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A New Generalization of Szász-Mirakjan Kantorovich Operators for Better Error Estimation

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Abstract

In this article, we construct a new sequence of Szász-Mirakjan-Kantorovich operators denoted as $K_{n,\gamma}(f;x)$, which depending on a parameter γ. We prove direct and local approximation properties of $K_{n,\gamma}(f;x)$. We obtain that, if $\gamma>1$, then the operators $K_{n,\gamma}(f;x)$ provide better approximation results than classical case for all $x\in[0,\infty)$. Furthermore, we investigate the approximation results of $K_{n,\gamma}(f;x)$, graphically and numerically. Moreover, we introduce new operators from $K_{n,\gamma}(f;x)$ that preserve affine functions and bivariate case of $K_{n,\gamma}(f;x)$. Then, we study their approximation properties and also illustrate the convergence of these operators comparing with their classical cases.

1. Introduction

The Weierstrass approximation theorem is a fundamental result in mathematical analysis which states that any continuous function on a closed interval can be uniformly approximated by a polynomial function (see [1]). Bernstein provides a simple and constructive proof to the Weierstrass Approximation Theorem for $C[0,1]$, where $C[0,1]$ is the set of all continuous functions (see [2], [3]). Because of the importance of the Bernstein Operators, many researchers lead to the discovery of their numerous generalizations such as [4], [5], [6], [7], [8], [9], [10]. For a function f belonging to the space $C[0,\infty)$, the Szász-Mirakjan operators are introduced by

$$S_n(f;x)=\sum_{k=0}^{\infty}s_{n,k}(x)f\left(\frac{k}{n}\right), \tag{1.1}$$

where,

$$s_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!}, \tag{1.2}$$

for any $x\in[0,\infty)$, in [11] and [12]. However, this kind of operators do not suitable for discontinuous functions. P. L. Butzer introduced the Kantorovich type Szász-Mirakjan operators for Lebesque-integrable function space, in [13] as:

$$S_n(f;x)=n\sum_{k=0}^{\infty}s_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(t)\,dt, \tag{1.3}$$

where $s_{n,k}(x)$ is defined in (1.2). Szász-Mirakjan operators, Kantorovich type Szász-Mirakjan operators and some of their generalizations have been the subject of extensive research by various scholars as documented [14], [15], [16], [17], [18], [19], [20]. For further developments in this area, interested readers are encouraged to explore the insights provided in [21], [22],

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[23], [24], [25], [26].
In this article, we introduce a new family of Kantorovich type Szász-Mirakjan operators $K_{n,\gamma}$ as:

$$K_{n,\gamma}(f;x) := \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 f\left(\frac{k+t^\gamma}{n}\right) dt, \tag{1.4}$$

where $s_{n,k}(x)$ is given in (1.2) and $\gamma \in \mathbb{R}^+$. Note that, $K_{n,\gamma}$ are positive and linear. One can easily obtain that, in (1.4) the classical Százs-Mirakjan Kantorovich operators can be produced, by choosing $\gamma = 1$. We observe that, the error estimation of $K_{n,\gamma}$ decreases by increasing the value γ. Therefore, in cases where we choose the γ value greater than 1, it can be seen that the error estimation is less than the classical case. Therefore, this modification gives better approximation results than classical one, when $\gamma > 1$. It should be stressed out that this kind of Kantorovich type operators for the Bernstein polynomials was defined and studied in [27] by Özarslan, and Duman.
After giving geometric properties and significant results of $K_{n,\gamma}$ in Section 2, direct and local approximation properties, theoretical proofs, and numerical examples for better error estimations are given for these operators in Section 3. Then, applying slight modification to the operators $K_{n,\gamma}$, a new family of these operators is introduced, that preserves affine functions. In Section 5, bivariate cases of these operators are introduced and studied.

2. Some basic results

In this part, we provide some geometric properties of $K_{n,\gamma}$ and significant results of $K_{n,\gamma}$ which will be used in the next sections.

Theorem 2.1. *Let $0 \le \gamma < \infty$ and $n \in \mathbb{N}$, then,*

1. *If the function f is increasing (or decreasing) on $[0,\infty)$, then $K_{n,\gamma}(f;x)$ is also increasing (or decreasing) on $[0,\infty)$.*
2. *If the function f is convex (or concave) function on $[0,\infty)$, then $K_{n,\gamma}(f;x)$ is also convex (or concave) on $[0,\infty)$.*

Proof. 1. Taking the first derivative of $K_{n,\gamma}(f;x)$ we get,

$$\begin{aligned}
\left(K_{n,\gamma}\right)'(f;x) &= \sum_{k=0}^{\infty} s'_{n,k}(x) \int_0^1 f\left(\frac{k+t^\gamma}{n}\right) dt \\
&= \sum_{k=0}^{\infty} \left[-ne^{-nx}\frac{(nx)^k}{k!} + e^{-nx}\frac{nk(nx)^{k-1}}{k!}\right] \int_0^1 f\left(\frac{k+t^\gamma}{n}\right) dt \\
&= \sum_{k=1}^{\infty} \left[ne^{-nx}\frac{k(nx)^{k-1}}{k!}\right] \int_0^1 f\left(\frac{k+t^\gamma}{n}\right) dt - n\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!} \int_0^1 f\left(\frac{k+t^\gamma}{n}\right) dt \\
&= n\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!} \int_0^1 f\left(\frac{k+1+t^\gamma}{n}\right) dt - n\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!} \int_0^1 f\left(\frac{k+t^\gamma}{n}\right) dt \\
&= n\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!} \left[\int_0^1 f\left(\frac{k+1+t^\gamma}{n}\right) dt - \int_0^1 f\left(\frac{k+t^\gamma}{n}\right) dt\right] \\
&= n\sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 \Delta_h^1 f\left(\frac{k+t^\gamma}{n}\right) dt,
\end{aligned} \tag{2.1}$$

where $h = \frac{1}{n}$ and $n = 1,2,\dots$.
For an increasing function f on $[0,\infty)$, we have

$$\Delta_h^1 f\left(\frac{k+t^\gamma}{n}\right) = f\left(\frac{k+1+t^\gamma}{n}\right) - f\left(\frac{k+t^\gamma}{n}\right) \ge 0, \tag{2.2}$$

where $k = 0,1,\dots$ and $t \in [0,1]$. Therefore, combining (2.1) and (2.2), we obtain

$$\left(K_{n,\gamma}\right)'(f;x) \ge 0 \text{ for each } x \in [0,\infty).$$

In other words, $K_{n,\gamma}(f;x)$ is increasing on $[0,\infty)$.

2. Similarly, the second derivative of $K_{n,\gamma}(f;x)$ is

$$\left(K_{n,\gamma}\right)''(f;x) = n^2 \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 \Delta_h^2 f\left(\frac{k+t^\gamma}{n}\right) dt \tag{2.3}$$

where $h = \frac{1}{n}$ for $n = 1,2,3,\dots$. Let f is convex on $[0,\infty)$, then for any $k = 0,1,\dots$ we have

$$0 \le \frac{k+t^\gamma}{n} \le \frac{k+t^\gamma+1}{n} \le \frac{k+t^\gamma+2}{n} \le 1.$$

Theorem 3.2.2 in [[28], p.59] implies that

$$\Delta_h^2 f\left(\frac{k+t^\gamma}{n}\right) \geq 0. \tag{2.4}$$

Therefore, combining (2.3) and (2.4) we get,

$$\left(K_{n,\gamma}\right)''(f;x) \geq 0,$$

$\forall x \in [0,\infty)$. As a conclusion, $K_{n,\gamma}(f;x)$ is convex on $[0,\infty)$.

□

Lemma 2.2. *Recall the first 3 moments of (1.1)*

1. $S_n(1;x) = 1$
2. $S_n(t;x) = x$
3. $S_n(t^2;x) = x^2 + \frac{x}{n}$.

Lemma 2.3. *Let $\gamma \in (0,\infty)$ and $x \in [0,\infty)$, then*

1. $K_{n,\gamma}(1;x) = 1$
2. $K_{n,\gamma}(t;x) = x + \frac{1}{(\gamma+1)n}$
3. $K_{n,\gamma}(t^2;x) = x^2 + \frac{(\gamma+3)x}{(\gamma+1)n} + \frac{1}{(2\gamma+1)n^2}$.

Proof. For each $\gamma \in (0,\infty)$ and $x \in [0,\infty)$, using Lemma 2.2, we get

1.

$$K_{n,\gamma}(1;x) = \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 dt = 1.$$

2.

$$\begin{aligned}
K_{n,\gamma}(t;x) &= \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 \frac{k+t^\gamma}{n} dt \\
&= \sum_{k=0}^{\infty} \frac{k}{n} s_{n,k}(x) \int_0^1 dt + \frac{1}{n} \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 t^\gamma dt \\
&= \sum_{k=0}^{\infty} s_{n,k}(x) \frac{k}{n} + \frac{1}{\gamma+1} \frac{1}{n} \sum_{k=0}^{\infty} s_{n,k}(x) \\
&= x + \frac{1}{(\gamma+1)n}.
\end{aligned}$$

3.

$$\begin{aligned}
K_{n,\gamma}(t^2;x) &= \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 \left(\frac{k+t^\gamma}{n}\right)^2 dt \\
&= \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 \left(\frac{k^2+2kt^\gamma+t^{2\gamma}}{n^2}\right) dt \\
&= \sum_{k=0}^{\infty} s_{n,k}(x) \frac{k^2}{n^2} \int_0^1 dt + \frac{2}{n} \sum_{k=0}^{\infty} s_{n,k}(x) \frac{k}{n} \int_0^1 t^\gamma dt + \frac{1}{n^2} \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 t^{2\gamma} dt \\
&= \sum_{k=0}^{\infty} s_{n,k}(x) \frac{k^2}{n^2} + \frac{2}{(\gamma+1)n} \sum_{k=0}^{\infty} \frac{k}{n} s_{n,k}(x) + \frac{1}{(2\gamma+1)n^2} \sum_{k=0}^{\infty} s_{n,k}(x) \\
&= x^2 + \frac{(\gamma+3)}{(\gamma+1)n} x + \frac{1}{(2\gamma+1)n^2}.
\end{aligned}$$

□

In the following lemma, we establish the connection between the moments of the operators $K_{n,\gamma}$ and S_n. Consequently, we can compute higher-order moments of $K_{n,\gamma}$ by utilizing classical Szász-Mirakjan operators S_n.

Lemma 2.4. *Consider $n \in \mathbb{N}$, $x \in [0,\infty)$, and $\gamma \in (0,\infty)$, we get*

$$K_{n,\gamma}(t^m;x) = \frac{1}{n^m}\sum_{i=0}^{m}\binom{m}{i}\frac{n^i}{\gamma(m-i)+1}S_n(t^i;x) \tag{2.5}$$

where S_n is defined in (1.1).

Proof. From (1.4), we get

$$\begin{aligned}
K_{n,\gamma}(t^m;x) &= \sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1\left(\frac{k+t^\gamma}{n}\right)^m dt \\
&= \frac{1}{n^m}\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1 (k+t^\gamma)^m\,dt \\
&= \frac{1}{n^m}\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1 \sum_{i=0}^{m}\binom{m}{i}k^i t^{\gamma(m-i)}dt \\
&= \frac{1}{n^m}\sum_{k=0}^{\infty} s_{n,k}(x)\sum_{i=0}^{m}\binom{m}{i}k^i\int_0^1 t^{\gamma(m-i)}dt \\
&= \frac{1}{n^m}\sum_{k=0}^{\infty} s_{n,k}(x)\sum_{i=0}^{m}\binom{m}{i}k^i\frac{1}{\gamma(m-i)+1} \\
&= \frac{1}{n^m}\sum_{i=0}^{m}\binom{m}{i}\frac{n^i}{\gamma(m-i)+1}\sum_{k=0}^{\infty} s_{n,k}(x)\frac{k^i}{n^i} \\
&= \frac{1}{n^m}\sum_{i=0}^{m}\binom{m}{i}\frac{n^i}{\gamma(m-i)+1}S_n(t^i;x).
\end{aligned}$$

□

Corollary 2.5. *We obtain,*

1. $K_{n,\gamma}(t-x;x) = \dfrac{1}{(\gamma+1)n}$
2. $K_{n,\gamma}((t-x)^2;x) = \frac{x}{n} + \dfrac{1}{(2\gamma+1)n^2}$.

3. Direct and local approximation properties of $K_{n,\gamma}$

We now turn our focus to direct and local approximation properties of $K_{n,\gamma}$. To begin, let's remember that $C_B[0,\infty)$ signifies the set of all real-valued functions f on $[0,\infty)$ that are both uniformly bounded and continuous. We measure the norm of such functions using $\|.\|$ defined as:

$$\|f\| = \sup_{x\in[0,\infty)} |f(x)|.$$

Theorem 3.1. *For any $A \in \mathbb{R}^+$, let $f \in C_B[0,A]$, and $\gamma \in (0,\infty)$, then $K_{n,\gamma}(f;x)$ is uniformly convergent to $f(x)$ on $[0,A]$.*

Proof. According to the Bohman-Korovkin Theorem (see [30]), it suffices to establish that

$$\lim_{n\to\infty}\sup_{x\in[0,A]}\left|K_{n,\gamma}(t^i;x)-t^i\right| = 0, \tag{3.1}$$

for $i = 0,1,2$. As a result of Lemma 2.3, one can easily see that (3.1) is hold for $i = 0,1,2$. So, the proof is done. □

For each $A \in \mathbb{R}^+$, the operators $K_{n,\gamma}$ on $C_B[0,A]$ satisfies $K_{n,\gamma}(1;x) = 1$. Therefore, for all $\varepsilon > 0$ we get

$$K_{n,\gamma}(f;x) \le \varepsilon + \frac{2\|f\|}{\delta^2}K_{n,\gamma}((t-x)^2;x)$$

where $x \in [0,A]$. Here δ comes from the uniform continuity of f. Therefore, the order of approximation of $K_{n,\gamma}(f;x)$ to f is much better controlled by the term $K_{n,\gamma}((t-x)^2;x)$.
Let $A \in (0,\infty)$ and $f \in C_B[0,A]$. If we consider $\gamma > 0$ and $x \in [0,A]$ such that

$$K_{n,\gamma}((t-x)^2;x) \leq K_n((t-x)^2;x). \tag{3.2}$$

We can compare how well the operators $K_{n,\gamma}(f;x)$ and $K_n(f;x)$ approximate the function f. From Lemma 2.3 and equation (3.2),

$$\begin{aligned} K_{n,\gamma}((t-x)^2;x) &\leq K_n((t-x)^2;x) \\ \frac{x}{n} + \frac{1}{(2\gamma+1)n^2} &\leq \frac{x}{n} + \frac{1}{3n^2} \\ \frac{1}{(2\gamma+1)n^2} &\leq \frac{1}{3n^2} \\ 1 &\leq \gamma. \end{aligned}$$

Hence, for every $\gamma > 1$, the accuracy of the approximation $K_{n,\gamma}(f;x)$ to $f(x)$ outperforms that of the classical Szász-Mirakjan Kantorovich operators for any $f \in C_B[0,A]$ and $x \in [0,A]$. Additionally, the approximation error of $K_{n,\gamma}(f;x)$ to $f(x)$ diminishes with increasing γ.
Now, we give some graphical and numerical results to illustrate that we have better error estimation by increasing the value γ.

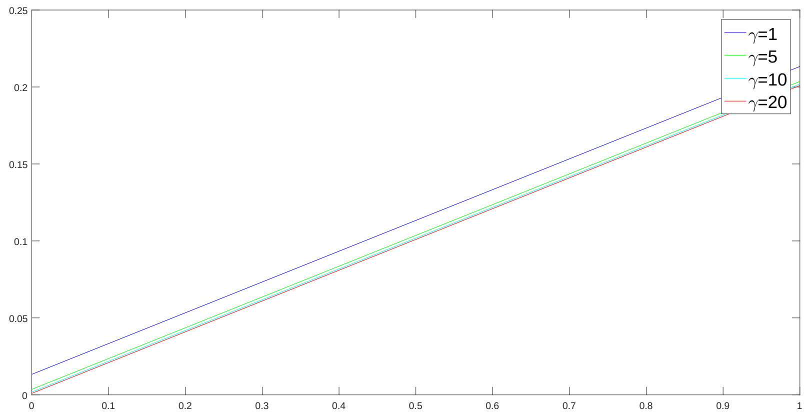

Figure 1: Error of approximation $K_{n,\gamma}((t-x)^2;x)$ for, $\gamma=1$, $\gamma=5$, $\gamma=10$, $\gamma=20$, when $n=5$ and $x \in [0,1]$.

x	$K_{5,1}((t-x)^2;x)$	$K_{5,5}((t-x)^2;x)$	$K_{5,10}((t-x)^2;x)$	$K_{5,20}((t-x)^2;x)$
0.00	0.0133	0.0036	0.0019	0.0010
0.10	0.0333	0.0236	0.0219	0.0210
0.20	0.0533	0.0436	0.0419	0.0410
0.30	0.0733	0.0636	0.0619	0.0610
0.40	0.0733	0.0836	0.0819	0.0810
0.50	0.1133	0.1036	0.1019	0.1010
0.60	0.1333	0.1236	0.1219	0.1210
0.70	0.1533	0.1436	0.1419	0.1410
0.80	0.1733	0.1636	0.1619	0.1610
0.90	0.1933	0.1836	0.1819	0.1810
1.00	0.2133	0.2036	0.2019	0.2010

Table 1: Table captions the different values of x.

Now, let's delve into the local approximation properties of $K_{n,\gamma}$. Recall that, in the case of $f \in C_B[0,\infty)$, the modulus of continuity (see [29]) is

$$w(f;\delta) = sup_{0<h\leq\delta} sup_{x\in[0,\infty)} |f(x+h) - f(x)|.$$

Theorem 3.2. *Let $f \in C_B[0,\infty)$, and $\gamma \in (0,\infty)$, we obtain,*

$$\left|K_{n,\gamma}(f;x) - f(x)\right| \leq 2\omega\left(f; \sqrt{\frac{x}{n} + \frac{1}{(2\gamma+1)n^2}}\right)$$

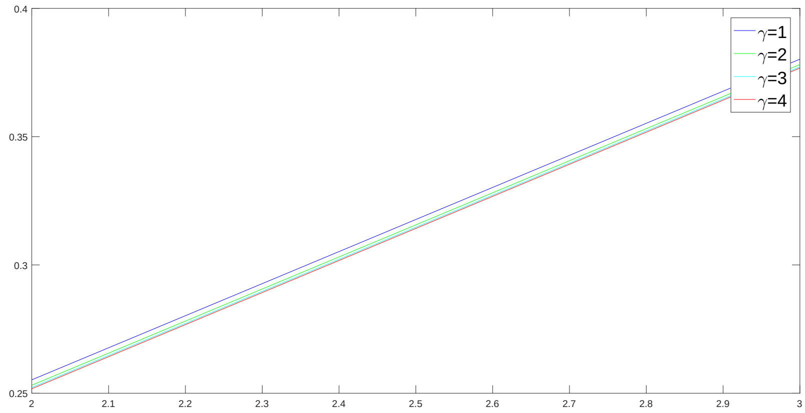

Figure 2: Error of approximation $K_{n,\gamma}((t-x)^2;x)$ for, $\gamma=1$, $\gamma=2$, $\gamma=3$, $\gamma=4$, when $n=8$ and $x\in[2,3]$.

x	$K_{8,1}((t-x)^2;x)$	$K_{8,2}((t-x)^2;x)$	$K_{8,3}((t-x)^2;x)$	$K_{8,4}((t-x)^2;x)$
2.00	0.2552	0.2531	0.2522	0.2517
2.10	0.2677	0.2656	0.2647	0.2642
2.20	0.2802	0.2781	0.2772	0.2767
2.30	0.2927	0.2906	0.2897	0.2892
2.40	0.3052	0.3031	0.3022	0.3017
2.50	0.3177	0.3156	0.3147	0.3142
2.60	0.3302	0.3281	0.3272	0.3267
2.70	0.3427	0.3406	0.3397	0.3392
2.80	0.3552	0.3531	0.3522	0.3517
2.90	0.3677	0.3656	0.3647	0.3642
3.00	0.3802	0.3781	0.3772	0.3767

Table 2: Table captions the different values of x.

for all $x\in[0,\infty)$.

Proof. According to the positivity of $K_{n,\gamma}$ and the equality $K_{n,\gamma}(1;x)=1$, we have,

$$\left|K_{n,\gamma}(f;x)-f(x)\right|\leq\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^1\left|f\left(\frac{k+t^{\gamma}}{n}\right)-f(x)\right|dt. \tag{3.3}$$

Applying the property of the modulus of continuity, which is

$$|f(\zeta)-f(\lambda)|\leq\left(1+\frac{|\zeta-\lambda|}{\delta}\right)\omega(f;\delta)$$

to (3.3). We obtain,

$$\left|K_{n,\gamma}(f;x)-f(x)\right|\leq\omega(f;\delta)\left(1+\frac{1}{\delta}\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^1\left|\frac{k+t^{\gamma}}{n}-x\right|dt\right).$$

Applying Cauchy-Schwarz inequality,

$$\begin{aligned}\left|K_{n,\gamma}(f;x)-f(x)\right| &\leq \omega(f;\delta)\left(1+\frac{1}{\delta}\sqrt{\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^1\left(\frac{k+t^{\gamma}}{n}-x\right)^2dt}\right)\\ &= \omega(f;\delta)\left(1+\frac{1}{\delta}\sqrt{\frac{x}{n}+\frac{1}{(2\gamma+1)n^2}}\right).\end{aligned}$$

Choosing $\delta=\sqrt{\frac{x}{n}+\frac{1}{(2\gamma+1)n^2}}$, we have the desired result. □

Lemma 3.3. *Let $\gamma \in (0,\infty)$, then for each $f \in C_B[0,\infty)$, we get*

$$\|K_{n,\gamma}(f;\cdot)\| \leq \|f\| \tag{3.4}$$

where $\|.\|$ denotes the uniform norm of $C_B[0,\infty)$.

The Peetre-K functional is

$$K_2(f;\delta) := inf_{\tau\in\varpi^2[0,\infty)} \left\{\|f-\tau\| + \delta\|\tau''\|\right\}, (\delta > 0)$$

where $\varpi^2[0,\infty) = \{\tau \in C_B[0,\infty) : \tau', \tau'' \in C_B[0,\infty)\}$.Furthermore, $\exists\, C > 0$ (See [31]) such that

$$K_2(f;\delta) \leq C\omega_2(f;\sqrt{\delta})$$

where $\omega_2(f;\sqrt{\delta})$ is the modulus of smoothness for $f \in C_B[0,\infty)$ defined as

$$\omega_2(f;\sqrt{\delta}) = sup_{0<h\leq\delta} sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|.$$

Theorem 3.4. *Assume that $n \in \mathbb{N}$, $\gamma \in (0,\infty)$ and $f \in C_B[0,\infty)$. Then, $\exists C \in \mathbb{R}^+$ such that*

$$|K_{n,\gamma}(f;x) - f(x)| \leq C\omega_2\left(f; \frac{1}{2}\sqrt{\frac{x}{n} + \frac{1}{(2\gamma+1)n^2} + \left(\frac{1}{(\gamma+1)n}\right)^2}\right) + \omega\left(f; \frac{1}{(\gamma+1)n}\right)$$

$\forall x \in [0,\infty)$.

Proof. Let

$$K^*_{n,\gamma}(f;x) := K_{n,\gamma}(f;x) + f(x) - f\left(x + \frac{1}{(\gamma+1)n}\right). \tag{3.5}$$

From Lemma 2.3, we obtain

$$K^*_{n,\gamma}(1;x) = 1,$$

and

$$K^*_{n,\gamma}(t-x;x) = 0.$$

Now, assume that $\tau \in \varpi^2[0,\infty)$. By the Taylor's expansion,

$$\tau(t) = \tau(x) + (t-x)\tau'(x) + \int_x^t (t-u)\tau''(u)du. \tag{3.6}$$

Applying the operators $K^*_{n,\gamma}$ for both sides, we get

$$\begin{aligned} K^*_{n,\gamma}(\tau;x) &= \tau(x) + K^*_{n,\gamma}\left(\int_x^t (t-u)\tau''(u)du; x\right) \\ &= \tau(x) + K_{n,\gamma}\left(\int_x^t (t-u)\tau''(u)du; x\right) - \int_x^{x+\frac{1}{(\gamma+1)n}} \left(x + \frac{1}{(\gamma+1)n} - u\right)\tau''(u)du. \end{aligned}$$

Hence;

$$K^*_{n,\gamma}(\tau;x) - \tau(x) = K_{n,\gamma}\left(\int_x^t (t-u)\tau''(u)du; x\right) - \int_x^{x+\frac{1}{(\gamma+1)n}} \left(x + \frac{1}{(\gamma+1)n} - u\right)\tau''(u)du.$$

By using above equation, we get

$$\begin{aligned} |K^*_{n,\gamma}(\tau;x) - \tau(x)| &\leq \left|K_{n,\gamma}\left(\int_x^t (t-u)\tau''(u)du; x\right)\right| + \left|\int_x^{x+\frac{1}{(\gamma+1)n}} (x + \frac{1}{(\gamma+1)n} - u)\tau''(u)du\right| \\ &\leq K_{n,\gamma}\left(\left|\int_x^t (t-u)\tau''(u)du\right|; x\right) + \int_x^{x+\frac{1}{(\gamma+1)n}} \left|x + \frac{1}{(\gamma+1)n} - u\right| |\tau''(u)|\, du \\ &\leq K_{n,\gamma}\left(\left|\int_x^t |(t-u)|du\right|; x\right)\|\tau''\| + \int_x^{x+\frac{1}{(\gamma+1)n}} \left|x + \frac{1}{(\gamma+1)n} - u\right| du\|\tau''\| \\ &\leq K_{n,\gamma}((t-x)^2;x)\|\tau''\| + \left(x + \frac{1}{(\gamma+1)n} - x\right)^2 \|\tau''\| \\ &= \left[\frac{x}{n} + \frac{1}{(2\gamma+1)n^2} + \left(\frac{1}{(\gamma+1)n}\right)^2\right]\|\tau''\|. \end{aligned}$$

So, we have

$$\left|K_{n,\gamma}^{*}(\tau;x)-\tau(x)\right| \leq \left[\frac{x}{n}+\frac{1}{(2\gamma+1)n^{2}}+\left(\frac{1}{(\gamma+1)n}\right)^{2}\right]\|\tau''\|. \tag{3.7}$$

Also, from Lemma 3.3 and the equation (3.5), we get

$$\left|K_{n,\gamma}^{*}(f;\cdot)\right| \leq 3\|f\| \tag{3.8}$$

for all $f \in C_B[0,\infty)$ and $x \in [0,\infty)$.

For $f \in C_B[0,\infty)$ and $\tau \in \varpi^2[0,\infty)$, using (3.7) and (3.8), we observe that

$$\begin{aligned}
\left|K_{n,\gamma}(f;x)-f(x)\right| &= \left|K_{n,\gamma}^{*}(f;x)-f(x)+f\left(x+\frac{1}{(\gamma+1)n}\right)-f(x)\right| \\
&= \left|K_{n,\gamma}^{*}(f;x)-K_{n,\gamma}^{*}(\tau;x)+K_{n,\gamma}^{*}(\tau;x)-\tau(x)+\tau(x)-f(x)+f\left(x+\frac{1}{(\gamma+1)n}\right)-f(x)\right| \\
&\leq \left|K_{n,\gamma}^{*}(f;x)-K_{n,\gamma}^{*}(\tau;x)\right|+\left|K_{n,\gamma}^{*}(\tau;x)-\tau(x)\right|+|\tau(x)-f(x)|+\left|f\left(x+\frac{1}{(\gamma+1)n}\right)-f(x)\right| \\
&\leq 4\|f-\tau\|+\left[\frac{x}{n}+\frac{1}{(2\gamma+1)n^{2}}+\left(\frac{1}{(\gamma+1)n}\right)^{2}\right]\|\tau''\|+\omega\left(f;\frac{1}{(\gamma+1)n}\right).
\end{aligned}$$

Hence, by taking the infimum on the right-hand side over all $\tau \in \varpi^2[0,\infty)$, we obtain:

$$\begin{aligned}
\left|K_{n,\gamma}(f;x)-f(x)\right| &\leq 4K_2\left(f;\frac{\frac{x}{n}+\frac{1}{(2\gamma+1)n^{2}}+\left(\frac{1}{(\gamma+1)n}\right)^{2}}{4}\right)+\omega\left(f;\frac{1}{(\gamma+1)n}\right) \\
&= C\omega_2\left(f;\frac{1}{2}\sqrt{\frac{x}{n}+\frac{1}{(2\gamma+1)n^{2}}+\left(\frac{1}{(\gamma+1)n}\right)^{2}}\right)+\omega\left(f;\frac{1}{(\gamma+1)n}\right).
\end{aligned}$$

So, the proof is completed. □

Recall that the usual Lipschitz class for $0<a\leq 1$ and $M>0$ is

$$Lip_M(a) := \{f \in C_B[0,\infty) : |f(\rho)-f(\sigma)| \leq M|\rho-\sigma|^{a}\}$$

$\forall \rho,\sigma \in [0,\infty)$.

Theorem 3.5. *For every $f \in Lip_M(a)$, we have*

$$\left|K_{n,\gamma}(f;x)-f(x)\right| \leq M\left[\frac{x}{n}+\frac{1}{(2\gamma+1)n^{2}}\right]^{\frac{a}{2}}.$$

Proof. Assume that $f \in Lip_M(a)$, then,

$$\begin{aligned}
\left|K_{n,\gamma}(f;x)-f(x)\right| &\leq \sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1\left|f\left(\frac{k+t^{\gamma}}{n}\right)-f(x)\right|dt \\
&\leq M\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1\left|\frac{k+t^{\gamma}}{n}-x\right|^{a}dt.
\end{aligned}$$

Utilizing Hölder's inequality, we obtain

$$\begin{aligned}
\left|K_{n,\gamma}(f;x)-f(x)\right| &\leq M\left[\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1\left(\frac{k+t^{\gamma}}{n}-x\right)^{2}dt\right]^{\frac{a}{2}}\left[\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1 dt\right]^{\frac{2-a}{2}} \\
&= M\left[\frac{x}{n}+\frac{1}{(2\gamma+1)n^{2}}\right]^{\frac{a}{2}}.
\end{aligned}$$

Therefore, the proof is completed. □

Now, we present graphical and numerical results to illustrate the convergence of $K_{n,\gamma}(f;x)$ to certain functions $f(x)$. Additionally, we compare the newly defined operators $K_{n,\gamma}(f;x)$ with the classical Szász-Mirakjan Kantorovich operators $K_n(f;x)$ for different values of γ and n. As anticipated, the results of these comparisons consistently demonstrate that, for any γ chosen to be greater than 1, the approximation of $K_{n,\gamma}(f;x)$ to $f(x)$ surpasses that of $K_n(f;x)$. Moreover, as the value of γ increases, the convergence of the operators $K_{n,\gamma}(f;x)$ to the functions $f(x)$ improves.

In the following Figure 3, we compare the approximation of the operators $K_{20,1}(f;x)$, $K_{20,4}(f;x)$, $K_{20,16}(f;x)$ to

$$f(x) = \begin{cases} 1+x(x-1)(x-2), & 0 \leq x \leq 2 \\ 1, & otherwise. \end{cases}$$

Here, $K_{20,1}(f;x)$ is the classical Százs-Mirakjan Kantorovich operators. Then, the graphics show that choosing $\gamma > 1$ we get better approximation results to the function. Furthermore Figure 4 gives that the graphics of the error of approximation and Table 3 shows the numerical results of the error of approximation of these operators.

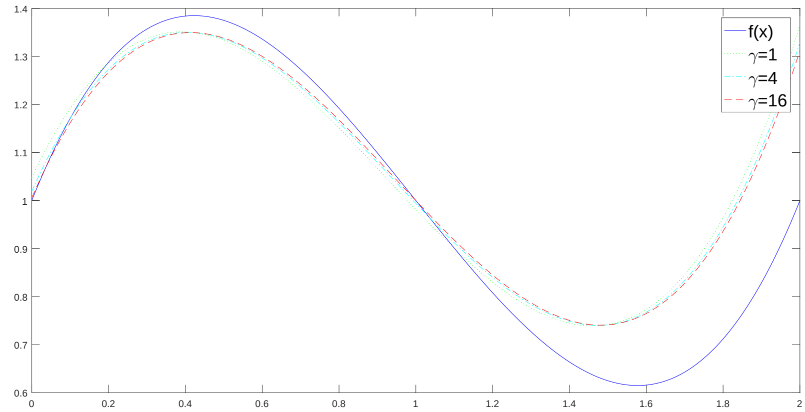

Figure 3: Approximation of $K_{n,\gamma}(f;x)$ to $f(x)$ for $\gamma=1$, $\gamma=4$ and $\gamma=16$ when $n=20$.

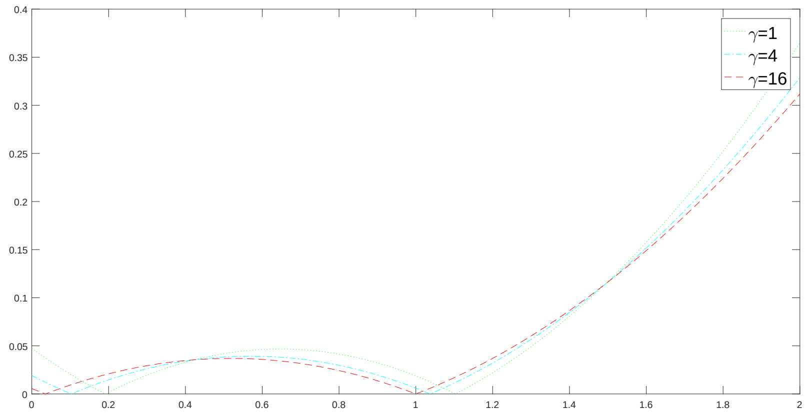

Figure 4: Error of approximation $\varepsilon_{n,\gamma}(f(x)) = \left|K_{n,\gamma}(f;x) - f(x)\right|$ for $\gamma=1$, $\gamma=4$ and $\gamma=16$ when $n=20$.

Now, in Figure 5, we compare the approximation of the operators $K_{50,3}(f;x)$, $K_{100,3}(f;x)$, $K_{150,3}(f;x)$ to the function $f(x)$ where

$$f(x) = \begin{cases} (x-\frac{1}{2})(x-\frac{1}{4})(x-\frac{15}{4})(x-1)(x-2)(x-3)(x-4), & 0 \leq x \leq 4 \\ 0, & x > 4. \end{cases}$$

As expected, increasing the value of n we get better approximation results. Moreover, we give error of the approximation for these operators in Figure 6, graphically. And Table 4 shows the error of approximation, numerically.

x	$\mid K_{20,1}(f;x)-f(x)\mid$	$\mid K_{20,4}(f;x)-f(x)\mid$	$\mid K_{20,16}(f;x)-f(x)\mid$
0.0	0.0475	0.0192	0.0057
0.2	0.0017	0.0147	0.0209
0.4	0.0330	0.0341	0.0347
0.6	0.0462	0.0391	0.0359
0.8	0.0415	0.0298	0.0243
1.0	0.0187	0.0060	0.0000
1.2	0.0220	0.0322	0.0370
1.4	0.0808	0.0847	0.0867
1.6	0.1575	0.1517	0.1491
1.8	0.2523	0.2331	0.2242
2.0	0.3650	0.3288	0.3120

Table 3: Table captions the different values of x.

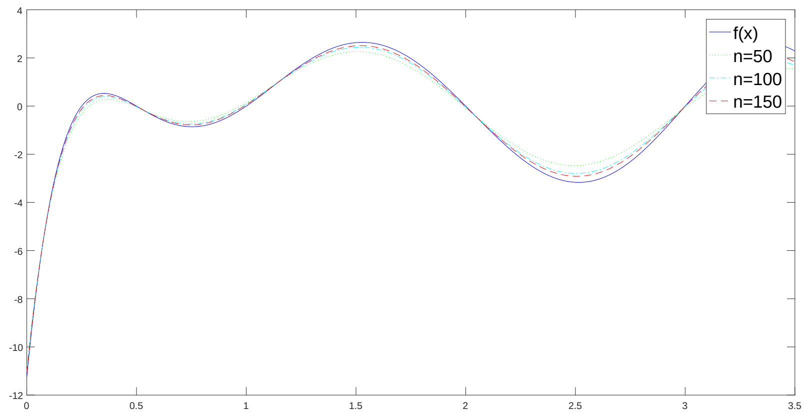

Figure 5: Approximation of $K_{n,\gamma}(f;x)$ to $f(x)$ for $n=50$, $n=100$ and $n=150$ when $\gamma=3$.

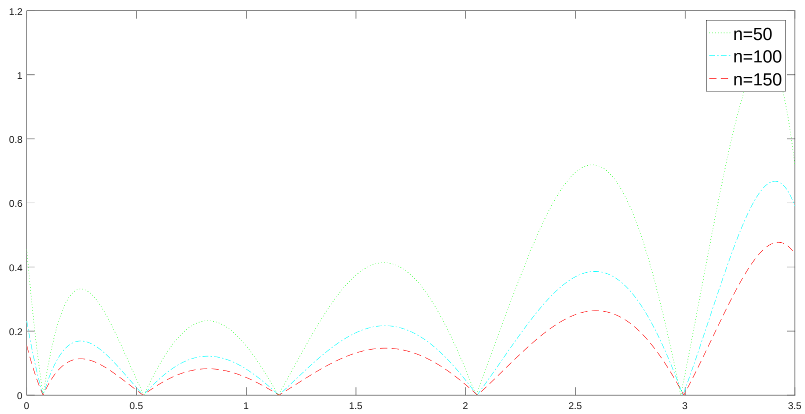

Figure 6: Error of approximation $\varepsilon_{n,\gamma}(f(x)) = \left|K_{n,\gamma}(f;x)-f(x)\right|$ for $n=50$, $n=100$, and $n=150$ when $\gamma=3$.

4. A new modification of $K_{n,\gamma}$ for preserving affine functions

Classical Szász–Mirakjan–Kantorovich operators do not preserve affine functions. But in 2020, Bustamante modified these operators by a new technique (see [32]) and this new family of operators preserve affine functions. In this section, we apply this kind of modification to $K_{n,\gamma}(f;x)$ so that they preserves affine functions. We set,

$$A_{n,\gamma}(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 f\left(a_k \frac{k+t^\gamma}{n}\right) dt, \tag{4.1}$$

where $a_k = \dfrac{(\gamma+1)k}{[(\gamma+1)k+1]}$.

x	$\mid K_{50,3}(f;x)-f(x)\mid$	$\mid K_{100,3}(f;x)-f(x)\mid$	$\mid K_{150,3}(f;x)-f(x)\mid$
0.00	0.4545	0.2310	0.1549
0.35	0.2641	0.1341	0.0899
0.70	0.1873	0.0984	0.0667
1.05	0.1073	0.0572	0.0390
1.40	0.2967	0.1535	0.1035
1.75	0.3755	0.1984	0.1347
2.10	0.0963	0.0457	0.0298
2.45	0.6568	0.3473	0.2358
2.80	0.4962	0.2794	0.1939
3.15	0.6018	0.3206	0.2163
3.50	0.7171	0.5902	0.4409

Table 4: Table captions the different values of x.

We also set,

$$\mu_{n,k}(t)=\frac{t^2}{[(\gamma+1)nt+1]^2}$$

to use in Lemma 4.2 to investigate the moments of the operator $A_{n,\gamma}$.

Lemma 4.1. *Let $k,n\in\mathbb{N}$ and $\gamma\in(0,\infty)$, then we have:*

1. $\int_0^1\left(a_k\frac{k+t^\gamma}{n}\right)dt=\frac{k}{n}$
2. $\int_0^1\left(a_k\frac{k+t^\gamma}{n}\right)^2dt=\frac{k^2}{n^2}+\frac{\gamma^2k^2}{(2\gamma+1)n^2\left[(\gamma+1)k+1\right]^2}$

where $a_k=\frac{(\gamma+1)k}{[(\gamma+1)k+1]}$.

Lemma 4.2. *For each $n\in\mathbb{N}$, $\gamma\in(0,\infty)$ and $x\in[0,\infty)$ we obtain,*

1. $A_{n,\gamma}(1;x)=1$
2. $A_{n,\gamma}(t;x)=x$
3. $A_{n,\gamma}(t^2;x)=x^2+\frac{x}{n}+\frac{\gamma^2}{2\gamma+1}S_n(\mu_{n,k}(t);x)$.

where S_n is defined in (1.1).

Remark 4.3. *The operators $A_{n,\gamma}(f;x)$ in (4.1) reproduce linear polynomials, that is*

$$A_{n,\gamma}(ct+d;x)=cx+d,$$

where $c,d\in\mathbb{R}$.

Remark 4.4. *If $\gamma=1$ in (4.1), then $A_{n,\gamma}$ reduce to the operators in [32], which is introduced and studied by Bustamante.*

Theorem 4.5. *Let $A>0$, then for any $f\in C_B[0,A]$, and $\gamma\in(0,\infty)$, we obtain that $A_{n,\gamma}(f;x)$ are uniformly convergent to $f(x)$ on $[0,A]$.*

Proof. From Korovkin Theorem, it sufficies to demonstrate that $\lim_{n\to\infty}A_{n,\gamma}(t^i;x)=x^i$ where $i=0,1,2$. Evidently, as a consequence of Lemma 4.2, $\lim_{n\to\infty}A_{n,\gamma}(1;x)=1$ and $\lim_{n\to\infty}A_{n,\gamma}(t;x)=x$. Moreover, we can establish that

$$A_{n,\gamma}(t^2;x)=x^2+\frac{x}{n}+\frac{\gamma^2}{2\gamma+1}S_n(\mu_{n,k}(t);x).$$

Taking limit for both sides as $n\to\infty$, we get

$$\lim_{n\to\infty}A_{n,\gamma}(t^2;x)=\lim_{n\to\infty}x^2+\lim_{n\to\infty}\frac{x}{n}+\lim_{n\to\infty}\frac{\gamma^2}{2\gamma+1}S_n(\mu_{n,k}(t);x).$$

From the convergence of classical Szász–Mirakjan operators $\lim_{n\to\infty} S_n(\mu_{n,k}(t);x) = 0$ because of $\lim_{n\to\infty} \mu_{n,k}(x) = 0$. As a result,

$$\lim_{n\to\infty} A_{n,\gamma}(t^2;x) = x^2.$$

Hence, Korovkin theorem conditions are hold for $A_{n,\gamma}$. Then, the proof is completed. □

Note that, $A_{n,1}$ are the operators which is defined by Bustamante. In the following graph, we compare the error estimation results of $A_{n,1}$ and $A_{n,\gamma}$ for different values of γ, by using central moments. One can easily observe that, $A_{n,\gamma}$ have better error estimation when decreasing the value γ. Therefore, for any $0 \leq \gamma \leq 1$, the operators $A_{n,\gamma}$ have better error estimation than $A_{n,1}$.

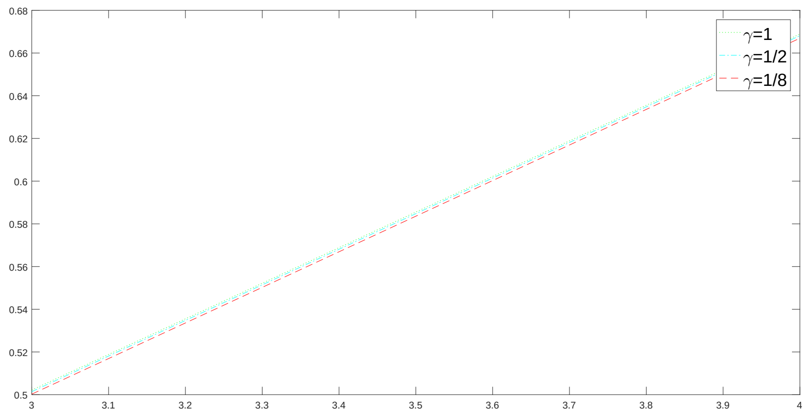

Figure 7: Error of approximation $A_{n,\gamma}((t-x)^2;x)$ for $\gamma = 1$, $\gamma = \frac{1}{2}$ and $\gamma = \frac{1}{8}$ when $n = 6$.

Now, we present graphical and numerical results to compare the convergence of the operators $A_{n,\gamma}(f;x)$ to $f(x)$, where

$$f(x) = \begin{cases} 1+x+x^2, & 0 \leq x \leq 2 \\ 7, & x > 2, \end{cases}$$

for different values of γ.

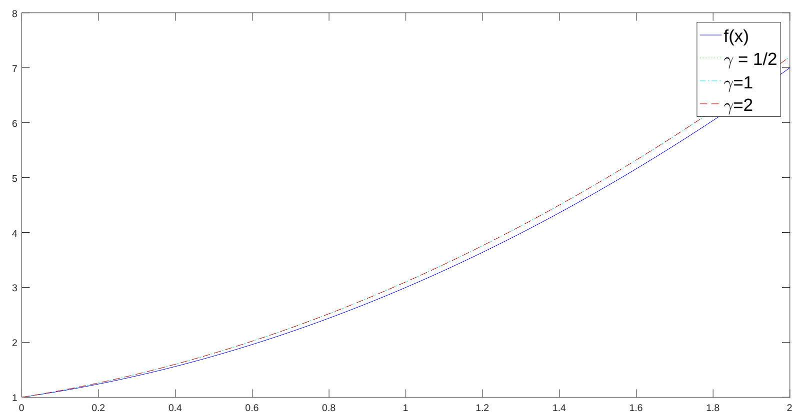

Figure 8: Approximation of $A_{n,\gamma}(f;x)$ to $f(x)$ for $\gamma = \frac{1}{2}$, $\gamma = 1$, and $\gamma = 2$ when $n = 10$.

As anticipated, Figure 8 and Table 5 illustrate that the approximation results of $A_{n,\gamma}(f;x)$ to a specific function $f(x)$ improve as the value of γ decreases. Now, in the upcoming figure, we compare the operators $A_{n,\gamma}(f;x)$ with the function $f(x)$, where

$$f(x) = \begin{cases} x(x-1)(x-\frac{1}{2}), & 0 \leq x \leq 1 \\ 0, & otherwise, \end{cases}$$

with different choices of n.

x	$\mid A_{10,\frac{1}{2}}(f;x)-f(x)\mid$	$\mid A_{10,1}(f;x)-f(x)\mid$	$\mid A_{10,2}(f;x)-f(x)\mid$
0.0	0.0000	0.0000	0.0000
0.2	0.0203	0.0205	0.0206
0.4	0.0404	0.0406	0.0407
0.6	0.0604	0.0607	0.0608
0.8	0.0805	0.0807	0.0808
1.0	0.1005	0.1007	0.1008
1.2	0.1205	0.1208	0.1208
1.4	0.1405	0.1408	0.1408
1.6	0.1605	0.1608	0.1609
1.8	0.1805	0.1808	0.1809
2.0	0.2005	0.2008	0.2009

Table 5: Table captions the different values of x.

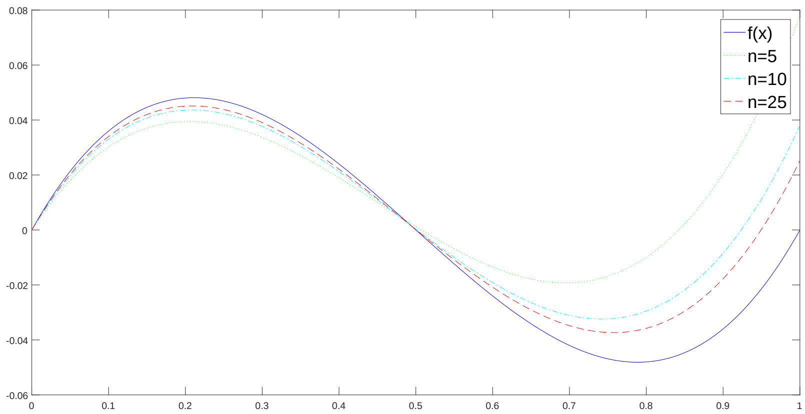

Figure 9: Approximation of $A_{n,\gamma}(f;x)$ to $f(x)$ for $n=5$, $n=10$, and $n=25$ when $\gamma=\frac{1}{3}$.

5. The bivariate case of $K_{n,\gamma}$

Favard [34] introduced and studied bivariate case of classical Szász–Mirakjan operators. Then, many researchers investigated these operators and their generalizations such as in [33] [35], [36], [37]. In this part, we define and investigate the bivariate case of $K_{n,\gamma}$. Consider, $C_B([0,\infty)\times[0,\infty))$ is the space of uniformly bounded and continuous bivariate functions on $[0,\infty)\times[0,\infty)$. We define the operators $K_{n_1,n_2}^{\gamma_1,\gamma_2}$ as

$$K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y) = \sum_{k_1=0}^{\infty} s_{n_1,k_1}(x)\sum_{k_2=0}^{\infty} s_{n_2,k_2}(y)\int_0^1\int_0^1 f\left(\frac{k_1+t_1^{\gamma}}{n_1},\frac{k_2+t_2^{\gamma}}{n_2}\right)dt_1dt_2, \tag{5.1}$$

where $(x,y)\in[0,\infty)\times[0,\infty)$ and $\gamma_1,\gamma_2\in(0,\infty)$ for an integrable functions $f:[0,\infty)\times[0,\infty)\to\mathbb{R}$.
Note that, the operators $K_{n_1,n_2}^{\gamma_1,\gamma_2}$ are linear and positive. Furthermore, choosing $\gamma_1=\gamma_2=1$ in (5.1), then we get the classical Bivariate Szász-Mirakjan Kantorovich operators.

Lemma 5.1. *For $(x,y)\in[0,\infty)\times[0,\infty)$ and $\gamma_1,\gamma_2\in(0,\infty)$, we have*

1. $K_{n_1,n_2}^{\gamma_1,\gamma_2}(1;x,y)=1$
2. $K_{n_1,n_2}^{\gamma_1,\gamma_2}(t_1;x,y)=x+\dfrac{1}{(\gamma_1+1)n_1}$
3. $K_{n_1,n_2}^{\gamma_1,\gamma_2}(t_2;x,y)=y+\dfrac{1}{(\gamma_2+1)n_2}$
4. $K_{n_1,n_2}^{\gamma_1,\gamma_2}(t_1^2;x,y)=x^2+\dfrac{(\gamma_1+3)x}{(\gamma_1+1)n_1}+\dfrac{1}{(2\gamma_1+1)n_1^2}$
5. $K_{n_1,n_2}^{\gamma_1,\gamma_2}(t_2^2;x,y)=y^2+\dfrac{(\gamma_2+3)y}{(\gamma_2+1)n_2}+\dfrac{1}{(2\gamma_2+1)n_2^2}.$

Corollary 5.2. *From Lemma 5.1, we have the following central moments:*

1. $K_{n_1,n_2}^{\gamma_1,\gamma_2}(t_1-x;x,y)=\dfrac{1}{(\gamma_1+1)n_1}=:\rho_{n_1,\gamma_1}(x)$
2. $K_{n_1,n_2}^{\gamma_1,\gamma_2}(t_2-y;x,y)=\dfrac{1}{(\gamma_2+1)n_2}=:\rho_{n_2,\gamma_2}(y)$
3. $K_{n_1,n_2}^{\gamma_1,\gamma_2}((t_1-x)^2;x,y)=\dfrac{x}{n_1}+\dfrac{1}{(2\gamma_1+1)n_1^2}=:\varphi_{n_1,\gamma_1}(x)$
4. $K_{n_1,n_2}^{\gamma_1,\gamma_2}((t_2-y)^2;x,y)=\dfrac{y}{n_2}+\dfrac{1}{(2\gamma_2+1)n_2^2}=:\varphi_{n_2,\gamma_2}(y).$

Theorem 5.3. *Let $A_1,A_2>0$, then for each $\gamma_1,\gamma_2\in(0,\infty)$ and $f\in C_B([0,A_1]\times[0,A_2])$, the operators $K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y)$ uniformly convergent to f as $n_1,n_2\to\infty$ on $[0,A_1]\times[0,A_2]$.*

Proof. Volkov in [38] gives the conditions for the uniformly convergence of bivariate positive linear operators to continous functions. Using Lemma 5.1, one can easily see that the conditions for Volkov's theorem are hold. So, proof is completed. □

For any $f\in C([0,\infty)\times[0,\infty))$, the modulus of continuity for the bivariate case is

$$\omega(f;\delta_1,\delta_2)=\sup_{|t_1-\rho|\le\delta_1,|t_2-\sigma|\le\delta_2}\{|f(t_1,t_2)-f(\rho,\sigma)|:(t_1,t_2),(\rho,\sigma)\in[0,\infty)\times[0,\infty)\}$$

where $\delta_1,\delta_2\in\mathbb{R}^+$.

Moreover, the function $\omega(f;\delta_1,\delta_2)$ has the following inequality,

$$|f(t_1,t_2)-f(\rho,\sigma)|\le\left(1+\frac{|t_1-\rho|}{\delta_1}\right)\left(1+\frac{|t_2-\sigma|}{\delta_2}\right)\omega(f;\delta_1,\delta_2).$$

Theorem 5.4. *Assume that $f\in C([0,\infty)\times[0,\infty))$ and $\gamma_1,\gamma_2\in(0,\infty)$. Then for each $(x,y)\in[0,\infty)\times[0,\infty)$ we get,*

$$\left|K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y)-f(x,y)\right|\le 4\omega\left(f;\sqrt{\varphi_{n_1,\gamma_1}(x)},\sqrt{\varphi_{n_2,\gamma_2}(y)}\right)$$

where $\varphi_{n_1,\gamma_1}(x)$ and $\varphi_{n_2,\gamma_2}(y)$ are given in Corollary 5.2.

Proof. Due to the linearity and positivity of $K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y)$, we are able to write

$$\begin{aligned}\left|K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y)-f(x,y)\right| &\le K_{n_1,n_2}^{\gamma_1,\gamma_2}(|f(t_1,t_2)-f(x,y)|;x,y)\\ &\le \left(1+\frac{K_{n_1,n_2}^{\gamma_1,\gamma_2}(|t_1-x|;x,y)}{\delta_1}\right)\left(1+\frac{K_{n_1,n_2}^{\gamma_1,\gamma_2}(|t_2-y|;x,y)}{\delta_2}\right)\omega(f;\delta_1,\delta_2).\end{aligned}$$

Using Cauchy-Schwarz inequality,

$$K_{n_1,n_2}^{\gamma_1,\gamma_2}(|t_1-x|;x,y)\le\left[K_{n_1,n_2}^{\gamma_1,\gamma_2}\left((t_1-x)^2;x,y\right)\right]^{\frac{1}{2}}$$

and

$$K_{n_1,n_2}^{\gamma_1,\gamma_2}(|t_2-y|;x,y)\le\left[K_{n_1,n_2}^{\gamma_1,\gamma_2}\left((t_2-y)^2;x,y\right)\right]^{\frac{1}{2}}.$$

Therefore,

$$\begin{aligned}\left|K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y)-f(x,y)\right| &\le \left(1+\frac{\sqrt{K_{n_1,n_2}^{\gamma_1,\gamma_2}((t_1-x)^2;x,y)}}{\delta_1}\right)\left(1+\frac{\sqrt{K_{n_1,n_2}^{\gamma_1,\gamma_2}((t_2-y)^2;x,y)}}{\delta_2}\right)\omega(f;\delta_1,\delta_2)\\ &= \left(1+\frac{\sqrt{\varphi_{n_1,\gamma_1}(x)}}{\delta_1}\right)\left(1+\frac{\sqrt{\varphi_{n_2,\gamma_2}(y)}}{\delta_2}\right)\omega(f;\delta_1,\delta_2).\end{aligned}$$

Finally, by choosing $\delta_1=\sqrt{\varphi_{n_1,\gamma_1}(x)}$ and $\delta_2=\sqrt{\varphi_{n_2,\gamma_2}(y)}$, we get the desired result. □

For $0<a_1,a_2\le 1$ the Lipschitz class $Lip_M(a_1,a_2)$ for bivariate case is defined as

$$Lip_M(a_1,a_2) := \left\{f(x,y)\in C([0,\infty)\times[0,\infty)):|f(t_1,t_2)-f(x,y)|\le M|t_1-x|^{a_1}|t_2-y|^{a_2}\right\}$$

where $M>0$ and $(x,y),(t_1,t_2)\in[0,\infty)\times[0,\infty)$.

Theorem 5.5. *If $f \in Lip_M(a_1, a_2)$, then we have,*

$$\left|K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y) - f(x,y)\right| \leq M\left[\varphi_{n_1,\gamma_1}(x)\right]^{\frac{a_1}{2}}\left[\varphi_{n_2,\gamma_2}(y)\right]^{\frac{a_2}{2}}$$

hold for all $(x,y) \in [0,\infty)\times[0,\infty)$, where $\varphi_{n_1,\gamma_1}(x)$ and $\varphi_{n_2,\gamma_2}(y)$ are given in Corollary 5.2.

Proof. Let $f \in Lip_M(a_1,a_2)$, then we have

$$\begin{aligned}\left|K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y) - f(x,y)\right| &\leq K_{n_1,n_2}^{\gamma_1,\gamma_2}(|f(t_1,t_2)-f(x,y)|;x,y)\\ &\leq MK_{n_1,n_2}^{\gamma_1,\gamma_2}(|t_1-x|^{a_1}|t_2-y|^{a_2};x,y)\\ &= MK_{n_1,n_2}^{\gamma_1,\gamma_2}(|t_1-x|^{a_1};x,y)K_{n_1,n_2}^{\gamma_1,\gamma_2}(|t_2-y|^{a_2};x,y).\end{aligned}$$

We apply Hölder's inequality with $p_1 = \dfrac{2}{a_1}$, $q_1 = \dfrac{2}{2-a_1}$ and $p_2 = \dfrac{2}{a_2}$, $q_2 = \dfrac{2}{2-a_2}$, to get

$$\begin{aligned}\left|K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y) - f(x,y)\right| &\leq MK_{n_1,n_2}^{\gamma_1,\gamma_2}((t_1-x)^2;x,y)^{\frac{a_1}{2}}K_{n_1,n_2}^{\gamma_1,\gamma_2}((t_2-y)^2;x,y)^{\frac{a_2}{2}}\\ &= M[\varphi_{n_1,\gamma_1}(x)]^{\frac{a_1}{2}}[\varphi_{n_2,\gamma_2}(y)]^{\frac{a_2}{2}}.\end{aligned}$$

□

Now, we illustrate the approximation of $K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y)$ to $f(x,y)$ for different choices of γ_1,γ_2 for, where

$$f(x,y) = \begin{cases} 4+(x-\frac{1}{2})(y-\frac{1}{2})(y-\frac{1}{4})(x-\frac{3}{4})(x-\frac{3}{2})(y-2), & 0 \leq x,y \leq 2\\ 4, & x,y > 2.\end{cases}$$

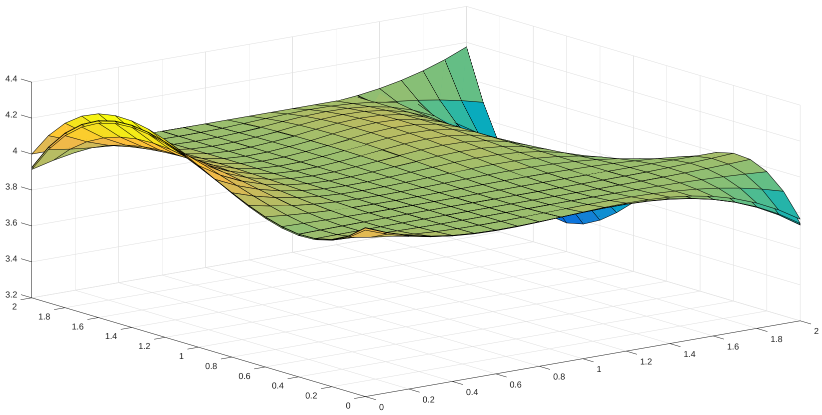

Figure 10: Approximation of $K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y)$ to $f(x,y)$ for $\gamma_1,\gamma_2 = 1$, $\gamma_1,\gamma_2 = 10$, $\gamma_1,\gamma_2 = 100$ when $n_1,n_2 = 50$.

6. Conclusion

We have introduced a novel generalization of the Szász-Mirakjan Kantorovich operators, denoted as $K_{n,\gamma}(f;x)$, defined by

$$K_{n,\gamma}(f;x) := \sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1 f\left(\frac{k+t^{\gamma}}{n}\right)dt.$$

It's important to note that when $\gamma = 1$, these operators reduce to the classical case.

The operators $K_{n,\gamma}$ has the following features:

- Uniformly convergent to any function $f \in C_B[0,A]$ on the interval $[0,A]$ for each $A,\gamma \in \mathbb{R}^+$.
- When γ is chosen to be greater than 1, these operators give improved error estimation compared to the classical case. Moreover, as the value of γ increases, the error estimation becomes smaller.
- Having shape preserving properties.

Furthermore, we introduced a new family of operators $A_{n,\gamma}(f;x)$. These operators reproduce linear(affine) polynomials. Note that, choosing $\gamma = 1$, the new operators $A_{n,\gamma}(f;x)$ reduce to classical case which is introduced and studied by Bustamante in [32]. These operators have better error estimation than classical case if γ is choosen less than 1. Moreover, decreasing the value γ, the error is getting smaller. Finally, we defined the bivariate case of $K_{n,\gamma}(f;x)$, and investigated their approximation properties. As expected, increasing the value of γ_1 and γ_2, we got better error estimation than classical bivariate case.

Declarations

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