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# On arithmetical properties of degenerate Cauchy polynomials and numbers

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## Abstract

In this paper, we study the arithmetical properties of degenerate Cauchy polynomials and numbers. We present some basic properties and recurrence relations, determine all the coefficients of degenerate Cauchy polynomials, and give some convolution identities. These results are particularly useful to deduce interesting and new divisibility properties for degenerate Cauchy polynomials and numbers.

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## 1. Introduction

The Cauchy numbers, defined firstly by Comtet [7], occur in many contexts like combinatorics, mathematical analysis, and number theory. They share a particular relationship with the Stirling numbers of the first kind, so they are related to combinatorics. The Laplace summation formula, which is an analogue of the Euler-Maclaurin summation formula that involves the Cauchy numbers and the difference operator instead of the Bernoulli numbers and differentiation, is used to approximate integrals, so the Cauchy numbers are related to mathematical analysis as well. The Cauchy numbers further appear in connection with the Bernoulli numbers of both kinds, the Nörlund numbers, the Euler-Mascheroni constant, and harmonic numbers, and hence they have been subjected to number theoretic studies.

Their appearance in such different and wide areas has led to the detailed investigations of Cauchy numbers (see, for example, [21]). Some of those studies cover several generalizations (c.f. [6, 15, 17–20, 23]). Their extensions in terms of degenerate number sequences were first studied by Carlitz [3, 4]. He defined the degenerate Cauchy polynomials $c_n(x|\lambda)$ by means of the generating function

$$\frac{\lambda t}{(1+t)^{\lambda}-1}(1+t)^{\lambda-x} = \sum_{n=0}^{\infty} c_n(x|\lambda)\frac{t^n}{n!}. \tag{1.1}$$

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When $x=\lambda\to 0$, we have $c_n=\lim_{\lambda\to 0} c_n(\lambda|\lambda)$ are the classical Cauchy numbers, defined by [7, 21]

$$\frac{t}{\log(1+t)}=\sum_{n=0}^{\infty} c_n\frac{t^n}{n!}. \tag{1.2}$$

The polynomial in (1.1) appears together with another polynomial $\beta_n(x|\lambda)$ defined also by Carlitz [4] (also see [5] and [24])

$$\frac{t}{(1+\lambda t)^{1/\lambda}-1}(1+\lambda t)^{x/\lambda}=\sum_{n=0}^{\infty}\beta_n(x|\lambda)\frac{t^n}{n!}. \tag{1.3}$$

When $\lambda\to 0$, we have $B_n(x)=\lim_{\lambda\to 0}\beta_n(x|\lambda)$ are the classical Bernoulli polynomials (see Section 2). $\beta_n(x|\lambda)$ are then called as the degenerate Bernoulli polynomials. Extensions of the degenerate Cauchy and Bernoulli polynomials and relationships with other number sequences are initially studied by Carlitz [4] and Howard [9].

It seems that the type (1.1) of the degenerate Cauchy polynomials has not been well studied but forgotten. Our goal in the present paper is to study arithmetical properties of the degenerate Cauchy polynomials and numbers. Particularly, we obtain some congruences, which, according to the authors' knowledge, seem to be new and reveal several interesting divisibility properties.

## 2. Degenerate Cauchy polynomials

We start with some basic properties of the degenerate Cauchy polynomials.

**Theorem 2.1.** We have

$$c_n(x|\lambda)=\sum_{m=0}^{n}\binom{n}{m}(-x+\lambda)_{n-m}c_m(\lambda),$$

where $(x)_n=x(x-1)\cdots(x-n+1)$ is the falling factorial for $n\ge 1$ with $(x)_0=1$.

***Proof.*** From (1.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}c_n(x|\lambda)\frac{t^n}{n!}&=\frac{\lambda t}{(1+t)^{\lambda}-1}(1+t)^{-x+\lambda}=\sum_{n=0}^{\infty}c_n(\lambda)\frac{t^n}{n!}\sum_{j=0}^{\infty}\binom{-x+\lambda}{j}t^j\\&=\sum_{n=0}^{\infty}c_n(\lambda)\frac{t^n}{n!}\sum_{j=0}^{\infty}(-x+\lambda)_j\frac{t^j}{j!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}(-x+\lambda)_{n-m}c_m(\lambda)\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $t^n/n!$ on both sides gives the result. □

The next result furnishes a formula for the derivative of the degenerate Cauchy polynomial with respect to its argument.

**Theorem 2.2.** For $n\ge 1$, we have

$$\frac{d}{dx}c_n(x|\lambda)=\sum_{m=0}^{n-1}(-1)^{n-m}\binom{n}{m}c_m(x|\lambda)(n-m-1)!.$$

***Proof.*** We have

$$\sum_{n=0}^{\infty}\frac{d}{dx}c_n(x|\lambda)\frac{t^n}{n!}=\frac{d}{dx}\frac{\lambda t}{((1+t)^\lambda-1)(1+t)^{x-\lambda}}$$
$$=\frac{\lambda t}{((1+t)^\lambda-1)(1+t)^{x-\lambda}}(-\log(1+t))$$
$$=\sum_{n=0}^{\infty}c_n(x|\lambda)\frac{t^n}{n!}\sum_{j=1}^{\infty}(-1)^j\frac{t^j}{j}=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n-1}\frac{c_m(x|\lambda)}{m!}\frac{(-1)^{n-m}}{n-m}\right)t^n\,.$$

Comparing the coefficients of both sides, we get the desired result. □

A difference formula can be stated as follows.

**Theorem 2.3.** For $n\ge 1$, we have

$$c_n(x+\lambda|\lambda)-c_n(x|\lambda)=\lambda(-1)^n n(x)^{(n-1)}\,,$$

where $(x)^{(n)}=x(x+1)\cdots(x+n-1)$ for $n\ge 1$ is the rising factorial with $(x)^{(0)}=1$.

***Proof.*** We have

$$\sum_{n=0}^{\infty}(c_n(x+\lambda|\lambda)-c_n(x|\lambda))\frac{t^n}{n!}=\frac{\lambda t}{((1+t)^\lambda-1)(1+t)^{x-\lambda}}\left(\frac{1}{(1+t)^\lambda}-1\right)$$
$$=-\lambda t\frac{1}{(1+t)^x}=-\lambda t\sum_{n=0}^{\infty}\binom{-x}{n}t^n$$
$$=-\lambda\sum_{n=0}^{\infty}(-x)_n\frac{t^{n+1}}{n!}=\lambda\sum_{n=1}^{\infty}(-1)^n n(x)^{(n-1)}\frac{t^n}{n!}\,.$$

Comparing the coefficients gives the result. □

Replacing $x$ by $x+\lambda$, $x+2\lambda$, $\dots$, $x+(k-1)\lambda$ in Theorem 2.3 and adding the resulting equations, we reach at a more general result that resembles sums of powers of integers and their generalizations (c.f. [14] and [16]).

**Corollary 2.4.** For $n\ge 0$ and $k\ge 1$, we have

$$\frac{c_{n+1}(x+k\lambda|\lambda)-c_{n+1}(x|\lambda)}{n+1}=(-1)^n\lambda\sum_{j=0}^{k-1}(x+j\lambda)^{(n)}\,.$$

The next result points out a multiplication theorem on $\lambda$.

**Theorem 2.5.** For $n\ge 0$ and $k\ge 1$, we have

$$kc_n\left(x|\lambda\right)=\sum_{m=0}^{k-1}c_n\left(x+\lambda m|k\lambda\right)\,.$$

***Proof.*** We have

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{k-1}c_n\left(x+\frac{\lambda m}{k}|\lambda\right)\right)\frac{t^n}{n!}&=\sum_{m=0}^{k-1}\left(\sum_{n=0}^{\infty}c_n\left(x+\frac{\lambda m}{k}|\lambda\right)\frac{t^n}{n!}\right)\\&=\sum_{m=0}^{k-1}\frac{\lambda t}{((1+t)^{\lambda}-1)(1+t)^{x-\lambda+\lambda m/k}}\\&=\frac{\lambda t}{((1+t)^{\lambda}-1)(1+t)^{x-\lambda}}\sum_{m=0}^{k-1}((1+t)^{-\lambda/k})^m\\&=\frac{\lambda t}{((1+t)^{\lambda}-1)(1+t)^{x-\lambda}}\frac{1-(1+t)^{-\lambda}}{1-(1+t)^{-\lambda/k}}\\&=k\frac{(\lambda/k)t}{((1+t)^{\lambda/k}-1)(1+t)^{x-\lambda/k}}=k\sum_{n=0}^{\infty}c_n\left(x|\frac{\lambda}{k}\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients and replacing $\lambda$ by $k\lambda$ give the result. □

Certain numbers and polynomials are found to be of special importance in connection with the study of the degenerate Cauchy polynomials. The Bernoulli polynomials $B_n(x)$ are monic polynomials defined by means of the generating function (c.f. [2])

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}.$$

When $x=0$, $B_n(0)=B_n$ are the Bernoulli numbers, which are rational numbers with $B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, $B_3=0$, and $B_{2k+1}=0$ for integers $k\geq 1$. The Stirling numbers of the second kind $S(n,m)$ are defined by means of the generating function (c.f. [7])

$$\frac{(e^t-1)^m}{m!}=\sum_{n=m}^{\infty}S(n,m)\frac{t^n}{n!}.$$

**Theorem 2.6.** For $n\geq 1$, we have

$$\sum_{m=0}^{n}c_m(x|\lambda)S(n,m)=\sum_{m=0}^{n}\binom{n}{m}\frac{\lambda^{n-m}}{m+1}B_{n-m}\left(1-\frac{x}{\lambda}\right).$$

***Proof.*** Replacing $t$ by $e^{t\lambda}-1$ in (1.1), the right-hand side turns into

$$\begin{aligned}\frac{\lambda(e^{t/\lambda}-1)}{e^{t(x/\lambda-1)}(e^t-1)}&=\frac{e^{t(1-x/\lambda)}}{e^t-1}\lambda(e^{t/\lambda}-1)=\frac{1}{t}\frac{te^{t(1-x/\lambda)}}{e^t-1}\lambda\sum_{j=1}^{\infty}\frac{t^j}{\lambda^j j!}\\&=\frac{1}{t}\left(\sum_{n=0}^{\infty}B_n\left(1-\frac{x}{\lambda}\right)\frac{t^n}{n!}\right)t\sum_{j=0}^{\infty}\frac{1}{\lambda^j(j+1)}\frac{t^j}{j!}\\&=\sum_{n=0}^{\infty}B_n\left(1-\frac{x}{\lambda}\right)\frac{t^n}{n!}\sum_{j=0}^{\infty}\frac{1}{\lambda^j(j+1)}\frac{t^j}{j!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\frac{1}{\lambda^m(m+1)}B_{n-m}\left(1-\frac{x}{\lambda}\right)\right)\frac{t^n}{n!}.\end{aligned}$$

On the other hand, the left-hand side is

$$\sum_{n=0}^{\infty} c_n(x|\lambda)\frac{(e^{t/\lambda}-1)^m}{m!} = \sum_{n=0}^{\infty} c_n(x|\lambda)\sum_{n=m}^{\infty} S(n,m)\frac{t^n}{\lambda^n n!} = \sum_{n=0}^{\infty}\left(c_m(x|\lambda)S(n,m)\lambda^{-n}\right)\frac{t^n}{n!},$$

so comparing the coefficients on both sides yields to the desired result. □

We round out the picture by introducing an expression between the degenerate Cauchy polynomials and degenerate Nörlund numbers $\beta_n^{(n)}(\lambda)$, which are generalizations of Nörlund's numbers (see [12] and [22]), defined by Howard [10] as

$$\frac{\lambda t}{((1+t)^{\lambda}-1)(1+t)^{1-\lambda}} = \sum_{n=0}^{\infty}\beta_n^{(n)}(\lambda)\frac{t^n}{n!}.$$

**Theorem 2.7.** For $n \ge 1$, we have

$$c_n(x|\lambda) = \sum_{m=0}^{n}\binom{n}{m}\beta_m^{(m)}(\lambda)(-x+1)_{n-m}.$$

***Proof.*** From (1.1), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} c_n(x|\lambda)\frac{t^n}{n!} &= \frac{\lambda t}{((1+t)^{\lambda}-1)(1+t)^{1-\lambda}}(1+t)^{-x+1}\\ &= \sum_{n=0}^{\infty}\beta_n^{(n)}(\lambda)\frac{t^n}{n!}\sum_{j=0}^{\infty}(-x+1)_j\frac{t^j}{j!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\beta_m^{(m)}(\lambda)(-x+1)_{n-m}\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $t^n/n!$ on both sides yields the result. □

## 3. Coefficients of degenerate Cauchy polynomials

Howard [13, Theorem 3.1] determined all the coefficients of the degenerate Bernoulli numbers $\beta_n(\lambda) := \beta(0|\lambda)$ as

$$\beta_n(\lambda) = c_n\lambda^n + \sum_{k=1}^{n}\frac{n}{k}(-1)^{n-k}s\,(n-1,k-1)\,B_k\lambda^{n-k} \quad (n \ge 1), \tag{3.1}$$

where $s\,(n,k)$ is the Stirling number of the first kind defined by means of the generating function (c.f. [7])

$$\frac{(\log(1+t))^k}{k!} = \sum_{n=k}^{\infty}(-1)^{n-k}\,s\,(n,k)\,\frac{t^n}{n!}. \tag{3.2}$$

In particular, the leading coefficient in (3.1) is the classical Cauchy numbers $c_n$ and the constant term is the classical Bernoulli numbers $B_n$.

In [19, Theorem 1] the coefficients of $c_n(\lambda)$ are also determined. In fact, the coefficients appear in the reverse order of those of $\beta_n(\lambda)$ as

$$c_n(\lambda) = c_n + \sum_{k=1}^{n}(-1)^{n-k}\frac{n}{k}s\,(n-1,k-1)\,B_k\lambda^k. \tag{3.3}$$

We can determine the coefficients of $c_n(x|\lambda)$ as the polynomial of $\lambda$.

**Theorem 3.1.** For $n \geq 1$, we have

$$c_n(x|\lambda) = c_n + \sum_{k=1}^{n} (-1)^{n-k} \frac{n}{k} s(n-1,k-1) \lambda^k B_k \left(1 - \frac{x}{\lambda}\right) \tag{3.4}$$
$$= c_n + \sum_{k=1}^{n} (-1)^{n} \frac{n}{k} s(n-1,k-1) x^k$$
$$+ \sum_{j=1}^{n} \sum_{k=j}^{n} \sum_{\ell=0}^{j} (-1)^{n-j} \frac{n}{k} s(n-1,k-1) \binom{k}{j} \binom{j}{\ell} B_\ell x^{k-j} \lambda^j .$$

***Proof.*** Let $f = \lambda \log(1+t)$. Then $(1+t)^\lambda = e^f$ and

$$\frac{f^k}{k!} = \lambda^k \sum_{n=k}^{\infty} (-1)^{n-k} s(n,k) \frac{t^n}{n!}$$

by (3.2). Thus

$$\sum_{n=0}^{\infty} c_n(x|\lambda) = \frac{\lambda t}{f} \frac{f}{e^f - 1} e^{f(1-\frac{x}{\lambda})} = \frac{\lambda t}{f} + \frac{\lambda t}{f} \sum_{k=1}^{\infty} B_k \left(1 - \frac{x}{\lambda}\right) \frac{f^k}{k!}$$
$$= \frac{t}{\log(1+t)} + \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{n}{k} B_k \left(1 - \frac{x}{\lambda}\right) \lambda^k (-1)^{n-k} s(n-1,k-1) \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} c_n \frac{t^n}{n!} + \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{n}{k} B_k \left(1 - \frac{x}{\lambda}\right) \lambda^k (-1)^{n-k} s(n-1,k-1) \frac{t^n}{n!},$$

where in the last line we use (1.2). Comparing the coefficients on both sides, we get (3.4). Now, using the expression for Bernoulli polynomials in terms of Bernoulli numbers

$$B_n(x) = \sum_{l=0}^{k} \binom{k}{l} B_l x^{k-l}$$

and the binomial expansion, we write (3.4) as

$$c_n(x|\lambda) = c_n + \sum_{k=1}^{n} (-1)^{n-k} \frac{n}{k} s(n-1,k-1) \sum_{\ell=0}^{k} \binom{k}{\ell} B_\ell \sum_{m=0}^{k-\ell} \binom{k-\ell}{m} (-x)^m \lambda^{k-m}.$$

After separating the terms of $j = \ell = 0$ and changing the order of summations with $k - m = j$, we have

$$\sum_{k=1}^{n} (-1)^{n-k} \frac{n}{k} s(n-1,k-1) \sum_{\ell=0}^{k} \binom{k}{\ell} B_\ell \sum_{m=0}^{k-\ell} \binom{k-\ell}{m} (-x)^m \lambda^{k-m}$$
$$= \sum_{k=1}^{n} (-1)^{n} \frac{n}{k} s(n-1,k-1) x^k$$
$$+ \sum_{j=1}^{n} \sum_{k=j}^{n} \sum_{\ell=0}^{j} (-1)^{n-j} \frac{n}{k} s(n-1,k-1) \binom{k}{\ell} \binom{k-\ell}{k-j} B_\ell x^{k-j} \lambda^j$$
$$= \sum_{k=1}^{n} (-1)^{n} \frac{n}{k} s(n-1,k-1) x^k$$
$$+ \sum_{j=1}^{n} \sum_{k=j}^{n} \sum_{\ell=0}^{j} (-1)^{n-j} \frac{n}{k} s(n-1,k-1) \binom{k}{j} \binom{j}{\ell} B_\ell x^{k-j} \lambda^j,$$

which completes the proof. □

*Remark.* When $x = \lambda$ in Theorem 3.1, we get the result in (3.3). The first several cases are given as follows.

$$
\begin{aligned}
c_0(x|\lambda) &= 1\,,\\
c_1(x|\lambda) &= \frac{1-2x}{2} + \frac{\lambda}{2}\,,\\
c_2(x|\lambda) &= \frac{-1+6x^2}{6} - x\lambda + \frac{\lambda^2}{6}\,,\\
c_3(x|\lambda) &= \frac{1-6x^2-4x^3}{4} + \frac{3x(1+x)}{2}\lambda - \frac{1+2x}{4}\lambda^2\,,\\
c_4(x|\lambda) &= \frac{-19+120x^2+120x^3+30x^4}{30} - 2x(2+3x+x^2)\lambda + \frac{2+6x+3x^2}{3}\lambda^2 - \frac{\lambda^4}{30}\,,\\
c_5(x|\lambda) &= \frac{27-180x^2-220x^3-90x^4-12x^5}{12} + \frac{5x(6+11x+6x^2+x^3)}{2}\lambda\\
&\quad + \frac{5(3+11x+9x^2+2x^3)}{6}\lambda^2 + \frac{3+2x}{12}\lambda^4\,.
\end{aligned}
$$

As a generalization of (3.1), we can also determine the coefficients of $\beta_n\left(x|\lambda\right)$.

**Theorem 3.2.** For $n \ge 1$, we have

$$\beta_n\left(x|\lambda\right) = c_n\lambda^n + \sum_{k=1}^{n}(-1)^{n-k}\frac{n}{k}s\left(n-1,k-1\right)B_k\left(x\right)\lambda^{n-k}\,.$$

***Proof.*** Let $f = \log(1+\lambda t)/\lambda$. Then $1+\lambda t = e^{f\lambda}$ and

$$\frac{f^k}{k!} = \sum_{n=k}^{\infty}(-1)^{n-k}s\left(n,k\right)\lambda^{n-k}\frac{t^n}{n!}$$

by (3.2). Thus

$$
\begin{aligned}
\sum_{n=0}^{\infty}\beta_n\left(x|\lambda\right) &= \frac{t\left(1+\lambda t\right)^{x/\lambda}}{(1+\lambda t)^{1/\lambda}-1} = \frac{t}{f} + t\sum_{k=1}^{\infty}B_k\left(x\right)\frac{f^{k-1}}{k!}\\
&= \frac{\lambda t}{\log(1+\lambda t)} + \sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\frac{n}{k}(-1)^{n-k}s\left(n-1,k-1\right)B_k\left(x\right)\lambda^{n-k}\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}c_n\lambda^n\frac{t^n}{n!} + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{n}{k}(-1)^{n-k}s\left(n-1,k-1\right)B_k\left(x\right)\lambda^{n-k}\frac{t^n}{n!}\,.
\end{aligned}
$$

Comparing the coefficients, the result follows. □

## 4. Convolutions

In this section we introduce some convolution identities for the degenerate Cauchy numbers. We start by noting the identity

$$\frac{1}{X^u-1}\frac{1}{X^v-1} = \frac{1}{X^{u+v}-1}\left(1+\frac{1}{X^u-1}+\frac{1}{X^v-1}\right) \quad (uv(u+v)\ne 0) \tag{4.1}$$

due to Agoh [1, Theorem 1 (a)].

Put $X = (1+t)^\lambda$ in (4.1). Then,

$$\frac{1}{(1+t)^{\lambda u}-1}\frac{1}{(1+t)^{\lambda v}-1} = \frac{1}{(1+t)^{\lambda(u+v)}-1}\left(1+\frac{1}{(1+t)^{\lambda u}-1}+\frac{1}{(1+t)^{\lambda v}-1}\right)\,.$$

Multiplying both sides by $uv\lambda^2t^2$ and expanding, we have

$$\frac{u\lambda t}{(1+t)^{\lambda u}-1}\frac{v\lambda t}{(1+t)^{\lambda v}-1}=\frac{uv\lambda t}{u+v}\frac{\lambda(u+v)t}{(1+t)^{\lambda(u+v)}-1}+\frac{v}{u+v}\frac{\lambda ut}{(1+t)^{\lambda u}-1}\frac{\lambda(u+v)t}{(1+t)^{\lambda(u+v)}-1}+\frac{u}{u+v}\frac{\lambda vt}{(1+t)^{\lambda v}-1}\frac{\lambda(u+v)t}{(1+t)^{\lambda(u+v)}-1}\,.$$

Hence,

$$\begin{aligned}&\left(\sum_{n=0}^{\infty}c_n(\lambda u)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}c_n(\lambda v)\frac{t^n}{n!}\right)\\&=\frac{uv}{u+v}\lambda t\left(\sum_{n=0}^{\infty}c_n(\lambda(u+v))\frac{t^n}{n!}\right)+\frac{v}{u+v}\left(\sum_{n=0}^{\infty}c_n(\lambda u)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}c_n(\lambda(u+v))\frac{t^n}{n!}\right)\\&+\frac{u}{u+v}\left(\sum_{n=0}^{\infty}c_n(\lambda v)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}c_n(\lambda(u+v))\frac{t^n}{n!}\right)\,.\end{aligned}\tag{4.2}$$

The right-hand side of (4.2) is equal to

$$\frac{uv}{u+v}\lambda\sum_{n=0}^{\infty}nc_{n-1}(\lambda(u+v))\frac{t^n}{n!}+\frac{v}{u+v}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}c_m(\lambda u)c_{n-m}(\lambda(u+v))\right)\frac{t^n}{n!}+\frac{u}{u+v}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}c_m(\lambda v)c_{n-m}(\lambda(u+v))\right)\frac{t^n}{n!}\,.$$

On the other hand, the left-hand side of (4.2) is

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}c_m(\lambda u)c_{n-m}(\lambda v)\right)\frac{t^n}{n!}\,.$$

Comparing the coefficients of $t^n/n!$ on both sides, we obtain the following convolution identity.

**Theorem 4.1.** For $n\ge 0$,

$$\sum_{m=0}^{n}\binom{n}{m}c_m(\lambda u)c_{n-m}(\lambda v)=\frac{v}{u+v}\sum_{m=0}^{n}\binom{n}{m}c_m(\lambda u)c_{n-m}(\lambda(u+v))+\frac{u}{u+v}\sum_{m=0}^{n}\binom{n}{m}c_m(\lambda v)c_{n-m}(\lambda(u+v))+\frac{uv}{u+v}\lambda nc_{n-1}(\lambda(u+v))\,.$$

Some special cases of Theorem 4.1 are of particular interest. For instance, when $u=v=1$, we obtain

$$\sum_{m=0}^{n}\binom{n}{m}c_m(\lambda)c_{n-m}(\lambda)-\sum_{m=0}^{n}\binom{n}{m}c_m(\lambda)c_{n-m}(2\lambda)=\frac{\lambda n}{2}c_{n-1}(2\lambda)\,.$$

We next consider an Euler-type convolution formula for the degenerate Cauchy numbers. For such an identity we utilize the higher-order generalization of the degenerate Cauchy polynomials defined by Howard [9] as

$$\left(\frac{\lambda t}{(1+t)^{\lambda}-1}\right)^k(1+t)^{\lambda-x}=\sum_{n=0}^{\infty}c_n^{(k)}(x|\lambda)\frac{t^n}{n!}\,.\tag{4.3}$$

If we differentiate both sides of (4.3) with respect to $t$ and compare the coefficients of $t^n$, we find that

$$kc_n^{(k+1)}(x|\lambda) = (k-n)\,c_n^{(k)}(x|\lambda) + n\,(k - k\lambda - n + 1 - \lambda - x)\,c_{n-1}^{(k)}(x|\lambda)\,.$$

In particular, letting $k=1$ and replacing $x$ by $x+y$, we have

$$c_n^{(2)}(x+y|\lambda) = (1-n)\,c_n(x+y|\lambda) + n\,(2-n-x-y)\,c_{n-1}(x+y|\lambda)\,. \qquad (4.4)$$

On the other hand, (4.3) gives

$$\begin{aligned}\sum_{n=0}^{\infty} c_n^{(2)}(x+y|\lambda)\frac{t^n}{n!} &= (1+t)^{-x-\frac{\lambda}{2}+\lambda}\frac{\lambda t}{(1+t)^{\lambda}-1}(1+t)^{-y-\frac{\lambda}{2}+\lambda}\frac{\lambda t}{(1+t)^{\lambda}-1}\\ &= \left(\sum_{n=0}^{\infty} c_n\left(x+\frac{\lambda}{2}|\lambda\right)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} c_n\left(y+\frac{\lambda}{2}|\lambda\right)\frac{t^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}c_m\left(x+\frac{\lambda}{2}|\lambda\right)c_{n-m}\left(y+\frac{\lambda}{2}|\lambda\right)\right)\frac{t^n}{n!}\,,\end{aligned}$$

from which we conclude that

$$c_n^{(2)}(x+y|\lambda) = \sum_{m=0}^{n}\binom{n}{m}c_m\left(x+\frac{\lambda}{2}|\lambda\right)c_{n-m}\left(y+\frac{\lambda}{2}|\lambda\right). \qquad (4.5)$$

Equating (4.4) and (4.5), we arrive at the following.

**Theorem 4.2.** We have

$$\begin{aligned}&\sum_{m=0}^{n}\binom{n}{m}c_m\left(x+\frac{\lambda}{2}|\lambda\right)c_{n-m}\left(y+\frac{\lambda}{2}|\lambda\right)\\ &\quad = (1-n)\,c_n(x+y|\lambda) + n\,(2-n-x-y)\,c_{n-1}(x+y|\lambda)\,.\end{aligned}$$

If we let $x=y=\frac{\lambda}{2}$, we obtain

$$\sum_{m=0}^{n}\binom{n}{m}c_m(\lambda)\,c_{n-m}(\lambda) = (1-n)\,c_n(\lambda) + n\,(2-n-\lambda)\,c_{n-1}(\lambda)\,,$$

an Euler-type convolution identity for the degenerate Cauchy numbers.

## 5. Congruences

In this section we present several congruences satisfied by the degenerate Cauchy polynomials and numbers modulo an odd prime. Although some congruences for the degenerate poly-Cauchy polynomials and numbers have been established (c.f. [6]), to the authors' knowledge, congruences for the degenerate Cauchy polynomials and numbers have not been investigated. The motivation rises from Theorem 2.6 and Theorem 2.7, where the degenerate Cauchy polynomials are represented in terms of the ordinary Bernoulli polynomials and degenerate Nörlund numbers, respectively.

Throughout we denote an odd prime number by $p$ and assume that $\lambda$ is a rational number $\lambda = a/b$ such that $b$ is not divisible by $p$, that is, $\lambda$ is an integer modulo $p$.

First, we consider congruences for the degenerate Cauchy polynomials when $\lambda \equiv 0 \pmod{p}$.

**Theorem 5.1.** For $\lambda \equiv 0 \pmod{p}$, we have

$$c_p(x|\lambda) \equiv 1 - x + (1-x)_p \pmod{p}$$

and

$$pc_{p-1}(x|\lambda) \equiv 1 \pmod{p}\,.$$

***Proof.*** We set $n=p$ in Theorem 2.7 to obtain

$$\begin{aligned} c_p(x|\lambda) &= \sum_{m=0}^{p} \binom{p}{m} \beta_m^{(m)}(\lambda)(-x+1)_{p-m} \\ &= (1-x)_p + \beta_p^{(p)}(\lambda)(1-x)_0 + p\beta_{p-1}^{(p-1)}(\lambda)(1-x)_1 \\ &\quad + \sum_{m=1}^{p-2} \binom{p}{m} \beta_m^{(m)}(\lambda)(1-x)_{p-m} \\ &= (1-x)_p + \beta_p^{(p)}(\lambda) + p\beta_{p-1}^{(p-1)}(\lambda)(1-x) \\ &\quad + \sum_{m=1}^{p-2} \frac{1}{m}\binom{p-1}{m-1} p\beta_m^{(m)}(\lambda)(1-x)_{p-m} \end{aligned}$$

since $(1-x)_0 = 1$ and $(1-x)_1 = 1-x$. Some congruences for the degenerate Nörlund numbers has been given by Howard in [10]. Particularly, we note that

$$\begin{aligned} p\beta_m^{(m)}(\lambda) &\equiv 0 \pmod p \quad \text{for} \quad m=1,2,\ldots,p-2\,, \\ p\beta_{p-1}^{(p-1)}(\lambda) &\equiv 1 \pmod p\,, \\ \beta_p^{(p)}(\lambda) &\equiv 0 \pmod p\,, \end{aligned}$$

for $\lambda \equiv 0 \pmod p$. Using these results above, we obtain the first congruence.

For the second congruence, we note that

$$\begin{aligned} pc_{p-1}(x|\lambda) &= \sum_{m=0}^{p-1} \binom{p-1}{m} p\beta_m^{(m)}(\lambda)(1-x)_{p-1-m} \\ &= p(1-x)_{p-1} + p\beta_{p-1}^{(p-1)}(\lambda) + \sum_{m=1}^{p-2} \binom{p-1}{m} p\beta_m^{(m)}(\lambda)(1-x)_{p-1-m}\,, \end{aligned}$$

from which the result follows. □

When $x=\lambda$ in Theorem 5.1, we have congruence relations for the degenerate Cauchy numbers when $\lambda \equiv 0 \pmod p$.

**Corollary 5.2.** For $\lambda \equiv 0 \pmod p$, we have

$$c_p(\lambda) \equiv 1 - \lambda + (1-\lambda)_p \pmod p$$

and

$$pc_{p-1}(\lambda) \equiv 1 \pmod p\,.$$

Next, we consider congruences for the degenerate Cauchy numbers when $\lambda \not\equiv 0 \pmod p$. We first set $x=\lambda$ in Theorem 2.6 to obtain

$$\sum_{m=0}^{n} c_m(\lambda) S(n,m) = \sum_{m=0}^{n} \binom{n}{m} \frac{\lambda^{n-m}}{m+1} B_{n-m}\,.$$

The inversion formula for the Stirling numbers, namely,

$$f_n = \sum_{m=0}^{n} (-1)^{n+m} s(n,m)\, g_m \quad \Leftrightarrow \quad g_n = \sum_{m=0}^{n} S(n,m)\, f_m\,,$$

which follows from the orthogonality relations of the Stirling numbers ([8, p.264]), yields to

$$c_n(\lambda) = \sum_{m=0}^{n} (-1)^{n+m} s(n,m) \sum_{k=0}^{m} \binom{m}{k} \frac{\lambda^{m-k}}{k+1} B_{m-k} \tag{5.1}$$

with $f_m = c_m(\lambda)$.

The Stirling numbers of the first kind satisfy the recurrence

$$s(n,m) = (n-1)\,s(n-1,m) + s(n-1,m-1)$$

together with the special values

$$\begin{aligned} s(n,0) &= 0 \quad \text{if} \quad n>0\,,\\ s(n,n) &= 1\,,\\ s(n,1) &= (n-1)! \quad \text{if} \quad n>0\,,\\ s(n,n-1) &= \binom{n}{2} \quad \text{if} \quad n>1\,,\\ s(n,m) &= 0 \quad \text{if} \quad m>n \text{ or } m<0. \end{aligned}$$

We note that (c.f. [11])

$$s(p,m) \equiv 0 \pmod p \quad \text{for} \quad m = 2,3,\dots,p-1 \tag{5.2}$$

and

$$s(p-1,m) \equiv 1 \pmod p \quad \text{for} \quad m = 1,2,\dots,p-1\,. \tag{5.3}$$

Divisibility properties of the Bernoulli numbers are mainly determined by the celebrated von Staudt-Clausen theorem, which can be stated as (see [2])

$$pB_n \equiv \begin{cases} 0 \pmod p, & \text{if } (p-1)\nmid n\,,\\ -1 \pmod p, & \text{if } (p-1)\mid n\,. \end{cases}$$

**Theorem 5.3.** For $\lambda \not\equiv 0 \pmod p$, we have

$$c_p(\lambda) \equiv 0 \pmod p\,.$$

***Proof.*** For $n = p$, (5.1) can be written as

$$\begin{aligned} c_p(\lambda) &= (p-1)!\left(-\frac{\lambda}{2}+\frac{1}{2}\right) + \sum_{k=0}^{p}\binom{p}{k}\frac{\lambda^{p-k}}{k+1}B_{p-k}\\ &+ \sum_{m=2}^{p-1}(-1)^{m+1}s(p,m)\sum_{k=0}^{m}\binom{m}{k}\frac{\lambda^{m-k}}{k+1}B_{m-k} \end{aligned}$$

separating out the terms with $m = 1$ and $m = p$, and using $s(p,1) = (p-1)!$, $s(p,p) = 1$, $B_0 = 1$, and $B_1 = -\frac{1}{2}$. We next separate out the terms with $k = 0$, $k = 1$, $k = p-1$, and $k = p$ in the first summation, and the terms with $m = p-1$ and $k = 0$, $k = p-1$ in the second summation, and note that $B_p = 0$ and $s(p,p-1) = \frac{p(p-1)}{2}$. We then obtain

$$\begin{aligned} c_p(\lambda) &= (p-1)!\frac{1-\lambda}{2} + pB_{p-1}\frac{\lambda^{p-1}}{2} + \frac{1}{p+1} - \frac{\lambda}{2} - (p-1)\,pB_{p-1}\frac{\lambda^{p-1}}{2} - \frac{p-1}{2}\\ &+ \frac{1}{p}\sum_{k=2}^{p-2}\binom{p}{k}\frac{\lambda^{p-k}}{k+1}pB_{p-k} - \frac{p-1}{2}\sum_{k=1}^{p-2}\binom{p-1}{k}\frac{\lambda^{p-1-k}}{k+1}pB_{p-1-k}\\ &+ \frac{1}{p}\sum_{m=2}^{p-2}(-1)^{m+1}s(p,m)\sum_{k=0}^{m}\binom{m}{k}\frac{\lambda^{m-k}}{k+1}pB_{m-k}. \end{aligned}$$

Now, $\binom{p}{k} \equiv 0 \pmod p$ and $pB_{p-k} \equiv 0 \pmod p$ for $k = 2,3,\dots,p-2$, so the first sum vanishes modulo $p$. Similarly, second sum vanishes modulo $p$ since $pB_{p-1-k} \equiv 0 \pmod p$ for $k = 1,2,\dots,p-2$. Finally, by (5.2) and $pB_{m-k} \equiv 0 \pmod p$ for $m = 2,3,\dots,p-2$

and $k=0,1,\ldots,m$, the last sum is zero modulo $p$. Hence

$$\begin{aligned} c_p(\lambda) &\equiv (p-1)!\frac{1-\lambda}{2}+pB_{p-1}\frac{\lambda^{p-1}}{2}+\frac{1}{p+1}-\frac{\lambda}{2}-(p-1)\,pB_{p-1}\frac{\lambda^{p-1}}{2}-\frac{p-1}{2}\\ &\equiv -1+\frac{1}{p+1}=-\frac{p}{p+1}\equiv 0 \pmod p \end{aligned}$$

since $pB_{p-1}\equiv -1 \pmod p$ and $\lambda\not\equiv 0 \pmod p$. □

**Theorem 5.4.** For $\lambda\not\equiv 0 \pmod p$, we have

$$pc_{p-1}(\lambda)\equiv 0 \pmod p\,.$$

***Proof.*** (5.1) and (5.3) give

$$\begin{aligned} c_{p-1}(\lambda) &= \sum_{m=0}^{p-1}(-1)^{p-1+m}\,s(p-1,m)\sum_{k=0}^{m}\binom{m}{k}\frac{\lambda^{m-k}}{k+1}B_{m-k}\\ &\equiv \sum_{m=1}^{p-1}(-1)^{m}\sum_{k=0}^{m}\binom{m}{k}\frac{\lambda^{m-k}}{k+1}B_{m-k} \pmod p. \end{aligned}$$

Arranging the right-hand side as

$$\lambda^{p-1}B_{p-1}+\frac{1}{p}+\sum_{k=1}^{p-2}\binom{p-1}{k}\frac{\lambda^{p-1-k}}{k+1}B_{p-1-k}+\sum_{m=1}^{p-2}(-1)^{m}\sum_{k=0}^{m}\binom{m}{k}\frac{\lambda^{m-k}}{k+1}B_{m-k}\,,$$

we obtain

$$\begin{aligned} pc_{p-1}(\lambda) &\equiv \lambda^{p-1}pB_{p-1}+1+\sum_{k=1}^{p-2}\binom{p-1}{k}\frac{\lambda^{p-1-k}}{k+1}pB_{p-1-k}\\ &\quad+\sum_{m=1}^{p-2}(-1)^{m}\sum_{k=0}^{m}\binom{m}{k}\frac{\lambda^{m-k}}{k+1}pB_{m-k} \pmod p\,, \end{aligned}$$

which yields the result by the von Staudt-Clausen and Fermat's theorems. □

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