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On ss-Lifting Modules In View of Singularity

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Research Article

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Abstract

In this essay we describe δ_{ss}-lifting modules as a singular version of ss-lifting ones. The focus of this study is to get a more general algebraic structure than ss-lifting modules. A module W is entitled δ_{ss}-lifting if for each $S \leq W$, there occurs a decomposition $W = X \oplus Y$ with $X \leq S$ and $S \cap Y \leq Soc_\delta(Y)$, where $Soc_\delta(Y) = \delta(Y) \cap Soc(Y)$. We examine the fundamental properties of this form of modules and also investigate a structure of a ring whose modules are all δ_{ss}-lifting. Finally, we give several characterizations for (projective) δ_{ss}-lifting modules and (amply) δ_{ss}-supplemented modules via δ_{ss}-perfect rings.

Keywords: Semisimple module, δ_{ss}-supplemented module, δ_{ss}-lifting module, Left δ_{ss}-perfect ring

Singülerlik Açısından ss-Lifting Modüller

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Öz

Bu makalede ss-yükseltilebilir modüllerin singüler versiyonu olan δ_{ss}-yükseltilebilir modülleri tanımlıyoruz. Çalışmanın amacı δ_{ss}-yükseltilebilir modüllerden daha genel bir cebirsel yapı elde etmektir. Bir W modülü, her $S \leq W$ alt modülü için, $Soc_\delta(Y) = \delta(Y) \cap Soc(Y)$ olmak üzere, $X \leq S$ ve $S \cap Y \leq Soc_\delta(Y)$, koşullarını gerçekleyen $W = X \oplus Y$ ayrışımına sahip ise W'ya δ_{ss}-yükseltilebilir modül denir. Bu modüllerin temel özelliklerini araştırıyor ve üzerindeki her modülü δ_{ss}-yükseltilebilir olan bir halka yapısı arıyoruz. Sonunda ise, δ_{ss}-mükemmel halkalar aracılığı ile (projektif) δ_{ss}-yükseltilebilir ve (bol) δ_{ss}-tümlenmiş modüllerin bir takım karakterizasyonlarını veriyoruz.

Anahtar Kelimeler: Yarı basit modül, δ_{ss}-tümlenmiş modül, δ_{ss}-yükseltilebilir modül, Sol δ_{ss}-mükemmel halka

Introduction

Throughout the paper, we presume that R is an associative ring with unit and W is a unitary left R-module and we use the representations $S \leq W$ and $S \leq_\oplus W$ which indicate that S is a submodule of W and S is a direct summand of W, respectively.

By $S \trianglelefteq W$, we point that S is an *essential submodule* of W, that is, the intersection of S with whole submodules of W is nonzero, except for $\{0\}$. The *socle* of a module W is signified by $Soc(W)$ as the intersection of whole essential submodules of W equivalently, the sum of whole simple submodules of W. By the notation $S \ll W$, a *small submodule* $S \leq W$ is pointed, that is, a proper submodule of W provided that $S + K \neq W$ for any proper submodule K of W. And the sum of whole small submodules of W is signified by $Rad(W)$. The *singular submodule* of a module W is signed by $Z(W)$ containing the elements of W whose annihilators are essential in R and W is called *singular* if $Z(W) = W$ [1].

In [1], Zhou contributed to literature the term of δ-small submodules as a generalized category of small ones. $S \leq W$ is called *δ-small* in W if $W = S+K$ implies $W = K$ for each $K \leq W$ with $\frac{W}{K}$ is singular and signed by $S \ll_\delta W$. And the sum of whole small submodules of W is signified by $\delta(W)$. As for *δ-hollow* modules are defined as the modules whose each proper submodule is δ-small [2]. Moreover in [3], $Soc_\delta(W)$ is identified as the intersection of $Soc(W)$ and $\delta(W)$. Hence $Soc_\delta(W)$ is the sum of each δ-small simple submodules of W.

A *supplement submodule* K of a submodule S in W is the minimal submodule with respect to $W = S + K$ which is equivalent to $W = S + K$ and $S \cap K \ll K$ and a *supplemented* module is defined as the module whose each submodule has a supplement. Also a submodule $S \leq W$ has *ample supplements* in W if each submodule $K \leq W$ with $W = S + K$, involves a supplement of S in W.

In [4], Koşan extended the notion of supplemented modules via singularity. A *δ-supplemented* module is defined as the module whose each submodule is of a δ-supplement. And a module W is named *amply δ-supplemented* if, S is of a δ-supplement K' in W with $K' \leq K$, where $W = S + K$ for any $K \leq W$.

In [5], Oshiro defined extending and lifting modules as a dual form of each other, which are the general forms of injective modules and projective supplemented modules, respectively. In order to obtain fundamental knowledge about supplements types one may refer to [6–10]. In summary, a module W is named *lifting* if, for each $S \leq W$, there occurs a decomposition $W = X \oplus Y$ such that $X \leq S$ and $S \cap Y \ll Y$. Motivated by this term, in [4] Koşan also defined δ-lifting modules as follows. A module W is called *δ-lifting* if, for each submodule $S \leq W$, there occurs a decomposition $W = X \oplus Y$ such that $X \leq S$ and $S \cap Y \ll_\delta Y$. A module W is named $\oplus$-δ-supplemented whose submodules are of a δ-supplement in W as a direct summand of W. For more information we refer to readers [1, 11–15].

In [3], the authors studied δ_{ss}-supplemented modules as a generalized type of ss-supplemented ones presented in [16]. A module W is named *δ_{ss}-supplemented* if for each submodule $S \leq W$ there occurs a δ_{ss}-supplement submodule K in W, where $S + K = W$, $S \cap K$ is semisimple and δ-small submodule of K. Moreover, amply δ_{ss}-supplemented modules are introduced. And by this way, the relations are indicated between these two new algebraic structures. The concept of δ_{ss}-perfect rings is contributed in the literature. The equivalent conditions are determined for a ring R to be δ_{ss}-perfect via some R-modules that has a projective δ_{ss}-cover. In [3] the authors restrict the definition of δ-supplemented modules to δ_{ss}-supplemented modules by replacing the condition of being δ-small submodule for δ-supplement submodules to the condition of being δ-small semisimple. Thus, a new module structure is constructed between ss-supplemented modules and δ-supplemented modules.

Inspired by [3, 17], in this paper, we define *δ_{ss}-lifting modules* and by this way a new structure is obtained among the category of ss-lifting modules and δ-lifting modules. Fundamental features are investigated for these modules. Firstly, we present matching conditions for a module to be *δ_{ss}-lifting* in Lemma 2. An another fact is that the direct summand of a δ_{ss}-lifting module saves the feature. δ_{ss}-lifting modules are clearly δ_{ss}-supplemented. However, a δ_{ss}-supplemented module W with $\delta(W) \leq Soc(W)$ is δ_{ss}-lifting. Suitable conditions are determined for the factor module of a δ_{ss}-lifting module to be

δ_{ss}-lifting in Proposition 3. Also some results on decompositions of δ_{ss}-lifting modules are obtained. Furthermore, some characterizations are given for projective (amply) δ_{ss}-supplemented modules and δ_{ss}-lifting modules via δ_{ss}-perfect rings in Theorem 4 and Theorem 5. As a consequence, a ring R is δ_{ss}-perfect if and only if ${}_RR$ is δ_{ss}-lifting if and only if ${}_RR$ is (amply) δ_{ss}-supplemented. A module W is δ_{ss}-lifting if and only if W is amply δ_{ss}-supplemented and each δ_{ss}-supplement S of W is of a decomposition $S = U \oplus V$ such that $U \leq_{\oplus} W$ and V is projective semisimple. In Proposition 7, it is proven that a projective module W is δ_{ss}-lifting if and only if $\frac{W}{Soc_{\delta}(W)}$ is semisimple and decompositions of $\frac{W}{Soc_{\delta}(W)}$ lift to decompositions of W.

δ_{ss}-Lifting Modules

Definition 1. A module W is named δ_{ss}-lifting if for each $S \leq W$, there occurs a decomposition $W = X \oplus Y$ with $X \leq S$ and $S \cap Y \leq Soc_{\delta}(Y)$, where $Soc_{\delta}(Y) = \delta(Y) \cap Soc(Y)$.

Owing to this concept, a new algebraic structure takes place between ss-lifting modules and δ-lifting modules.
Now the matching provisions for a module W to be δ_{ss}-lifting are given.

Lemma 1. The statements given below are equivalent:

1. For each $S \leq W$, there is a decomposition $W = X \oplus Y$ such that $X \leq S$ and $S \cap Y \leq Soc_{\delta}(W)$.
2. Each $S \leq W$ has the form $S = A \oplus B$ with $A \leq_{\oplus} W$ and $B \leq Soc_{\delta}(W)$.

Proof. $(1) \Rightarrow (2)$ is evident.
$(2) \Rightarrow (1)$ is similar to that of $(3) \Rightarrow (2)$ of Lemma 3.3 in [1].

Lemma 2.

1. The implications given below are equivalent for a module W:

a. W is δ_{ss}-lifting.

b. For each $S \leq W$, there occurs submodules $X, Y \leq S$ provided $S = X \oplus Y$, $X \leq_{\oplus} W$ and $Y \leq Soc_{\delta}(W)$.

c. For each $S \leq W$, there occurs a submodule $X \leq_{\oplus} W$ provided $X \leq S$ and $\frac{S}{X} \leq Soc_{\delta}\left(\frac{W}{X}\right)$.

(2) Every direct summand of a δ_{ss}-lifting module is δ_{ss}-lifting.

Proof. 1. $(1a) \Rightarrow (1b)$: It is obvious from Lemma 1.
$(1b) \Rightarrow (1c)$: Let $S \leq W$. By supposition, there occurs a decomposition of S provided $S = X \oplus Y$ with $X \leq_{\oplus} W$ and $Y \leq Soc_{\delta}(W)$. For the natural homomorphism $\pi : W \to \frac{W}{X}$, we have $\pi(Y) = \frac{Y+X}{X} = \frac{S}{X} \leq Soc_{\delta}(\frac{W}{X})$, since $Y \ll_{\delta} W$ since [1, Lemma 1.3] and [15, 20.3].
$(1c) \Rightarrow (1a)$: Let $S \leq W$. By $(1c)$ there occurs a decomposition of W, such that $W = X \oplus Y$ with $X \leq S$ and $\frac{S}{X} \leq Soc_{\delta}\left(\frac{W}{X}\right)$. Therefore, we have $W = S + Y$ and $S = X \oplus (Y \cap S)$. Because $\frac{W}{X} \cong Y$ and $\frac{S}{X} \cong S \cap Y$, then we get $S \cap Y \leq Soc_{\delta}(W)$. Hence, W is a δ_{ss}-lifting module.

2. Let W be δ_{ss}-lifting and $S \leq_{\oplus} W$. Thus, there occurs some $T \leq W$ such that $W = S \oplus T$. For any $X \leq S \leq W$, since W is δ_{ss}-lifting, there occurs a decomposition of W such that $W = Z \oplus Y$ with $Z \leq X$ and $X \cap Y \ll_{\delta} Y$. Therefore $S = Z \oplus (S \cap Y)$ is obtained such that $S \cap (X \cap Y) = X \cap (S \cap Y) = X \cap Y \ll_{\delta} S \cap Y$, since $S \leq_{\oplus} W$ and $S \cap Y \leq_{\oplus} S$.

A module W is called *strongly δ-local* if it is δ-local with a semisimple δ-radical [3].

Proposition 1. A strongly δ-local module is δ_{ss}-lifting.

Proof. Let W be a strongly δ-local module and $S \leq W$.
$Case$ 1 : Let $S \leq \delta(W)$. Thus, S is semisimple as a submodule of $\delta(S)$. Therefore, $S \ll_{\delta} W$ by [3, Lemma 2.2]. Clearly, W has the decomposition $W = 0 \oplus W$ and $0 \leq S$, $S \cap W = S \ll_{\delta} W$.
$Case$ 2 : Let $S \nleq \delta(W)$. Then we have $W = S + \delta(W)$ from the maximality of $\delta(W)$. Because $\delta(W) \ll_{\delta} W$, there occurs a projective semisimple submodule D of $\delta(W)$ with $W = S \oplus D$. Hence W is δ_{ss}-lifting.

Proposition 2. A δ-lifting module W with $\delta(W) \leq Soc(W)$ is δ_{ss}-lifting.

Proof. For each $S \leq W$ we have $W = X \oplus Y$ such that $X \leq S$ and $S \cap Y \ll_{\delta} Y$. As a result, $S \cap Y$ is semisimple since $S \cap Y \leq \delta(W) \leq Soc(W)$. This verifies that W is δ_{ss}-lifting.

Remember that a module W is named *distributive* if for any submodules X, Y and Z of W, $X \cap (Y + Z) = (X \cap Y) + (X \cap Z)$. If for each $f \in End(W)$, $f(X) \leq X$, we say that X is a *fully invariant* submodule of W.
Now, the conditions are investigated to obtain when the factor module of a δ_{ss}-lifting module is δ_{ss}-lifting.

Proposition 3. Let W be a δ_{ss}-lifting module. For any $X \leq W$ the factor module $\frac{W}{X}$ is δ_{ss}-lifting if one of the statements given below are satisfied:

1. For any $S \leq_{\oplus} W$, $\frac{S+X}{X} \leq_{\oplus} \frac{W}{X}$.
2. W is a distributive module.
3. $f(X) \subseteq X$ for any idempotent $f = f^2 \in End(W)$. Particularly, X is a fully invariant submodule of W.

Proof. 1. Let $\frac{K}{X} \leq \frac{W}{X}$. Since $K \leq W$ and W is δ_{ss}-lifting there occurs a direct summand T of W with $T \leq K$ and $\frac{K}{T} \leq Soc_{\delta}(\frac{W}{T})$. It is clear that $\frac{T+X}{X} \leq_{\oplus} \frac{W}{X}$ and $\frac{T+X}{X} \leq \frac{K}{X} \leq \frac{W}{X}$. Since $\frac{K}{T} \leq Soc_{\delta}(\frac{W}{T})$, then $\frac{K}{T+X} \leq Soc_{\delta}(\frac{W}{T+X})$ by Lemma [1, Lemma 1.3]. Hence $\frac{W}{X}$ is δ_{ss}-lifting.
2. This condition is proved by using (1). Let $W = Y \oplus Z$. We have $\frac{W}{X} = \frac{Y+X}{X} + \frac{Z+X}{X}$ and by the assumption $\frac{Y+X}{X} \cap \frac{Z+X}{X} = \frac{(Y\cap Z)+X}{X} = 0_{\frac{W}{X}}$. Hence $\frac{Y+X}{X} \leq_{\oplus} \frac{W}{X}$ and so $\frac{W}{X}$ is δ_{ss}-lifting.
3. Let $W = A \oplus B$. By (1), we will show that the factor module $\frac{A+X}{X}$ is a direct summand of $\frac{W}{X}$. Let $\pi : A \oplus B \to A$ be the projection map with the kernel $(1 - \pi)(W) = B$. Then $\pi^2 = \pi \in End(W)$ and $\pi(W) = A$. From assumption $\pi(X) \leq X$ and $(1 - \pi)(X) \leq X$ is obtained. Thus, we have $\pi(X) = X \cap A$ and $(1 - \pi)(X) = X \cap B$. So we have $X = \pi(X) \oplus (1 - \pi)X = (X \cap A) \oplus (X \cap B)$. Then, $\frac{A+X}{X} = \frac{A\oplus(X\cap B)}{X}$ and $\frac{B+X}{X} = \frac{B\oplus(X\cap A)}{X}$ which implies $\frac{W}{X} = \frac{A\oplus(X\cap B)}{X} + \frac{B\oplus(X\cap A)}{X}$. In addition to these, $[A\oplus(X\cap B)]\cap[B\oplus(X\cap A)] = \{[A\oplus(X\cap B)]\cap B\}\oplus(X\cap A) = (X\cap B)\oplus(A\cap B)\oplus(X\cap A) = (X \cap B) \oplus (X \cap A) = X$, we have $\frac{A+X}{X} \leq_{\oplus} \frac{W}{X}$. Hence, W is δ_{ss}-lifting by (1).

In Lemma 2, we proved that being δ_{ss}-lifting is transferred to direct summands. But generally, the converse is not true. By Theorem 1, we present a way to verify this claim by adding suitable conditions. But firstly, we give the following useful lemma (see in [15, 41.14]).

Lemma 3. Let $W = X \oplus Y$. Then the implications given above are equivalent.

1. X is Y-projective.

2. For every $S \leq W$ with $W = T + Y$, there occurs a submodule $T^{\acute{}} \leq T$ such that $W = T^{\acute{}} \oplus Y$.

Theorem 1. Let $W = X \oplus Y$ be a module such that X is both self and Y-projective. If X and Y are δ_{ss}-lifting modules, then so is W.

Proof. Let $S \leq W$. Thus, for $X \cap (S + Y) \leq W$, as X is δ_{ss}-lifting, there occurs direct summands $D, D^{\acute{}}$ of X such that $D \leq X \cap (S + Y)$ and $X \cap (S + Y) \cap D^{\acute{}} = (S + Y) \cap D^{\acute{}} \ll_\delta X$. So we get $W = X \oplus Y = D \oplus D^{\acute{}} \oplus Y = S + (D^{\acute{}} \oplus Y)$. Since X is self and Y-projective, it is clear that X is W-projective. By taking into account the exact sequence $D \longrightarrow D \oplus (D^{\acute{}} \oplus Y) \longrightarrow D^{\acute{}} \oplus Y$, it is apparent that D is $D^{\acute{}} \oplus Y$-projective [15, 18.1/18.2]. Therefore by Lemma 3 there occurs some $S^{\acute{}} \leq S$ such that $W = S^{\acute{}} \oplus (D^{\acute{}} \oplus Y)$. Thus, we can say $S \cap (W + D^{\acute{}}) = W \cap (S + D^{\acute{}})$ for each submodule $W \leq Y$. Furthermore, since Y is δ-lifting, there exists $Y_1 \leq Y \cap (S + D^{\acute{}}) = S \cap (Y + D^{\acute{}})$ such that $Y = Y_1 \oplus Y_2$ and $S \cap (Y_2 + D^{\acute{}}) = Y_2 \cap (S + D^{\acute{}}) \ll_\delta Y_2$ for any $Y_2 \leq Y$. Therefore that $W = S^{\acute{}} \oplus (D^{\acute{}} \oplus Y) = S^{\acute{}} \oplus (D^{\acute{}} \oplus Y_1 \oplus Y_2) = (S^{\acute{}} \oplus Y_1) \oplus (Y_2 \oplus D^{\acute{}})$ is obtained easily. Because $S^{\acute{}} \leq S$ and $X \leq S \cap (D^{\acute{}} \oplus Y) \leq S$, we get $S^{\acute{}} \oplus Y_1 \leq S$ and so $W = S + (D^{\acute{}} \oplus Y)$. In addition, $S \cap (Y_2 \oplus D^{\acute{}}) = Y_2 \cap (S \oplus D^{\acute{}}) \ll_\delta Y_2 \leq Y_2 \oplus D^{\acute{}}$.

Corollary 1. Let X be a semisimple module and Y be a δ_{ss}-lifting module which is relatively projective with X, then $W = X \oplus Y$ is δ_{ss}-lifting.

Example 1. Let us consider the $\mathbb{Z}$-module $W = \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}}$. Since the simple $\mathbb{Z}$-module $\mathbb{Z}_2$ and $\mathbb{Z}$-module $\mathbb{Z}_4$ are strongly local, they are also ss-supplemented and so δ_{ss}-supplemented. Thus, W is a δ_{ss}-supplemented module as a finite direct sum of δ_{ss}-supplemented modules [3, Proposition 4.9]. Otherwise W is a δ_{ss}-lifting module since $\delta(W) \leq Soc(W)$ and W is δ-lifting [4, Lemma 2.6] although $\frac{\mathbb{Z}}{2\mathbb{Z}}$ is not $\frac{\mathbb{Z}}{4\mathbb{Z}}$-projective.

Now we give some results on decomposition of a δ_{ss}-lifting module.

Proposition 4. The implications given below hold for a δ_{ss}-lifting module W.

1. $\frac{W}{Soc_\delta(W)}$ is semisimple.

2. Any $S \leq W$ with $S \cap Soc_\delta(W) = 0$ is semisimple.

3. W has a decomposition $W = X \oplus Y$ such that X is semisimple, Y is δ_{ss}-lifting module and $\delta(Y) \trianglelefteq Y$.

Proof. 1. It is clear from [3, Proposition 4.7].
2. Because $S \cong \frac{S \oplus Soc_\delta(W)}{Soc_\delta(W)} \leq \frac{W}{Soc_\delta(W)}$ is semisimple from (1), then S is semisimple by [18].
3. It is clear from [4] and Lemma 2(2).

Theorem 2. Let W is a δ_{ss}-lifting module. Then W is of a decomposition $W = X \oplus Y$ such that $\delta(X) = X$, $\delta(Y) = Soc_\delta(Y)$ and X, Y are δ_{ss}-lifting modules.

Proof. As W is δ_{ss}-lifting, there occurs a decomposition $W = X \oplus Y$ for the submodules $\delta(W)$ of W such that $\delta(W) \leq X$ and $\delta(W) \cap Y \leq Soc_\delta(Y)$. Thus, we have $\delta(W) \cap Y = [\delta(X) \oplus \delta(Y)] \cap Y = \delta(Y) \oplus [\delta(X) \cap Y] = \delta(Y) = Soc_\delta(Y)$. Moreover, $X = \delta(W) \cap X = \delta(X) \oplus (X \cap \delta(Y)) = \delta(X)$ is got. Also, X and Y are δ_{ss}-lifting by Lemma 2(2).

Proposition 5. Let W be a δ_{ss}-lifting module and $\delta(W)$ is of an ss-supplement in W. Then W has a decomposition $W = X \oplus Y$ such that X, Y are δ_{ss}-lifting modules and $Soc_\delta(Y) = Soc(Y)$.

Proof. Let W be a ss-supplement of $\delta(W)$ in W. In this case $\delta(W) + W = W$ and $\delta(W) \cap W \leq Soc_s(W)$. As W is δ_{ss}-lifting, there exists a decomposition $W = X \oplus Y$ for the submodule W with $X \leq W$ and $W \cap Y \leq Soc_\delta(Y)$. Then $W = X \oplus (W \cap Y)$. For any submodule S of X, from assumption we get a decomposition $X = A \oplus B$ such that $A \leq_\oplus S$ and $S \cap B \ll_\delta B$ by Lemma 1. Hence, we get $S = A \oplus (B \cap S)$. Since $\delta(W) \cap W \ll W$ and $B \cap S \leq \delta(W) \cap W$, it is obtained that $S \cap B \ll W$. Therefore, $B \cap S$ is semisimple and small in B by [13]. Thus, X is an ss-lifting module. In addition to these, $W = \delta(W) + W = \delta(X) + \delta(Y) + X + (Y \cap W) = \delta(X) + \delta(Y) + X = \delta(Y) \oplus X = Y \oplus X$ and so $Soc_\delta(Y) = Soc(Y) \cap \delta(Y) = Soc(Y) \cap Y = Soc(Y)$ is obtained.

In [3], the authors defined the (projective) δ_{ss}-cover of a module as follows.

Definition 2. Let W be a module and P be a (projective) module. P is named a *(projective) δ_{ss}-cover* of W if there exists an epimorphism from P to W with a semisimple and δ-small kernel in P.

Theorem 3. Let $W = X + Y$. If $\frac{W}{X}$ is of a projective δ_{ss}-cover, then Y includes a δ_{ss}-supplement of X.

Proof. Let $\pi : Y \longrightarrow \frac{Y}{X \cap Y} \cong \frac{X+Y}{X}$ be the natural homomorphism and let $f : P \longrightarrow \frac{X+Y}{X}$ be a projective δ_{ss}-cover. Because of the projectivity P, there occurs a homomorphism $g : P \longrightarrow Y$ satisfying $\pi g = f$, $Ker(f)$ is semisimple and δ-small in P. Then, it is clear that $W = X + g(P)$ and $X \cap g(P) = g(Ker(f))$. As $Ker(f) \ll_\delta P$ and $Ker(f)$ is semisimple, then $X \cap g(P) \ll_\delta g(P)$ and $X \cap g(P)$ is semisimple by [1, Lemma 1.3] and [18, Corollary 8.1.5].

Proposition 6. A projective module P is δ_{ss}-supplemented if and only if it is δ_{ss}-lifting.

Proof. It is clear from [3, Theorem 5.6].

Theorem 4. The implications given below are equivalent for a ring R:

1. R is a δ_{ss}-perfect ring.
2. Every R-module is δ_{ss}-supplemented.
3. Every projective R-module is δ_{ss}-supplemented.
4. Every projective R-module is δ_{ss}-lifting.
5. Every finitely generated projective R-module is δ_{ss}-lifting.

6. Every finitely generated projective R-module is δ_{ss}-supplemented.

7. Every finitely generated R-module is δ_{ss}-supplemented.

8. ${}_RR$ is δ_{ss}-supplemented.

Proof. $(1) \Longrightarrow (2)$: It is clear from [3, Theorem 5.3].
$(2) \Longrightarrow (3)$: It is obvious.
$(3) \Longrightarrow (4)$: It is clear from Proposition 6.
$(4) \Longrightarrow (5)$: It is clear.
$(5) \Longrightarrow (6)$: It is clear from [3, Theorem 5.6].
$(6) \Longrightarrow (7)$: Let W be a finitely generated module. Then W is a homomorphic image of a finitely generated free R-module, that is, $W \cong f(R^n)$ where $f : R^n \longrightarrow W$ is epic for some $n \geq 0$. Hence, W is δ_{ss}-supplemented from [3, Proposition 4.9, Proposition 4.14].
$(7) \Longrightarrow (8)$: It is apparent from implications.
$(8) \Longrightarrow (1)$: If ${}_RR$ is δ_{ss}-supplemented, then R is a δ_{ss}-perfect ring by [3, Theorem 5.3].

Example 2. Let $Q = \Pi_{i=1}^{\infty}\mathbb{Z}_2$ and R be the subring of Q generated by $\oplus_{i=1}^{\infty}\mathbb{Z}_2$ and 1_Q. Since $Soc(R) = \delta(R)$ is semisimple and δ-small in W [1, Example 4.1], then ${}_RR$ is strongly δ-local and δ_{ss}-supplemented from [3, Lemma 4.1]. Hence, ${}_RR$ is a δ_{ss}-lifting module by from Theorem 4 as ${}_RR$ is projective. Also it is not ss-lifting by [17, Theorem 5], as R is not semiperfect by [1, Example 4.1]

Example 3. Let R be a ring of polynomials over a field F in countably many commuting indeterminates $x_1, x_2, ...$ modulo the ideal generated by $\{x_1^2, x_2^2 - x_1, x_3^2 - x_2, ...\}$ with $Rad(R) = \frac{\langle x_1, x_2, ...\rangle}{\langle x_1^2, x_2^2 - x_1, x_3^2 - x_2, ...\rangle}$. Since R is local and R has no minimal ideal we have $Rad(R) = \delta(R) \neq Soc(R) = 0$. Nevertheless, R is a δ-semiperfect ring [1] which is not δ_{ss}-perfect [3]. Finally, the R-module R is δ-lifting [4, Theorem 3.3] but not δ_{ss}-lifting.

In [13, Theorem 4.44], it was shown that a projective module P such that $\frac{P}{Rad(P)}$ is semisimple and $Rad(P) \ll P$ is semiperfect if and only if decompositions of $\frac{P}{Rad(P)}$ lift to decompositions of P. Motivated by this reality, we give the following useful proposition.

Proposition 7. The statements given below are equivalent for a projective module W.

1. W is δ_{ss}-lifting.

2. $\frac{W}{Soc_\delta(W)}$ is semisimple and for any $\overline{X} = \frac{A+Soc_\delta(W)}{Soc_\delta(W)} \leq_\oplus \frac{W}{Soc_\delta(W)}$, there exists a direct summand A of W such that $\overline{X} = \overline{A}$.

Proof. $(1 \Longrightarrow 2)$: Let W be a δ_{ss}-lifting module. Since W is also δ_{ss}-supplemented, then from Proposition 4 $\frac{W}{Soc_\delta(W)}$ is semisimple. From assumption, there exists direct summands A, B of X with $X = A \oplus B$, with $A \leq_\oplus W$ and $B \leq Soc_\delta(W)$. Hence $\frac{X+Soc_\delta(W)}{Soc_\delta(W)} = \frac{A+Soc_\delta(W)}{Soc_\delta(W)}$ is obtained, i.e., $\overline{X} = \overline{A}$.
$(2 \Longrightarrow 1)$: Let $S \leq W$. As $\frac{W}{Soc_\delta(W)}$ is semisimple, we have $\frac{S+Soc_\delta(W)}{Soc_\delta(W)} \leq_\oplus \frac{W}{Soc_\delta(W)}$ and there occurs a submodule $X \leq_\oplus W$ from assumption satisfying $\frac{S+Soc_\delta(W)}{Soc_\delta(W)} = \frac{X+Soc_\delta(W)}{Soc_\delta(W)}$. In this case $W = X \oplus Y$ for some $Y \leq (W)$ and so $\frac{W}{Soc_\delta(W)} = \frac{(X+Y+Soc_\delta(W))}{Soc_\delta(W)} = \frac{(S+Y+Soc_\delta(W))}{Soc_\delta(W)}$. Since $Soc_\delta(W) \ll_\delta W$ from [3,

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Proposition 3.1(2)], then there exists a projective semisimple submodule of P of $Soc_\delta(W)$ such that $W = (S+Y) \oplus P$. Then, $S+Y$ is projective as a direct summand of W. From Lemma 3, $S+Y = S' \oplus Y$ with $S' \leq S$ is got. Thus, $W = S' \oplus (Y \oplus P)$ and even as $\frac{W}{Soc_\delta(W)} = \frac{(S+Soc_\delta(W))}{Soc_\delta(W)} \oplus \frac{(Y+Soc_\delta(W))}{Soc_\delta(W)}$, we have $S \cap (Y \oplus Soc_\delta(W)) \leq Soc_\delta(W)$. Hence, we get $S \cap (Y \oplus P) = S \cap Y \leq S \cap (Y \oplus Soc_\delta(W)) \leq Soc_\delta(W) \ll_\delta W$.

Definition 3 (*see from [3]*)**.** A module W is called *amply δ_{ss}-supplemented* if T includes a δ_{ss}-supplement of S in W whenever $W = S + T$ for any $T \leq W$.

In [3, Proposition 4.2], it is given that a projective strongly δ-local module is amply δ_{ss}-supplemented. In that manner, it is possible to get a relation between amply δ_{ss}-supplemented modules and δ_{ss}-lifting modules.

Lemma 4. Let W be a δ_{ss}-lifting module. Then, W is amply δ_{ss}-supplemented.

Proof. Let $X, Y \leq W$ be submodules of W with $W = X + Y$ By Lemma 2, it is obtained that $Y = Y' \oplus Y''$, $Y' \leq_\oplus W$ and $Y'' \leq Soc_\delta(W)$. So we get $W = X + Y' + Y''$. Then, there occurs a projective semisimple submodule Y''' of Y'' with, $W = (X + Y') \oplus Y'''$ as $Y'' \leq Soc_\delta(W)$. Take $W = X + T$, where $T = Y' \oplus Y'''$. Then, $W = Y' \oplus Y^*$ and $X \cap T = T_1 \oplus S$ where $T_1 \leq_\oplus W$ and $S \leq Soc_\delta(W)$. Let $\pi' : W \longrightarrow Y'$ the projection map. Then, $\pi'(S) \leq Soc_\delta(Y')$ and $S \leq T = Y' \oplus Y'''$. Thus, we have $S \leq \pi'(S) \oplus Y''' \ll_\delta Y' \oplus Y''' = T$. Let assume $T = T_1 \oplus T_2$ with the projection map $\pi : T \longrightarrow T_2$. So we get $X \cap T = T_1 \oplus (T_2 \cap X)$ by modular law, as $T_1 \leq T_1 \oplus S \leq X$. Hence, $X \cap T_2 = X \cap T \cap T_2 \leq \pi(X \cap T) = \pi(T_1 \oplus S) = \pi(S) \leq Soc_\delta(T_2)$ as $S \leq Soc_\delta(W)$ and so $X \cap T_2 \leq \pi(X \cap T) = \pi(S)$ is semisimple and δ-small in T_2. It follows that $W = X + T_2$ and $X \cap T_2 \leq Soc_\delta(T_2)$ where $T_2 \leq T \leq Y$. This means Y contains a δ_{ss}-supplement T_2 of X in T, that is, W is an amply δ_{ss}-supplemented module.

Remark 1. As a result of above lemma, we get the following relation for a module W.

W is δ_{ss}-lifting $\Longrightarrow$ W is amply δ_{ss}-supplemented $\Longrightarrow$ W is δ_{ss}-supplemented

Theorem 5. The statements given above are equivalent for a projective R-module W.

1. W is δ_{ss}-lifting.
2. W is amply δ_{ss}-supplemented.
3. W is δ_{ss}-supplemented.

Proof. (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3) are clear by Remark 1.
(3) $\Longrightarrow$ (1) is clear by [3, Theorem 5.6], Proposition 6 and Lemma 4.

Corollary 2. The following implications are equivalent for a ring R.

1. ${}_RR$ is δ_{ss}-lifting.
2. ${}_RR$ is amply δ_{ss}-supplemented.
3. ${}_RR$ is δ_{ss}-supplemented.

In [19], a submodule K of $S \leq W$ is named δ-cosmall submodule of S in W if $\frac{S}{K} \ll_\delta \frac{W}{K}$. And S is called δ-*coclosed* if S does not include a proper δ-*cosmall* submodule in W, that is, if there occurs a submodule $K \leq S$ with $\frac{S}{K} \ll_\delta \frac{W}{K}$, this implies $S = K$.

Proposition 8. Any singular δ-coclosed submodule of a δ_{ss}-lifting module is a direct summand.

Proof. Suppose S be any singular δ-coclosed submodule of W. As W is δ_{ss}-lifting, S includes a direct summand K of W with $\frac{S}{K} \leq Soc_\delta(\frac{W}{K})$. Therefore, we have $\frac{S}{K} \ll_\delta \frac{W}{K}$ and so $S = K$ is obtained as S is δ-coclosed.

Proposition 9. Let W be an amply δ_{ss}-supplemented module whose δ_{ss}-supplement submodules are direct summands, then W is a δ_{ss}-lifting module.

Proof. Note that W is δ_{ss}-supplemented as it is amply δ_{ss}-supplemented. Thus, for any $S \leq W$, there occurs a δ_{ss}-supplement K in W satisfying $S + K = W$ and $S \cap K \leq Soc_\delta(K)$. Then, K is of a δ_{ss}-supplement contained in S with $K + L = W$ and $K \cap L \leq Soc_\delta(L)$. From assumption, L is also a direct summand of W for some $D \leq W$, that is, $W = L \oplus D$. Following, we have $S = L \oplus (S \cap D) = L + (S \cap K)$. Let us assume the projection map $\pi : W \longrightarrow \frac{W}{L}$. Then we have $\pi(S \cap K) = \frac{(S \cap K)+L}{L} = \frac{(L+K)\cap S}{L} = \frac{S}{L} \cong S \cap D = \pi(S)$. Therefore, $\pi(S \cap K) \cong S \cap D \leq Soc_\delta(D)$ is got by [1, Lemma 1.3(2)] and [18, Corollary 8.1.5].

Proposition 10. Let W be an amply δ_{ss}-supplemented module whose δ_{ss}-supplement submodules are δ-coclosed. In that case, W is a δ_{ss}-lifting module if and only if each δ_{ss}-supplement submodule of W is a direct summand.

Proof. It is clear by Proposition 8 and Proposition 9.

Proposition 11. A module W is δ_{ss}-lifting iff W is amply δ_{ss}-supplemented and each δ_{ss}-supplement S of W is of a decomposition $S = U \oplus V$ satisfying $U \leq_\oplus W$ and V is projective semisimple.

Proof. $(\Longrightarrow)$: It is clear from the necessity part of Proposition 3.1 given in [20].
$(\Longleftarrow)$: As W is amply δ_{ss}-supplemented, each S of W is of a δ_{ss}-supplement K such that $S + K = W$ and $S \cap K \leq Soc_\delta(K)$. Therefore, there occurs a δ_{ss}-supplement $K^{'}$ of K included in S. In this case $K + K^{'} = W$ and $K \cap K^{'} \leq Soc_\delta(K^{'})$. From assumption, $K^{'}$ is of a decomposition $K^{'} = U \oplus V$ where $U \leq_\oplus W$ and V is projective semisimple. Thus, there occurs some $U^{'} \leq (W)$ with $W = U \oplus U^{'}$. By modular law, as $K' \leq S$ and $U \leq K^{'} \leq S$, we have $S = S \cap W = S \cap (K + K^{'}) = K' + (S \cap K)$ and $S = S \cap W = S \cap (U \oplus U^{'}) = U \oplus (U^{'} \cap S)$. Therefore, for the projection map $\pi : U \oplus U^{'} \longrightarrow U^{'}$, $U^{'} \cap S = \pi(S) = \pi(K') + \pi(S \cap K) = \pi(V) + \pi(S \cap K)$ and $\pi(K') + \pi(S \cap K) \leq Soc_\delta(U^{'})$ since V is projective semisimple and $S \cap K \leq Soc_\delta(K)$ by [[1, Lemma 2.2]; [18, Cor. 8.1.5] and [19, Lemma 1.2]]. Hence, W is δ_{ss}-lifting.

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