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On Tangent Bundles of Submanifolds of a Riemannian Manifold Endowed with a Quarter-Symmetric Non-metric Connection

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Abstract. The object of this article is to study a quarter-symmetric non-metric connection in the tangent bundle and induced metric and connection on a submanifold of co-dimension 2 and hypersurface concerning the quarter-symmetric non-metric connection in the tangent bundle. The Weingarten equations concerning the quarter-symmetric non-metric connection on a submanifold of co-dimension 2 and the hypersurface in the tangent bundle are obtained. Finally, we deduce the Riemannian curvature tensor and Gauss and Codazzi equations on a submanifold of co-dimension 2 and hypersurface of the Riemannian manifold concerning the quarter-symmetric non-metric connection in the tangent bundle.

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1. Introduction

A fundamental concept in differential geometry is a linear connection. A linear connection which is a metric connection and torsion-free is known as Levi-Civita connection. As a generalization of Levi-Civita connection, connection with non-zero torsion was introduced by Hayden in 1932 [20]. On the other hand, S. Golab [19] introduced non-metric linear connection which was symmetric.

A linear connection ∇ is said to be a quarter-symmetric connection if its torsion tensor T satisfies $T(\mathcal{X},\mathcal{Y}) = \eta(\mathcal{Y})\phi\mathcal{X} - \eta(\mathcal{X})\phi\mathcal{Y}$, where $\mathcal{X},\mathcal{Y}$ are arbitrary vector fields, η is a 1-form and ϕ is a (1,1) tensor field. The study of semi-symmetric metric connection on a differentiable manifold M was initiated and developed by Friedmann and Schouten [18] in 1924. It is well known that a linear connection is called semi-symmetric connection if its torsion tensor T is of the form $T(\mathcal{X},\mathcal{Y}) = \omega(\mathcal{Y})\mathcal{X} - \omega(\mathcal{X})\mathcal{Y}$, where the 1-form ω is defined by $\omega(\mathcal{X}) = g(\mathcal{X},\mathcal{U})$ and $\mathcal{U}$ is a vector field. Han at el [21] studied the characteristics of quarter-symmetric metric connections and some invariants under the projective transformation are obtained. The connections such as symmetric, semi-symmetric, quarter-symmetric non-metric connection have been recently discussed by [4, 6, 12–14, 21, 25, 26, 37, 38].

On the other hand, in the foundation of the differentiable geometry of tangent bundles, it is classical to study some geometrical structures and connections deploy natural operations transforming structures and connections on base manifold M_n to its tangent bundle TM_n. Tani introduced the notion of prolongations of surfaces to TM_n and develop

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the theory of the surface prolonged to TM_n [41]. Dida and Hathout investigated Ricci soliton structures on tangent bundles of Riemannian manifolds [15–17]. Numerous researchers have discussed various connections and geometric structures on the tangent bundle and determined their basic geometric properties [1,2,5,11,22,26,27,29,35,36].

Submanifold theory is an important topic in differential geometry. Gauss Codazzi and Weingarten equations are fundamentals of submanifold theory. We have deduce the relation between the connection of the ambient manifold and that of the submanifold in the tangent bundle. Also, we have deduced Weingarten, Gauss and Codazzi equations for a hypersurface of a Riemannian manifold with quarter symmetric non-metric connection in the tangent bundle.

The paper is organized as follows. In Section 2, we give a brief account of tangent bundle, vertical and complete lifts. The Section 3 deals with the study of the submanifold of codimension 2 concerning quarter-symmetric non-metric connection in the tangent bundle. Moreover, We establish Weingarten equations concerning quarter-symmetric non-metric connection on a submanifold of codimension 2 and hypersurface in the tangent bundle in Section 4. Finally, Riemannian curvature tensor, Gauss and Codazzi equations concerning quarter-symmetric non-metric connection on a submanifold of codimension 2 and hypersurface in TM_n are deduced.

2. Preliminaries

Let M_n be n-dimensional differentiable manifold of class C^∞ and TM_n its tangent bundle. Let $\mathcal{X}^C, \eta^C, \phi^C, \nabla^C$ and $\mathcal{X}^V, \eta^V, \phi^V, \nabla^V$ be the complete and vertical lifts of $\mathcal{X}, \eta, \phi, \nabla$, respectively. If $\mathcal{X} = \mathcal{X}^i \frac{\partial}{\partial x^i}$ is a local vector field on M, then its vertical, complete and horizontal lifts in terms of partial differential equations are given by

$$\begin{aligned} \mathcal{X}_1^V &= \mathcal{X}^i \frac{\partial}{\partial y^i}, \\ \mathcal{X}_1^C &= \mathcal{X}^i \frac{\partial}{\partial x^i} + \frac{\partial \mathcal{X}^i}{\partial x^j} y^j \frac{\partial}{\partial y^i}, \\ \mathcal{X}_1^H &= \mathcal{X}^i \frac{\partial}{\partial x^i} - \Gamma^i_{js} y^j \mathcal{X}^s \frac{\partial}{\partial y^i}, \end{aligned}$$

where Γ^i_{js} represent components of the affine connection. Then, [8,9,24,32,42]

$$\begin{gathered} \eta^V(\mathcal{X}^C) = \eta^C(\mathcal{X}^V) = \eta(\mathcal{X})^V, \eta^C(\mathcal{X}^C) = \eta(\mathcal{X})^C, \\ \phi^V \mathcal{X}^C = (\phi\mathcal{X})^V, \phi^C \mathcal{X}^C = (\phi\mathcal{X})^C, \\ [\mathcal{X},\mathcal{Y}]^V = [\mathcal{X}^C,\mathcal{Y}^V] = [\mathcal{X}^V,\mathcal{Y}^C], [\mathcal{X},\mathcal{Y}]^C = [\mathcal{X}^C,\mathcal{Y}^C]. \\ \nabla^C_{\mathcal{X}^C}\mathcal{Y}^C = (\nabla_{\mathcal{X}}\mathcal{Y})^C, \qquad \nabla^C_{\mathcal{X}^C}\mathcal{Y}^V = (\nabla_{\mathcal{X}}\mathcal{Y})^V. \end{gathered}$$

2.1. Vertical and Complete Lifts of $\Im^r_s(M_{n-1}, M_{n+1})$ to TM_{n+1}.

Let $\overline{f}$ be a function on M_{n-1}. The complete lift $\overline{f}^{\overline{C}}$ and vertical lift $\overline{f}^{\overline{V}}$ of $\overline{f}$ to TM_{n+1}. Let U be neighborhood of p in M_{n+1}. Then, the function $\hat{f}$ fits with $\overline{f}$ in $U \cup M_{n+1}$ containing p. If $\overline{\mathcal{X}}$ is an element of $\Im^r_s(M_{n-1}, M_{n+1})$. The vertical lift $\overline{\mathcal{X}}^{\overline{V}}$ to TM_{n+1} is defined by $\overline{\mathcal{X}}^{\overline{V}} \hat{f}^C = (\overline{\mathcal{X}}\hat{f})^{\overline{V}}$ and complete lift $\overline{\mathcal{X}}^{\overline{C}}$ to TM_{n+1} is defined as $\overline{\mathcal{X}}^{\overline{C}} \hat{f}^C = (\overline{\mathcal{X}}\hat{f})^{\overline{C}}$, for each $\hat{f} \in \Im^0_0(M_{n+1})$ along M_{n-1}. Similarly, if $\overline{\eta}$ is an element of $\Im^0_1(M_{n-1}, M_{n+1})$. The vertical lift $\overline{\eta}^{\overline{V}}$ and complete lift $\overline{\omega}^{\overline{C}}$ to TM_{n+1} are defined by $\overline{\eta}^{\overline{V}}(\overline{\mathcal{X}}^{\overline{C}}) = (\overline{\eta}(\overline{\mathcal{X}}))^{\overline{V}}$ and $\overline{\eta}^{\overline{C}}(\overline{\mathcal{X}}^{\overline{C}}) = (\overline{\eta}(\overline{\mathcal{X}}))^{\overline{C}}$ for each $\overline{\mathcal{X}} \in \Im^0_1(M_{n+1})$ respectively. Similarly, $\overline{\phi}^{\overline{V}}(\overline{\mathcal{X}}^{\overline{C}}) = (\overline{\phi}(\overline{\mathcal{X}}))^{\overline{V}}$ and $\overline{\phi}^{\overline{C}}(\overline{\mathcal{X}}^{\overline{C}}) = (\overline{\phi}(\overline{\mathcal{X}}))^{\overline{C}}$ for each $\overline{\mathcal{X}} \in \Im^1_1(M_{n+1})$ respectively [7,23,30].

3. Submanifold of Codimension 2

Let M_{n+1} be a differentiable manifold (dim=$n+1$)and M_{n-1} be the submanifold of M_{n+1} with mapping $\tau : M_{n-1} \to M_{n+1}$. Let B represents differentiability $d\tau$ of τ. Then, the vector field $\mathcal{X}$ belong to M_{n-1} and the vector field $B\mathcal{X}$ to M_{n+1} [3,28,31].

Let $\overset{\bullet}{\nabla}$ Riemannian connection induced on the submanifold M_{n-1} from connection $\tilde{\overset{\bullet}{\nabla}}$ on the enveloping manifold with respect to normals N_1 and N_2, then [3,33]

$$\tilde{\overset{\bullet}{\nabla}}_{B\mathcal{X}} B\mathcal{Y} = B(\overset{\bullet}{\nabla}_{\mathcal{X}}\ \mathcal{Y}) + h(\mathcal{X},\mathcal{Y})N_1 + k(\mathcal{X},\mathcal{Y})N_2, \tag{3.1}$$

where N_1 and N_2 are normal vector fields, $\mathcal{X}, \mathcal{Y}$ vector fields , h and k are the second fundamental tensors of M_{n-1}. Similarly, if ∇ be connection induced on M_{n-1} from the quarter symmetric non-metric connection $\tilde{\nabla}$ on M_{n-1}, then

$$\tilde{\nabla}_{B\mathcal{X}} B\mathcal{Y} = B(\nabla_{\mathcal{X}}\mathcal{Y}) + m(\mathcal{X},\mathcal{Y})N_1 + n(\mathcal{X},\mathcal{Y})N_2, \tag{3.2}$$

m and n being tensor fields of type (0,2) of submanifold M_{n-1} [3].

A quarter-symmetric non-metric connection (briefly, QSNM) $\tilde{\nabla}$ is given by ([10], [11], [40])

$$\tilde{\nabla}_{\tilde{\mathcal{X}}}\tilde{\mathcal{Y}} = \tilde{\overset{\bullet}{\nabla}}_{\tilde{\mathcal{X}}}\tilde{\mathcal{Y}} + \tilde{\eta}(\tilde{\mathcal{Y}})\tilde{\phi}\tilde{\mathcal{X}}, \tag{3.3}$$

for arbitrary vector fields $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$ tangents to M_{n+1}, where $\tilde{\overset{\bullet}{\nabla}}$ be Levi-Civita connection, $\tilde{\eta}$ is a 1-form, $\tilde{\phi}$ is a tensor of type (1,1) [34, 39].

Let TM_{n-1} and TM_{n+1} be the tangent bundles of M_{n-1} and M_{n+1}, respectively. Applying complete lifts on the equation (3.3), we acquire

$$\begin{aligned} \tilde{\nabla}^C_{\tilde{B}\mathcal{X}^C}\tilde{B}\mathcal{Y}^C &= \tilde{\overset{\bullet}{\nabla}}{}^C_{\tilde{B}\mathcal{X}^C}\tilde{B}\mathcal{Y}^C + (\tilde{\eta}(B\mathcal{Y})(B\phi\mathcal{X}))^C \\ &= \tilde{\overset{\bullet}{\nabla}}{}^C_{\tilde{B}\mathcal{X}^C}\tilde{B}\mathcal{Y}^C + (\tilde{\eta}^C(\tilde{B}\mathcal{Y}^C)(\tilde{B}\phi\mathcal{X})^V) + (\tilde{\eta}^V(\tilde{B}\mathcal{Y}^C)(\tilde{B}\phi\mathcal{X})^C), \end{aligned} \tag{3.4}$$

where $\mathcal{X}^C, \mathcal{Y}^C$ are vector fields on TM_{n-1}, $\tilde{\overset{\bullet}{\nabla}}{}^C$, $\tilde{\eta}^C$ and $\tilde{\phi}^C$ are complete lifts of $\tilde{\overset{\bullet}{\nabla}}$, $\tilde{\eta}$, $\tilde{\phi}$, respectively. Now, we are going the prove the following theorem:

Theorem 3.1. *The connection* $\overset{\bullet}{\nabla}{}^C$ *induced on the submanifold* $T(M_{n-1})$ *from* $\tilde{\overset{\bullet}{\nabla}}{}^C$ *of a Riemannian manifold with a QSNM connection is also a QSNM connection.*

Proof: Let $\overset{\bullet}{\nabla}{}^C$ be the induced connection from $\tilde{\overset{\bullet}{\nabla}}{}^C$ on the submanifold $T(M_{n-1})$ from the connection $\tilde{\overset{\bullet}{\nabla}}{}^C$ on the enveloping manifold concerning the unit normals N_1 and N_2 whose complete and vertical lifts are $N_1^{\bar{C}}, N_1^{\bar{V}}, N_2^{\bar{C}}$ and $N_2^{\bar{V}}$ respectively.

Applying complete lifts on equation (3.1), we have

$$\begin{aligned} \tilde{\overset{\bullet}{\nabla}}{}^C_{\tilde{B}\mathcal{X}^C}\tilde{B}\mathcal{Y}^C &= B(\overset{\bullet}{\nabla}{}^C_{\mathcal{X}^C}\mathcal{Y}^C) + h^C(\mathcal{X}^C,\mathcal{Y}^C)N_1^{\bar{V}} + h^V(\mathcal{X}^C,\mathcal{Y}^C)N_1^{\bar{C}} \\ &+ k^C(\mathcal{X}^C,\mathcal{Y}^C)N_2^{\bar{V}} + k^V(\mathcal{X}^C,\mathcal{Y}^C)N_2^{\bar{C}}, \end{aligned} \tag{3.5}$$

where h^C, h^V, k^C and k^V are complete and vertical lifts of h and k respectively on submanifold M_{n-1}.

Applying complete lifts on equation (3.2), we have

$$\begin{aligned} \tilde{\nabla}^C_{\tilde{B}\mathcal{X}^C}\tilde{B}\mathcal{Y}^C &= B(\nabla^C_{\mathcal{X}^C}\mathcal{Y}^C) + m^C(\mathcal{X}^C,\mathcal{Y}^C)N_1^{\bar{V}} + m^V(\mathcal{X}^C,\mathcal{Y}^C)N_1^{\bar{C}} \\ &+ n^C(\mathcal{X}^C,\mathcal{Y}^C)N_2^{\bar{V}} + n^V(\mathcal{X}^C,\mathcal{Y}^C)N_2^{\bar{C}}, \end{aligned} \tag{3.6}$$

where ∇^C be connection induced on submanifold $T(M_{n-1})$ from the QSNM connection $\tilde{\nabla}^C$ on $T(M_{n-1})$ and m^C, m^V, n^C and n^V are complete and vertical lifts of second fundamental tensors m and n of type (0,2) respectively on M_{n-1}.

In the view of equations (3.4), (3.5) and (3.6) , we have

$$\begin{aligned} B(\nabla^C_{\mathcal{X}^C}\mathcal{Y}^C) &+ m^C(\mathcal{X}^C,\mathcal{Y}^C)N_1^{\bar{V}} + m^V(\mathcal{X}^C,\mathcal{Y}^C)N_1^{\bar{C}} + n^C(\mathcal{X}^C,\mathcal{Y}^C)N_2^{\bar{V}} + n^V(\mathcal{X}^C,\mathcal{Y}^C)N_2^{\bar{C}} \\ &= B(\overset{\bullet}{\nabla}{}^C_{\mathcal{X}^C}\mathcal{Y}^C) + h^C(\mathcal{X}^C,\mathcal{Y}^C)N_1^{\bar{V}} + h^V(\mathcal{X}^C,\mathcal{Y}^C)N_1^{\bar{C}} + k^C(\mathcal{X}^C,\mathcal{Y}^C)N_2^{\bar{V}} + k^V(\mathcal{X}^C,\mathcal{Y}^C)N_2^{\bar{C}} \\ &+ (\tilde{\eta}^C(\tilde{B}\mathcal{Y}^C)(\tilde{B}\phi\mathcal{X})^V) + (\tilde{\eta}^V(\tilde{B}\mathcal{Y}^C)(\tilde{B}\phi\mathcal{X})^C). \end{aligned}$$

Comparison of tangential and normal vector fields, we get

$$\nabla^C_{\mathcal{X}^C}\mathcal{Y}^C = \overset{\bullet}{\nabla}{}^C_{\mathcal{X}^C}\mathcal{Y}^C + \tilde{\eta}^C(\tilde{B}\mathcal{Y}^C)(\tilde{B}\phi\mathcal{X})^V + \tilde{\eta}^V(\tilde{B}\mathcal{Y}^C)(\tilde{B}\phi\mathcal{X})^C$$

and

$$
\begin{aligned}
m^C(\mathcal{X}^C,\mathcal{Y}^C) &= h^C(\mathcal{X}^C,\mathcal{Y}^C),\\
m^V(\mathcal{X}^C,\mathcal{Y}^C) &= h^V(\mathcal{X}^C,\mathcal{Y}^C),\\
n^C(\mathcal{X}^C,\mathcal{Y}^C) &= k^C(\mathcal{X}^C,\mathcal{Y}^C),\\
n^V(\mathcal{X}^C,\mathcal{Y}^C) &= k^V(\mathcal{X}^C,\mathcal{Y}^C).
\end{aligned}
$$

Thus,

$$\nabla^C_{\mathcal{X}^C}\mathcal{Y}^C - \nabla^C_{\mathcal{Y}^C}\mathcal{X}^C - [\mathcal{X}^C,\mathcal{Y}^C] = \tilde{\eta}^C(\tilde{B}\mathcal{Y}^C)(\tilde{B}\phi\mathcal{X})^V + \tilde{\eta}^V(\tilde{B}\mathcal{Y}^C)(\tilde{B}\phi\mathcal{X})^C - \tilde{\eta}^C(\tilde{B}\mathcal{X}^C)(\tilde{B}\phi\mathcal{Y})^V - \tilde{\eta}^V(\tilde{B}\mathcal{X}^C)(\tilde{B}\phi\mathcal{Y})^C. \quad (3.7)$$

Hence, the connection ∇^C induced on $T(M_{n-1})$ is the QSNM connection. Hence, the proof is completed.

Let M_{n+1} be a differentiable manifold (dim=$n+1$) and M_n be hypersurface in M_{n+1} by immersion $\tau : M_{n+1} \to M_n$ and by B the mapping induced by τ from $T(M_n)$ to $T(M_{n+1})$, where $T(M_n)$ and $T(M_{n+1})$ denote tangent bundles of manifold M_n and M_{n+1} respectively.

Corollary 3.2. *The connection induced on the hypersurface $T(M_n)$ of a Riemannian manifold with a QSNM connection with the unit normals $N^{\bar{C}}$ and $N^{\bar{V}}$ is also a QSNM connection.*

Proof: Let $\overset{\bullet}{\nabla}{}^C$ be the induced connection from $\tilde{\overset{\bullet}{\nabla}}{}^C$ on $T(M_n)$ concerning the unit normal N whose complete and vertical lifts are $N^{\bar{C}}$ and $N^{\bar{V}}$. Making use of equation (3.1), we have

$$\tilde{\overset{\bullet}{\nabla}}{}^C_{\tilde{B}\mathcal{X}^C}\tilde{B}\mathcal{Y}^C = B(\overset{\bullet}{\nabla}{}^C_{\mathcal{X}^C}\mathcal{Y}^C) + h^C(\mathcal{X}^C,\mathcal{Y}^C)N^{\bar{V}} + h^V(\mathcal{X}^C,\mathcal{Y}^C)N^{\bar{C}}, \quad (3.8)$$

where $\mathcal{X}^C,\mathcal{Y}^C$ are vector fields on TM_n and h is the second fundamental tensor of the hypersurface M_n whose complete and vertical lifts are h^C and h^V respectively on $T(M_n)$.

Let ∇^C be connection induced on hyprsurface from $\tilde{\nabla}^C$ concerning the unit normal N whose complete and vertical lifts are $N^{\bar{C}}$ and $N^{\bar{V}}$. From equation (3.2), we get

$$\tilde{\nabla}^C_{\tilde{B}\mathcal{X}^C}\tilde{B}\mathcal{Y}^C = B(\nabla^C_{\mathcal{X}^C}\mathcal{Y}^C) + m^C(\mathcal{X}^C,\mathcal{Y}^C)N^{\bar{V}} + m^V(\mathcal{X}^C,\mathcal{Y}^C)N^{\bar{C}}, \quad (3.9)$$

where m^C and m^V are complete and vertical lifts of m of on M_n.

From equation (3.4), we have

$$
\begin{aligned}
B(\tilde{\nabla}^C_{\mathcal{X}^C}\mathcal{Y}^C) &= \tilde{\overset{\bullet}{\nabla}}{}^C_{\tilde{B}\mathcal{X}^C}\tilde{B}\mathcal{Y}^C + (\tilde{\eta}^C(\tilde{B}\mathcal{Y}^C)(\tilde{B}\phi\mathcal{X})^V)\\
&+ (\tilde{\eta}^V(\tilde{B}\mathcal{Y}^C)(\tilde{B}\phi\mathcal{X})^C).
\end{aligned}
$$

Using equations (3.8) and (3.9) in the above equation, we get

$$
\begin{aligned}
B(\nabla^C_{\mathcal{X}^C}\mathcal{Y}^C) &+ m^C(\mathcal{X}^C,\mathcal{Y}^C)N^{\bar{V}} + m^V(\mathcal{X}^C,\mathcal{Y}^C)N^{\bar{C}}\\
&= B(\overset{\bullet}{\nabla}{}^C_{\mathcal{X}^C}\mathcal{Y}^C) + h^C(\mathcal{X}^C,\mathcal{Y}^C)N^{\bar{V}} + h^V(\mathcal{X}^C,\mathcal{Y}^C)N^{\bar{C}} + (\tilde{\eta}^C(\tilde{B}\mathcal{Y}^C)(\tilde{B}\phi\mathcal{X})^V) + (\tilde{\eta}^V(\tilde{B}\mathcal{Y}^C)(\tilde{B}\phi\mathcal{X})^C).
\end{aligned}
$$

Comparison of tangential and normal vector fields, we get

$$
\begin{aligned}
\nabla^C_{\mathcal{X}^C}\mathcal{Y}^C &= \overset{\bullet}{\nabla}{}^C_{\mathcal{X}^C}\mathcal{Y}^C + \tilde{\eta}^C(\tilde{B}\mathcal{Y}^C)(\tilde{B}\phi\mathcal{X})^V + \tilde{\eta}^V(\tilde{B}\mathcal{Y}^C)(\tilde{B}\phi\mathcal{X})^C,\\
m^C(\mathcal{X}^C,\mathcal{Y}^C) &= h^C(\mathcal{X}^C,\mathcal{Y}^C),\\
m^V(\mathcal{X}^C,\mathcal{Y}^C) &= h^V(\mathcal{X}^C,\mathcal{Y}^C).
\end{aligned}
$$

Thus,

$$\nabla^C_{\mathcal{X}^C}\mathcal{Y}^C - \nabla^C_{\mathcal{Y}^C}\mathcal{X}^C - [\mathcal{X}^C,\mathcal{Y}^C] = \tilde{\eta}^C(\tilde{B}\mathcal{Y}^C)(\tilde{B}\phi\mathcal{X})^V + \tilde{\eta}^V(\tilde{B}\mathcal{Y}^C)(\tilde{B}\phi\mathcal{X})^C - \tilde{\eta}^C(\tilde{B}\mathcal{X}^C)(\tilde{B}\phi\mathcal{Y})^V - \tilde{\eta}^V(\tilde{B}\mathcal{X}^C)(\tilde{B}\phi\mathcal{Y})^C. \quad (3.10)$$

Thus, the connection ∇^C induced on TM_n is QSNM connection. Hence, the proof is completed.

4. Weingarten Equations for the QSNM Connection

In this section, we shall investigate Weingarten Equations concerning the QSNM connection $\tilde{\nabla}^C$ on the submanifold TM_{n-1} in TM_{n+1}.

The Weingarten equations for the Riemannian connection $\overset{\bullet}{\nabla}{}^C$ are given by

$$\begin{aligned}
(a)\ \overset{\bullet}{\nabla}{}^C_{\tilde{B}\mathcal{X}^C}\ N_1^{\bar{V}} &= -\tilde{B}H^V\mathcal{X}^C + l(\mathcal{X}^C)N_2^{\bar{V}},\\
(b)\ \overset{\bullet}{\nabla}{}^C_{\tilde{B}\mathcal{X}^C}\ N_1^{\bar{C}} &= -\tilde{B}H^C\mathcal{X}^C + l(\mathcal{X}^C)N_2^{\bar{C}},\\
(c)\ \overset{\bullet}{\nabla}{}^C_{\tilde{B}\mathcal{X}^C}\ N_2^{\bar{V}} &= -\tilde{B}K^V\mathcal{X}^C + l(\mathcal{X}^C)N_1^{\bar{V}},\\
(d)\ \overset{\bullet}{\nabla}{}^C_{\tilde{B}\mathcal{X}^C}\ N_2^{\bar{C}} &= -\tilde{B}K^C\mathcal{X}^C + l(\mathcal{X}^C)N_1^{\bar{C}},
\end{aligned} \tag{4.1}$$

where H^C, H^V, K^C and K^V are complete and vertical lifts of tensor fields H and K of type (1,1) such that

$$\begin{aligned}
(a)\tilde{g}^C(H^C\mathcal{X}^C,\mathcal{Y}^C) &= h^C(\mathcal{X}^C,\mathcal{Y}^C),\\
(b)\tilde{g}^C(K^C\mathcal{X}^C,\mathcal{Y}^C) &= k^C(\mathcal{X}^C,\mathcal{Y}^C),\\
(c)\tilde{g}^V(H^V\mathcal{X}^C,\mathcal{Y}^C) &= h^V(\mathcal{X}^C,\mathcal{Y}^C),\\
(d)\tilde{g}^V(K^V\mathcal{X}^C,\mathcal{Y}^C) &= k^V(\mathcal{X}^C,\mathcal{Y}^C).
\end{aligned}$$

Replacing $\tilde{B}\mathcal{Y}^C$ by $N^{\bar{C}}$ in equation (3.4), we have

$$\tilde{\nabla}^C_{\tilde{B}\mathcal{X}^C}N_1^{\bar{C}} = \tilde{\overset{\bullet}{\nabla}}{}^C_{\tilde{B}\mathcal{X}^C}N_1^{\bar{C}} + \tilde{\eta}^C(\phi N_1)^{\bar{C}}(\tilde{B}\mathcal{X}^V) + \tilde{\eta}^V(\phi N_1)^{\bar{C}}(\tilde{B}\mathcal{X}^C).$$

Setting $\phi N_1 = B\xi_1, \xi_1$ is a vector field on M_n.

$$\tilde{\nabla}^C_{\tilde{B}\mathcal{X}^C}N_1^{\bar{C}} = \tilde{\overset{\bullet}{\nabla}}{}^C_{\tilde{B}\mathcal{X}^C}N_1^{\bar{C}} + \tilde{B}(\eta^C(\xi_1^C)\mathcal{X}^V + \eta^V(\xi_1^C)\mathcal{X}^C). \tag{4.2}$$

Making use of equations (4.1a) and (4.2), we get

$$\begin{aligned}
\tilde{\nabla}^C_{\tilde{B}\mathcal{X}^C}N_1^{\bar{C}} &= -\tilde{B}H^C\mathcal{X}^C + l(\mathcal{X}^C)N_2^{\bar{C}} + \tilde{B}(\eta^C(\xi_1^C)\mathcal{X}^V + \eta^V(\xi_1^C)\mathcal{X}^C)\\
&= \tilde{B}M_1\mathcal{X}^C + l(\mathcal{X}^C)N_2^{\bar{C}},
\end{aligned} \tag{4.3}$$

where $M_1^C\mathcal{X}^C = -H^C\mathcal{X}^C + \eta^C(\xi_1^C)\mathcal{X}^V + \eta^V(\xi_1^C)\mathcal{X}^C$

$$\begin{aligned}
\tilde{\nabla}^C_{\tilde{B}\mathcal{X}^C}N_2^{\bar{C}} &= -\tilde{B}K^C\mathcal{X}^C + l(\mathcal{X}^C)N_1^{\bar{C}} + \tilde{B}(\eta^C(\xi_2^C)\mathcal{X}^V + \eta^V(\xi_2^C)\mathcal{X}^C)\\
&= \tilde{B}M\mathcal{X}^C + l(\mathcal{X}^C)N_1^{\bar{C}},
\end{aligned} \tag{4.4}$$

where $M_2^C\mathcal{X}^C = -K^C\mathcal{X}^C + \eta^C(\xi_2^C)\mathcal{X}^V + \eta^V(\xi_2^C)\mathcal{X}^C$.

Similarly,

$$\begin{aligned}
\tilde{\nabla}^C_{\tilde{B}\mathcal{X}^C}N_1^{\bar{V}} &= -\tilde{B}H^V\mathcal{X}^C + l(\mathcal{X}^C)N_2^{\bar{V}} + \tilde{B}(\eta^C(\xi_1^C))\mathcal{X}^V\\
&= \tilde{B}M_1\mathcal{X}^C + l(\mathcal{X}^C)N_2^{\bar{V}},
\end{aligned} \tag{4.5}$$

where $M_1^V\mathcal{X}^C = -H^V\mathcal{X}^C + \eta^C(\xi_1^C)\mathcal{X}^V$

$$\begin{aligned}
\tilde{\nabla}^C_{\tilde{B}\mathcal{X}^C}N_2^{\bar{V}} &= -\tilde{B}K^V\mathcal{X}^C + l(\mathcal{X}^C)N_1^{\bar{V}} + \tilde{B}(\eta^C(\xi_1^C))\mathcal{X}^V\\
&= \tilde{B}M_2\mathcal{X}^C + l(\mathcal{X}^C)N_1^{\bar{V}},
\end{aligned} \tag{4.6}$$

where $M_2^V\mathcal{X}^C = -H^V\mathcal{X}^C + \eta^C(\xi_1^C)\mathcal{X}^V$.

The equations (4.3), (4.4), (4.5) and (4.6) are Weingarten equations concerning the QSNM connection. We have the following theorem:

Theorem 4.1. *The connection $\overset{\bullet}{\nabla}{}^C$ induced on the submanifold $T(M_{n-1})$ from $\tilde{\overset{\bullet}{\nabla}}{}^C$ of a Riemannian manifold with a QSNM connection. The Weingarten equations concerning the QSNM connection are given by (4.3), (4.4), (4.5) and (4.6).*

Corollary 4.2. *The Weingarten equations concerning the QSNM connection $\tilde{\nabla}^C$ on TM_n in TM_{n+1} are given by (4.9) and (4.11).*

Proof: The Weingarten equations are given in the following form

$$\overset{\bullet}{\nabla}{}^C_{\tilde{B}\mathcal{X}^C} N^{\bar{V}} = -\tilde{B}H^V\mathcal{X}^C,$$
$$\overset{\bullet}{\nabla}{}^C_{\tilde{B}\mathcal{X}^C} N^{\bar{C}} = -\tilde{B}H^C\mathcal{X}^C, \tag{4.7}$$

where H is a tensor field of type (1,1) of M_n defined by

$$\tilde{g}^C(H^C\mathcal{X}^C, \mathcal{Y}^C) = h^C(\mathcal{X}^C, \mathcal{Y}^C),$$
$$\tilde{g}^V(H^V\mathcal{X}^C, \mathcal{Y}^C) = h^V(\mathcal{X}^C, \mathcal{Y}^C).$$

Replacing $\tilde{B}\mathcal{Y}^C$ by $N^{\bar{C}}$ in equation (3.4), we have

$$\tilde{\nabla}^C_{\tilde{B}\mathcal{X}^C} N^{\bar{C}} = \overset{\tilde{\bullet}}{\nabla}{}^C_{\tilde{B}\mathcal{X}^C} N^{\bar{C}} + \tilde{\eta}^C(\phi N)^{\bar{C}}(\tilde{B}\mathcal{X}^V) + \tilde{\eta}^V(\phi N)^{\bar{C}}(\tilde{B}\mathcal{X}^C).$$

Setting $\phi N = B\xi$, where ξ is a vector field on M_n.

$$\tilde{\nabla}^C_{\tilde{B}\mathcal{X}^C} N^{\bar{C}} = \overset{\tilde{\bullet}}{\nabla}{}^C_{\tilde{B}\mathcal{X}^C} N^{\bar{C}} + \tilde{B}((\eta^C\xi^C)B\mathcal{X}^V + (\eta^V\xi^C)B\mathcal{X}^C). \tag{4.8}$$

Using equation (4.7) in equation (4.8), we get

$$\tilde{\nabla}^C_{\tilde{B}\mathcal{X}^C} N^{\bar{C}} = -\tilde{B}H^C\mathcal{X}^C + \tilde{B}(\eta^C(\xi^C)\mathcal{X}^V + \eta^V(\xi^C)\mathcal{X}^C)$$
$$\tilde{\nabla}^C_{\tilde{B}\mathcal{X}^C} N^{\bar{C}} lde BM^C\mathcal{X}^C, \tag{4.9}$$

where

$$M^C\mathcal{X}^C = -H^C\mathcal{X}^C + \eta^C(\xi^C)\mathcal{X}^V - \eta^V(\xi^C)\mathcal{X}^C,$$

for arbitrary vector field $\mathcal{X}$ on M_n.

Similarly,

$$\tilde{\nabla}^C_{\tilde{B}\mathcal{X}^C} N^{\bar{V}} = \overset{\tilde{\bullet}}{\nabla}{}^C_{\tilde{B}\mathcal{X}^C} N^{\bar{V}} + \tilde{B}(\eta^C\xi^C)\mathcal{X}^V. \tag{4.10}$$

Using equation (4.7) in equation (4.10), we get

$$\tilde{\nabla}^C_{\tilde{B}\mathcal{X}^C} N^{\bar{V}} = -\tilde{B}H^V\mathcal{X}^C + \tilde{B}(\eta^C(\xi^C))\mathcal{X}^V$$
$$= \tilde{B}M^V\mathcal{X}^C, \tag{4.11}$$

where

$$M^V\mathcal{X}^C = -H^V\mathcal{X}^C + \eta^C(\xi^C)\mathcal{X}^V.$$

The equations (4.9) and (4.11) are Weingarten equations concerning the QSNM connection on TM_n in TM_{n+1}. Hence, the proof of corollary is completed.

5. Riemannian Curvature Tensor and Gauss and Codazzi Equations for the QSNM Connection

In this section, we shall investigate Riemannian curvature and equations of Gauss and Codazzi concerning the QSNM connection on TM_{n-1} in TM_{n+1}.

Let $\tilde{K}^C$ and K^C be the curvature tensors of TM_{n+1} and TM_{n-1} concerning the mathematical operators $\overset{\tilde{\bullet}}{\nabla}{}^C$ and $\overset{\bullet}{\nabla}{}^C$ respectively. Thus,

$$\tilde{K}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C)\tilde{B}\mathcal{Z}^C = \overset{\tilde{\bullet}}{\nabla}{}^C_{\tilde{B}\mathcal{X}^C}\overset{\tilde{\bullet}}{\nabla}{}^C_{\tilde{B}\mathcal{Y}^C}\tilde{B}\mathcal{Z}^C - \overset{\tilde{\bullet}}{\nabla}{}^C_{\tilde{B}\mathcal{Y}^C}\overset{\tilde{\bullet}}{\nabla}{}^C_{\tilde{B}\mathcal{X}^C}\tilde{B}\mathcal{Z}^C - \overset{\tilde{\bullet}}{\nabla}{}^C_{[\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C]}\tilde{B}\mathcal{Z}^C$$

and

$$K^C(\mathcal{X}^C, \mathcal{Y}^C)\mathcal{Z}^C = \nabla^C_{\mathcal{X}^C}\nabla^C_{\mathcal{Y}^C}\mathcal{Z}^C - \nabla^C_{\mathcal{Y}^C}\nabla^C_{\mathcal{X}^C}\mathcal{Z}^C - \nabla^C_{[\mathcal{X},\mathcal{Y}]^C}\mathcal{Z}^C.$$

Then, the equation of Gauss is given by

$$\tilde{K}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C, \tilde{B}\mathcal{Z}^C, \tilde{B}\mathcal{U}^C) = K^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C, \tilde{B}\mathcal{Z}^C, \tilde{B}\mathcal{U}^C) + h^V(\mathcal{X}^C, \mathcal{U}^C)h^C(\mathcal{Y}^C, \mathcal{Z}^C) + h^C(\mathcal{X}^C, \mathcal{U}^C)h^V(\mathcal{Y}^C, \mathcal{Z}^C)$$
$$- h^V(\mathcal{Y}^C, \mathcal{U}^C)h^C(\mathcal{X}^C, \mathcal{Z}^C) - h^C(\mathcal{Y}^C, \mathcal{U}^C)h^V(\mathcal{X}^C, \mathcal{Z}^C),$$

where $\tilde{K}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C, \tilde{B}\mathcal{Z}^C, \tilde{B}\mathcal{U}^C) = \tilde{g}^C(\tilde{K}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C, \tilde{B}\mathcal{Z}^C, \tilde{B}\mathcal{U}^C)$ and the similar expression for $K^C(\mathcal{X}^C, \mathcal{Y}^C, \mathcal{Z}^C, \mathcal{U}^C)$ for TM_{n-1}.

The equation of Codazzi is given by

$$\begin{aligned}
\tilde{K}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C)N^{\bar{V}} &= \tilde{B}(\overset{\bullet}{\nabla}{}^C_{\mathcal{X}^C} H^V\mathcal{Y}^C - \overset{\bullet}{\nabla}{}^C_{\mathcal{X}^C} H^V\mathcal{X}^C),\\
\tilde{K}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C)N^{\bar{C}} &= \tilde{B}(\overset{\bullet}{\nabla}{}^C_{\mathcal{X}^C} H^C\mathcal{Y}^C - \overset{\bullet}{\nabla}{}^C_{\mathcal{X}^C} H^C\mathcal{X}^C),\\
\tilde{K}^C(N^{\bar{V}}, N^{\bar{C}})\tilde{B}\mathcal{X}^C &= 0.
\end{aligned}$$

Let $\tilde{R}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C)\tilde{B}\mathcal{Z}^C$ be the Riemannian curvature tensor field of the enveloping manifold TM_{n+1} concerning the QSNM connection $\tilde{\nabla}^C$, then

$$\tilde{R}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C)\tilde{B}\mathcal{Z}^C = \tilde{\nabla}^C_{\tilde{B}\mathcal{X}^C}\tilde{\nabla}^C_{\tilde{B}\mathcal{Y}^C}\tilde{B}\mathcal{Z}^C - \tilde{\nabla}^C_{\tilde{B}\mathcal{Y}^C}\tilde{\nabla}^C_{\tilde{B}\mathcal{X}^C}\tilde{B}\mathcal{Z}^C - \tilde{\nabla}^C_{[\tilde{B}\mathcal{X}^C,\tilde{B}\mathcal{Y}^C]}\tilde{B}\mathcal{Z}^C.$$

Using the equations (3.6), (4.3), (4.4), (4.5), (4.6) and (3.7), we get

$$\begin{aligned}
\tilde{R}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C)\tilde{B}\mathcal{Z}^C &= B\{R^C(\mathcal{X}^C, \mathcal{Y}^C)\mathcal{Z}^C + m^V\{(-\tilde{\eta}(\mathcal{Y}))^V(\phi\mathcal{X})^V + (\tilde{\eta}(\mathcal{X}))^V(\phi\mathcal{Y})^V, \mathcal{Z}^C)\}N_1^{\bar{C}}\\
&+ m^V\{-(\tilde{\eta}(\mathcal{Y}))^V(\phi\mathcal{X})^C - \tilde{\eta}(\mathcal{Y}))^C(\phi\mathcal{X})^V + (\tilde{\eta}(\mathcal{X}))^V(\phi\mathcal{Y})^C + \tilde{\eta}(\mathcal{X}))^V(\phi\mathcal{Y})^C, \mathcal{Z}^C)\}N_1^{\bar{V}}\\
&+ m^V\{-(\tilde{\eta}(\mathcal{Y}))^V(\phi\mathcal{X})^V + (\tilde{\eta}(\mathcal{X}))^V(\phi\mathcal{Y})^V, \mathcal{Z}^C)\}N_1^{\bar{V}} + B\{-m^V(\mathcal{X}^C, \mathcal{Z}^C)H^C\mathcal{Y}^C\\
&- m^C(\mathcal{X}^C, \mathcal{Z}^C)H^V\mathcal{Y}^C + m^V(\mathcal{Y}^C, \mathcal{Z}^C)H^C\mathcal{X}^C + m^C(\mathcal{Y}^C, \mathcal{Z}^C)H^V\mathcal{X}^C\\
&- n^V(\mathcal{X}^C, \mathcal{Z}^C)H^C\mathcal{Y}^C - n^C(\mathcal{X}^C, \mathcal{Z}^C)H^V\mathcal{Y}^C + n^V(\mathcal{Y}^C, \mathcal{Z}^C)H^C\mathcal{X}^C + n^C(\mathcal{Y}^C, \mathcal{Z}^C)H^V\mathcal{X}^C\}\\
&- B\{m^V(\mathcal{X}^C, \mathcal{Z}^C)\tilde{\eta}^C(\mathcal{Y}^C) + m^C(\mathcal{X}^C, \mathcal{Z}^C)\tilde{\eta}^V(\mathcal{Y}^C) - m^V(\mathcal{Y}^C, \mathcal{Z}^C)\tilde{\eta}^C(\mathcal{X}^C)\\
&- m^C(\mathcal{Y}^C, \mathcal{Z}^C)\tilde{\eta}^V(\mathcal{X}^C)\}\xi_1^V - B\{m^V(\mathcal{X}^C, \mathcal{Z}^C)\tilde{\eta}^V(\mathcal{Y}^C) - m^V(\mathcal{Y}^C, \mathcal{Z}^C)\tilde{\eta}^V(\mathcal{X}^C)\}\xi_1^C\\
&- B\{n^V(\mathcal{X}^C, \mathcal{Z}^C)\tilde{\eta}^C(\mathcal{Y}^C) + n^C(\mathcal{X}^C, \mathcal{Z}^C)\tilde{\eta}^V(\mathcal{Y}^C) - n^V(\mathcal{Y}^C, \mathcal{Z}^C)\tilde{\eta}^C(\mathcal{X}^C) - n^C(\mathcal{Y}^C, \mathcal{Z}^C)\tilde{\eta}^V(\mathcal{X}^C)\}\xi_2^V\\
&- B\{n^V(\mathcal{X}^C, \mathcal{Z}^C)\tilde{\eta}^V(\mathcal{Y}^C) - n^V(\mathcal{Y}^C, \mathcal{Z}^C)\tilde{\eta}^V(\mathcal{X}^C)\}\xi_2^C + n^V\{-(\tilde{\eta}(\mathcal{X}))^V(\phi\mathcal{Y})^V\\
&+ (\tilde{\eta}(\mathcal{Y}))^V(\phi\mathcal{X})^V, \mathcal{Z}^C)\}N_2^{\bar{C}} + n^V\{-(\tilde{\eta}(\mathcal{X}))^V(\phi\mathcal{Y})^C - (\tilde{\eta}(\mathcal{X}))^C(\phi\mathcal{Y})^V + (\tilde{\eta}(\mathcal{Y}))^V(\phi\mathcal{X})^C\\
&+ (\tilde{\eta}(\mathcal{Y}))^C(\phi\mathcal{X})^V, \mathcal{Z}^C)\}N_2^{\bar{V}}\\
&+ n^C\{-(\tilde{\eta}(\mathcal{X}))^V(\phi\mathcal{Y})^V + (\tilde{\eta}(\mathcal{Y}))^V(\phi\mathcal{X})^V, \mathcal{Z}^C)\}N_2^{\bar{V}} + \{(\nabla^C_{\mathcal{X}^C}m^V)(\mathcal{Y}^C, \mathcal{Z}^C) - (\nabla^C_{\mathcal{Y}^C}m^V)(\mathcal{X}^C, \mathcal{Z}^C)\}N_1^{\bar{C}}\\
&+ \{(\nabla^C_{\mathcal{X}^C}m^C)(\mathcal{Y}^C, \mathcal{Z}^C) - (\nabla^C_{\mathcal{Y}^C}m^C)(\mathcal{X}^C, \mathcal{Z}^C)\}N_1^{\bar{V}} + \{(\nabla^C_{\mathcal{X}^C}n^V)(\mathcal{Y}^C, \mathcal{Z}^C) - (\nabla^C_{\mathcal{Y}^C}n^V)(\mathcal{X}^C, \mathcal{Z}^C)\}N_2^{\bar{C}}\\
&+ \{(\nabla^C_{\mathcal{X}^C}n^C)(\mathcal{Y}^C, \mathcal{Z}^C) - (\nabla^C_{\mathcal{Y}^C}n^C)(\mathcal{X}^C, \mathcal{Z}^C)\}N_2^{\bar{V}} + l(\mathcal{X}^C)\{m^C(\mathcal{Y}^C, \mathcal{Z}^C)N_2^{\bar{V}} + m^V(\mathcal{Y}^C, \mathcal{Z}^C)N_2^{\bar{C}}\\
&- n^C(\mathcal{Y}^C, \mathcal{Z}^C)N_1^{\bar{V}} - n^V(\mathcal{Y}^C, \mathcal{Z}^C)N_1^{\bar{C}}\} - l(\mathcal{Y}^C)\{m^C(\mathcal{X}^C, \mathcal{Z}^C)N_2^{\bar{V}} + m^V(\mathcal{X}^C, \mathcal{Z}^C)N_2^{\bar{C}}\\
&- n^C(\mathcal{X}^C, \mathcal{Z}^C)N_1^{\bar{V}} - n^V(\mathcal{X}^C, \mathcal{Z}^C)N_1^{\bar{C}}\},
\end{aligned} \tag{5.1}$$

where $R^C(\mathcal{X}^C, \mathcal{Y}^C)\mathcal{Z}^C$ being the Riemannian curvature tensor of the submanifold TM_{n-1} with QSNM connection ∇^C. Thus:

Theorem 5.1. *Let $R^C(\mathcal{X}^C, \mathcal{Y}^C)\mathcal{Z}^C$ be the Riemannian curvature tensor of submanifold TM_{n-1} with QSNM connection ∇^C, then the Riemannian curvature tensor $\tilde{R}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C)\tilde{B}\mathcal{Z}^C$ $\tilde{\nabla}^C$ of the enveloping manifold TM_{n+1} with the QSNM connection $\tilde{\nabla}^C$ is given by equation (5.1).*

Substituting

$$\tilde{R}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C, \tilde{B}\mathcal{Z}^C, \tilde{B}\mathcal{U}^C) = \tilde{g}^C(\tilde{R}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C)\tilde{B}\mathcal{Z}^C, \tilde{B}\mathcal{U}^C)$$

and

$$R^C(\mathcal{X}^C, \mathcal{Y}^C, \mathcal{Z}^C, \mathcal{U}^C) = g^C(R^C(\mathcal{X}^C, \mathcal{Y}^C)\mathcal{Z}^C, \mathcal{U}^C).$$

Then, from equation (5.1), we can easily show that

$$\begin{aligned}
\hat{R}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C, \tilde{B}\mathcal{Z}^C, \tilde{B}\mathcal{U}^C) &= R^C(\mathcal{X}^C, \mathcal{Y}^C, \mathcal{Z}^C, \mathcal{U}^C) - m^C(\mathcal{X}^C, \mathcal{Z}^C)g(H^V\mathcal{Y}^C, \mathcal{U}^C)\\
&+ m^V(\mathcal{Y}^C, \mathcal{Z}^C)g(H^C\mathcal{X}^C, \mathcal{U}^C) + m^C(\mathcal{Y}^C, \mathcal{Z}^C)g(H^V\mathcal{X}^C, \mathcal{U}^C),
\end{aligned} \tag{5.2}$$

$$
\begin{aligned}
\tilde{R}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C, \tilde{B}\mathcal{Z}^C, N^{\bar{C}}) &= m^V(\mathcal{X}^C, \mathcal{Z}^C)\tilde{\eta}^C(\mathcal{Y}^C) + m^C(\mathcal{X}^C, \mathcal{Z}^C)\tilde{\eta}^V\mathcal{Y}^C - m^V(\mathcal{Y}^C, \mathcal{Z}^C)\tilde{\eta}^C(\mathcal{X}^C) \\
&- m^C(\mathcal{Y}^C, \mathcal{Z}^C)\tilde{\eta}^V(\mathcal{X}^C) + m^V\{-(\tilde{\eta}(\mathcal{X}))^V(\phi\mathcal{Y})^C - (\tilde{\eta}(\mathcal{X}))^C(\phi\mathcal{Y})^V \\
&+ (\tilde{\eta}(\mathcal{Y}))^V(\phi\mathcal{X})^C + (\tilde{\eta}(\mathcal{Y}))^C(\phi\mathcal{X})^V, \mathcal{Z}^C)\} \\
&+ m^C\{-(\tilde{\eta}(\mathcal{X}))^V(\phi\mathcal{Y})^V + (\tilde{\eta}(\mathcal{Y}))^V(\phi\mathcal{X})^V, \mathcal{Z}^C)\}
\end{aligned} \tag{5.3}
$$

The equations (5.2) and (5.3) are equations of Gauss and Codazzi concerning the QSNM connection. We have the following theorem:

Theorem 5.2. *Let $\tilde{K}^C$ and K^C be the curvature tensors of TM_{n+1} and TM_{n-1} concerning $\overset{\tilde{\bullet}}{\nabla}^C$ and $\overset{\bullet}{\nabla}^C$ respectively. The Gauss and Codazzi equations endowed with the QSNM connection are given by equations (5.2) and (5.3).*

Let $\tilde{K}^C$ and K^C be the curvature tensors of TM_n and TM_{n+1} concerning the mathematical operators $\overset{\tilde{\bullet}}{\nabla}^C$ and $\overset{\bullet}{\nabla}^C$ respectively. Thus,

$$
\tilde{K}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C)\tilde{B}\mathcal{Z}^C = \overset{\tilde{\bullet}}{\nabla}^C_{\tilde{B}\mathcal{X}^C}\overset{\tilde{\bullet}}{\nabla}^C_{\tilde{B}\mathcal{Y}^C}\tilde{B}\mathcal{Z}^C - \overset{\tilde{\bullet}}{\nabla}^C_{\tilde{B}\mathcal{Y}^C}\overset{\tilde{\bullet}}{\nabla}^C_{\tilde{B}\mathcal{X}^C}\tilde{B}\mathcal{Z}^C - \overset{\tilde{\bullet}}{\nabla}^C_{[\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C]}\tilde{B}\mathcal{Z}^C
$$

and

$$
K^C(\mathcal{X}^C, \mathcal{Y}^C)\mathcal{Z}^C = \nabla^C_{\mathcal{X}^C}\nabla^C_{\mathcal{Y}^C}\mathcal{Z}^C - \nabla^C_{\mathcal{Y}^C}\nabla^C_{\mathcal{X}^C}\mathcal{Z}^C - \nabla^C_{[\mathcal{X},\mathcal{Y}]^C}\mathcal{Z}^C.
$$

Then, the equation of Gauss is given by

$$
\begin{aligned}
\tilde{K}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C, \tilde{B}\mathcal{Z}^C, \tilde{B}\mathcal{U}^C) &= K^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C, \tilde{B}\mathcal{Z}^C, \tilde{B}\mathcal{U}^C) + h^V(\mathcal{X}^C, \mathcal{U}^C)h^C(\mathcal{Y}^C, \mathcal{Z}^C) \\
&+ h^C(\mathcal{X}^C, \mathcal{U}^C)h^V(\mathcal{Y}^C, \mathcal{Z}^C) - h^V(\mathcal{Y}^C, \mathcal{U}^C)h^C(\mathcal{X}^C, \mathcal{Z}^C) \\
&- h^C(\mathcal{Y}^C, \mathcal{U}^C)h^V(\mathcal{X}^C, \mathcal{Z}^C),
\end{aligned}
$$

where $\tilde{K}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C, \tilde{B}\mathcal{Z}^C, \tilde{B}\mathcal{U}^C) = \tilde{g}^C(\tilde{K}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C, \tilde{B}\mathcal{Z}^C, \tilde{B}\mathcal{U}^C)$ and the similar expression for $K^C(\mathcal{X}^C, \mathcal{Y}^C, \mathcal{Z}^C, \mathcal{U}^C)$ for TM_n.

The equation of Codazzi is given by

$$
\begin{aligned}
\tilde{K}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C)N^{\bar{V}} &= \tilde{B}(\overset{\bullet}{\nabla}^C_{\mathcal{X}^C}\ H^V\mathcal{Y}^C - \overset{\bullet}{\nabla}^C_{\mathcal{X}^C}\ H^V\mathcal{X}^C), \\
\tilde{K}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C)N^{\bar{C}} &= \tilde{B}(\overset{\bullet}{\nabla}^C_{\mathcal{X}^C}\ H^C\mathcal{Y}^C - \overset{\bullet}{\nabla}^C_{\mathcal{X}^C}\ H^C\mathcal{X}^C), \\
\tilde{K}^C(N^{\bar{V}}, N^{\bar{C}})\tilde{B}\mathcal{X}^C &= 0.
\end{aligned}
$$

The curvature tensor concerning the QSNM connection $\tilde{\nabla}^C$ of TM_{n+1} is

$$
\tilde{R}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C)\tilde{B}\mathcal{Z}^C = \tilde{\nabla}^C_{\tilde{B}\mathcal{X}^C}\tilde{\nabla}^C_{\tilde{B}\mathcal{Y}^C}\tilde{B}\mathcal{Z}^C - \tilde{\nabla}^C_{\tilde{B}\mathcal{Y}^C}\tilde{\nabla}^C_{\tilde{B}\mathcal{X}^C}\tilde{B}\mathcal{Z}^C - \tilde{\nabla}^C_{[\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C]}\tilde{B}\mathcal{Z}^C
$$

by virtue of (3.9), (3.10) and (4.9), we get

$$
\begin{aligned}
\tilde{R}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C)\tilde{B}\mathcal{Z}^C &= B\{R^C(\mathcal{X}^C, \mathcal{Y}^C)\mathcal{Z}^C - m^V(\mathcal{X}^C, \mathcal{Z}^C)H^C\mathcal{Y}^C - m^C(\mathcal{X}^C, \mathcal{Z}^C)H^V\mathcal{Y}^C \\
&+ m^V(\mathcal{Y}^C, \mathcal{Z}^C)H^C\mathcal{X}^C + m^C(\mathcal{Y}^C, \mathcal{Z}^C)H^V\mathcal{X}^C\} - B\{m^V(\mathcal{X}^C, \mathcal{Z}^C)\tilde{\eta}^C(\mathcal{Y}^C) \\
&+ m^C(\mathcal{X}^C, \mathcal{Z}^C)\tilde{\eta}^V(\mathcal{Y})^C - m^V(\mathcal{Y}^C, \mathcal{Z}^C)\tilde{\eta}^C(\mathcal{X}^C) - m^C(\mathcal{Y}^C, \mathcal{Z}^C)\tilde{\eta}^V(\mathcal{X}^C)\}\xi^V \\
&- B\{m^V(\mathcal{X}^C, \mathcal{Z}^C)\tilde{\eta}^V(\mathcal{Y}^C) - m^V(\mathcal{Y}^C, \mathcal{Z}^C)\tilde{\eta}^V(\mathcal{X}^C)\}\xi^C + m^V\{-(\tilde{\eta}(\mathcal{X}))^V(\phi\mathcal{Y})^V \\
&+ (\tilde{\eta}(\mathcal{Y}))^V(\phi\mathcal{X})^V, \mathcal{Z}^C)\}N^{\bar{C}} + m^V\{(-\tilde{\eta}(\mathcal{X}))^V(\phi\mathcal{Y})^C - (\tilde{\eta}(\mathcal{X}))^C(\phi\mathcal{Y})^V \\
&+ (\tilde{\eta}(\mathcal{Y}))^V(\phi\mathcal{X})^C + (\tilde{\eta}(\mathcal{Y}))^C(\phi\mathcal{X})^V, \mathcal{Z}^C)\}N^{\bar{V}} \\
&+ m^C\{-(\tilde{\eta}(\mathcal{X}))^V(\phi\mathcal{Y})^V + (\tilde{\eta}(\mathcal{Y}))^V(\phi\mathcal{X})^V, \mathcal{Z}^C)\}N^{\bar{V}}
\end{aligned} \tag{5.4}
$$

where $R^C(\mathcal{X}^C, \mathcal{Y}^C)\mathcal{Z}^C = \nabla^C_{\mathcal{X}^C}\nabla^C_{\mathcal{Y}^C}\mathcal{Z}^C - \nabla^C_{\mathcal{Y}^C}\nabla^C_{\mathcal{X}^C}\mathcal{Z}^C - \nabla^C_{[\mathcal{X}^C, \mathcal{Y}^C]}\mathcal{Z}^C$ is curvature tensor of the QSNM connection.

As an immediate consequence of the theorem (5.1) and theorem (5.2), we have the following corollaries:

Corollary 5.3. *Let $R^C(\mathcal{X}^C, \mathcal{Y}^C)\mathcal{Z}^C$ be the Riemannian curvature tensor of hypersurface TM_n with QSNM connection ∇^C, then $\tilde{R}^C(\tilde{B}\mathcal{X}^C, \tilde{B}\mathcal{Y}^C)\tilde{B}\mathcal{Z}^C$ of the enveloping manifold TM_n with the QSNM connection $\tilde{\nabla}^C$ is given by equation (5.4).*

Corollary 5.4. *Let $\tilde{K}^C$ and K^C be the curvature tensors of TM_n and TM_{n+1} concerning $\overset{\tilde{\bullet}}{\nabla}{}^C$ and $\overset{\bullet}{\nabla}{}^C$ respectively. The Gauss and Codazzi equations endowed with the QSNM connection are similar equations obtained from Theorem 5.2.*

Conflict of interest

The authors declare no conflicts of interest in this paper.

Authors Contribution Statement

All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

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