),\phi(p(x,y))).$$

So, $\varphi(p(x,y)) = 0$ or $\phi(p(x,y)) = 0$, which implies that $p(x,y) = 0$, that is, $x = y$.

2. Consider $\max\{F(\varphi(p(x,y)),\phi(p(x,y))),F(\varphi(0),\phi(0))\} = F(\varphi(0),\phi(0))$. Then from (3.9), we get

$$\varphi(p(x,y)) \leq F(\varphi(0),\phi(0)) \leq \varphi(0),$$

which implies that $p(x,y) \leq 0$. Therefore, $p(x,y) = 0$, that is, $x = y$.

Hence in both the cases $x = y$, that is, the fixed point is unique. This completes the proof of the Theorem 3.1. □

From Theorem 3.1, we have the following corollaries.

Corollary 3.2. *Let (X,p) be a complete partial metric space and $T : X \to X$ be a self map such that there exists functions $\varphi \in \Psi, \phi \in \Phi_u, F \in C$ satisfying*

$$\varphi(p(Tx,Ty)) \leq F(\varphi(p(x,y)),\phi(p(x,y))),$$

for all $x,y \in X$. Then T has a unique fixed point in X.

Corollary 3.3. *Let (X,p) be a complete partial metric space and $T : X \to X$ be a self map such that there exists functions $\varphi \in \Psi, \phi \in \Phi_u, F \in C$ satisfying*

$$\varphi(p(Tx,Ty)) \leq F\Big(\varphi\Big(p(y,Ty)\frac{1+p(x,Tx)}{1+p(x,y)}\Big),\phi\Big(p(y,Ty)\frac{1+p(x,Tx)}{1+p(x,y)}\Big)\Big),$$

for all $x,y \in X$. Then T has a unique fixed point in X.

Remark 3.4. The contractive condition (1.1) in Theorem 1.1 in [6] implies that

$$\begin{aligned}d(Tx,Ty) &\leq (\alpha+\beta)\max\Big\{d(x,y),\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}\Big\} \\ &\leq \max\Big\{(\alpha+\beta)d(x,y),(\alpha+\beta)\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}\Big\}\end{aligned}$$

for all $x,y \in X$.

This condition is a particular case of the contractive condition appearing in the Theorem 3.1 with the functions $F(s,t) = (\alpha+\beta)t, \alpha+\beta < 1$ given by $\varphi(t) = t$. That is, Theorem 3.1 is considered as an extension and generalization of Theorem 1.1 in complete partial metric spaces.

Example 3.5. Let $F(s,t) = \frac{s}{1+t}$, $X = [0,1]$ and $p(x,y) = \max\{x,y\}$, then it is clear that (X,p) is a complete partial metric space. Assume $T : X \to X$ such that $Tx = \frac{x}{7}$ for all $x \in X$. Let $\varphi,\phi : [0,\infty) \to [0,\infty)$ as follows:

$$\varphi(x) = \ln\Big(\frac{1}{12}+\frac{7x}{12}\Big), \quad \text{and} \quad \phi(x) = \frac{\ln\left(\frac{1}{12}+\frac{7x}{12}\right)-\ln\left(\frac{1}{12}+\frac{3x}{12}\right)}{\ln\left(\frac{1}{12}+\frac{3x}{12}\right)}, \quad \text{for all} \quad x \in [0,\infty).$$

Without loss of generality, suppose that $x \geq y$. Then we have

$$\begin{aligned}\varphi(p(Tx, Ty)) &= \ln\Big(\frac{1}{12} + \frac{7}{12} p(Tx, Ty)\Big)\\ &= \ln\Big(\frac{1}{12} + \frac{1}{12}\max\{x, y\}\Big)\\ &= \ln\Big(\frac{1}{12} + \frac{1}{12}x\Big).\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{\varphi(p(x, y))}{1 + \phi(p(x, y))} &= \ln\Big(\frac{1}{12} + \frac{3}{12} p(x, y)\Big)\\ &= \ln\Big(\frac{1}{12} + \frac{3}{12}\max\{x, y\}\Big)\\ &= \ln\Big(\frac{1}{12} + \frac{3}{12}x\Big)\end{aligned}$$

and

$$\begin{aligned}\frac{\varphi\Big(p(y, Ty)\frac{1+p(x,Tx)}{1+p(x,y)}\Big)}{1 + \phi\Big(p(y, Ty)\frac{1+p(x,Tx)}{1+p(x,y)}\Big)} &= \ln\Big(\frac{1}{12} + \frac{3}{12}\Big(p(y, Ty)\frac{1 + p(x, Tx)}{1 + p(x, y)}\Big)\Big)\\ &= \ln\Big(\frac{1}{12} + \frac{3}{12}\Big(y\frac{1 + x}{1 + x}\Big)\Big)\\ &= \ln\Big(\frac{1}{12} + \frac{3}{12}y\Big).\end{aligned}$$

So,

$$\begin{aligned}\max\Big\{\frac{\varphi(p(x, y))}{1 + \phi(p(x, y))}, \frac{\varphi\Big(p(y, Ty)\frac{1+p(x,Tx)}{1+p(x,y)}\Big)}{1 + \phi\Big(p(y, Ty)\frac{1+p(x,Tx)}{1+p(x,y)}\Big)}\Big\} &= \max\Big\{\ln\Big(\frac{1}{12} + \frac{3}{12}x\Big), \ln\Big(\frac{1}{12} + \frac{3}{12}y\Big)\Big\}\\ &= \ln\Big(\frac{1}{12} + \frac{3}{12}x\Big).\end{aligned}$$

From the observations above, we obtain

$$\begin{aligned}\varphi(p(Tx, Ty)) &= \ln\Big(\frac{1}{12} + \frac{1}{12}x\Big)\\ &\leq \ln\Big(\frac{1}{12} + \frac{3}{12}x\Big)\\ &= \max\Big\{\frac{\varphi(p(x, y))}{1 + \phi(p(x, y))}, \frac{\varphi\Big(p(y, Ty)\frac{1+p(x,Tx)}{1+p(x,y)}\Big)}{1 + \phi\Big(p(y, Ty)\frac{1+p(x,Tx)}{1+p(x,y)}\Big)}\Big\}.\end{aligned}$$

This shows that the conditions of Theorem (3.1) are satisfied. Hence T has a unique fixed point, that is, $x = 0$ is the required fixed point.

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