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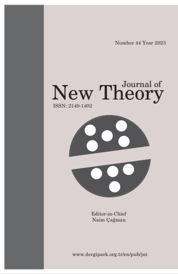

An Alternative Method for Determination of the Position Vector of a Slant Helix

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Research Article

Abstract — In this paper, we provide an alternative method to determine the position vector of a slant helix with the help of an alternative moving frame. We then construct a vector differential equation in terms of the principal normal vector of a slant helix using an alternative moving frame. By solving this vector differential equation, we determine the position vector of the slant helix. Afterward, we obtain parametric representations of some examples of slant helices for chosen curvature and torsion functions as an application of the proposed method. Finally, we discuss the method and whether further research should be conducted or not.

Keywords *Alternative moving frame, intrinsic equations, position vector, slant helix*

Mathematics Subject Classification (2020) 53A04, 34A05

1. Introduction

In differential geometry, the theory of curves is one of the main study areas. The theory of curves is generally studied with the well-known Frenet frame. Many geometric properties of differentiable curves can be defined with the help of this frame. In addition, the determination of the characterization of some special curves can be achieved by the curvatures of the Frenet frame. Among these special curves, various types of helices, including general helices, circular helices, and slant helices, are the curves that attract the most attention from researchers. A general helix (formerly called cylindrical helix) is defined by the property that all tangent vectors along the curve make a constant angle with a fixed direction. A necessary and sufficient condition for a curve to be a general helix is that the ratio of torsion to curvature is a constant [1]. There are numerous uses for general helices in different branches of science, such as biology, fractal geometry, computer-aided geometric design, engineering, and architecture [2–5]. After a long time since the concept of general helix hs been introduced, a new curve called slant helix has been defined with a similar idea. A slant helix is a curve whose principal normal vectors make a constant angle with a fixed straight line, which is the axis of the slant helix [6]. A necessary and sufficient condition for a curve to be a slant helix is that the ratio $\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'$ is constant, where κ and τ are curvature and torsion functions of the curve, respectively [6].

According to the fundamental theorem for curves, given two continuous functions of one parameter, a space curve can be determined uniquely up to rigid motion for which the two functions are its curvature and torsion [7]. The problem of determining the position vector of this curve is known as

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solving natural or intrinsic equations [1]. Although the problem has a solution for an arbitrary curve in Galilean space, it remains an open problem for Euclidean and Minkowski spaces [8, 9]. For the problem of determining the position vector of a general helix, a method depending on solving a vector differential equations constructed with the help of the Frenet frame has been proposed in [10]. Using this method, the parametric representation of the position vector of a general helix with the known curvature and torsion functions has been found. Then, a similar method has been used to determine a slant helix's position vector in [11]. After that, similar techniques have been applied for determining position vectors of some special curves with the help of various moving frames, such as the Frenet frame, the type-2 Bishop frame, the Darboux frame, and the alternative moving frame, in various spaces, Euclidean space, Minkowski space, and Galilean space [9, 12–19].

In this paper, we propose an alternative method to the existing methods in the literature to determine the position vector of a slant helix. We first rewrite the derivative formulae of the alternative moving frame to obtain a simpler differential equation. Then, we construct a vector differential equation in terms of the principal normal vector with the help of the new derivative formulae of the alternative moving frame. By solving this vector differential equation, we find the principal normal vector of the slant helix and thus determine the position vector of the curve. Finally, applying the proposed method, we obtain parametric representations of some examples of slant helices for chosen curvature and torsion functions.

2. Preliminaries

Let $\alpha = \alpha(s)$ be a unit speed Frenet curve in E^3, that is, $\langle \alpha'(s), \alpha'(s) \rangle = 1$ and $\alpha''(s) \neq 0$. The Frenet frame along the curve α consists of three mutually orthonormal vectors defined by

$$\boldsymbol{T}(s) = \alpha'(s), \quad \boldsymbol{N}(s) = \frac{1}{\|\boldsymbol{T}'(s)\|}\boldsymbol{T}'(s), \quad \text{and} \quad \boldsymbol{B}(s) = \boldsymbol{T}(s) \times \mathbf{N}(s) \tag{1}$$

where $\boldsymbol{T}(s)$, $\boldsymbol{N}(s)$, and $\boldsymbol{B}(s)$ are called tangent vector, principal normal vector, and binormal vector, respectively. The derivative formulae of the Frenet frame also known as Frenet formulae can be provided as follows:

$$\begin{bmatrix} \boldsymbol{T}'(s) \\ \boldsymbol{N}'(s) \\ \boldsymbol{B}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}(s) \\ \boldsymbol{N}(s) \\ \boldsymbol{B}(s) \end{bmatrix} \tag{2}$$

where κ and τ are called curvature and torsion functions of the curve α, respectively. These functions, also called Frenet curvatures, are defined by

$$\kappa(s) = \|\boldsymbol{T}'(s)\| \quad \text{and} \quad \tau(s) = -\langle \boldsymbol{B}'(s), \boldsymbol{N}(s) \rangle$$

In Euclidean 3-space, apart from the Frenet frame, many moving frames has been defined to study the differential geometric properties of curves. One of them is the alternative moving frame. The alternative moving frame consists of three mutually orthonormal vectors. These vectors are the unit principal normal vector $\boldsymbol{N}(s)$, also included in the Frenet frame, the unit vector $\boldsymbol{C}(s)$ defined by $\boldsymbol{C}(s) = \frac{\boldsymbol{N}'(s)}{\|\boldsymbol{N}'(s)\|}$, and the unit vector $\boldsymbol{W}(s)$ defined by $\boldsymbol{W}(s) = \boldsymbol{N}(s) \times \boldsymbol{C}(s)$, also normalized instantaneous rotation vector of the Frenet frame [20, 21]. The derivative formulae of the alternative moving frame are as follows [20]:

$$\begin{bmatrix} \boldsymbol{N}'(s) \\ \boldsymbol{C}'(s) \\ \boldsymbol{W}'(s) \end{bmatrix} = \begin{bmatrix} 0 & f(s) & 0 \\ -f(s) & 0 & g(s) \\ 0 & -g(s) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{N}(s) \\ \boldsymbol{C}(s) \\ \boldsymbol{W}(s) \end{bmatrix} \tag{3}$$

where the functions

$$f = \sqrt{\kappa^2 + \tau^2} \tag{4}$$

and

$$g = \frac{\kappa^2}{\kappa^2 + \tau^2}\left(\frac{\tau}{\kappa}\right)' \tag{5}$$

are called the first and the second alternative curvatures of the curve α, respectively. From Equalities 4 and 5, we have the followings [22]:

$$\kappa(s) = f(s)\cos\left(\int g(s)ds\right) \tag{6}$$

and

$$\tau(s) = f(s)\sin\left(\int g(s)ds\right) \tag{7}$$

The alternative curvatures play a major role in the characterizations of curves. The following theorem supports this idea.

Theorem 2.1. [20] Let α be a curve in E^3 with alternative curvatures f and g. The curve α is a slant helix if and only if the function

$$\sigma = \frac{g}{f} \tag{8}$$

is a constant.

3. Determination of the Position Vector of a Slant Helix

In this section, the problem of determining a slant helix's position vector is solved using a method based on the alternative moving frame. To achieve this, we first rewrite the derivative formulae of the alternative moving frame with a new parameter. Then, we construct a vector differential equation in terms of the principal normal vector by using these new formulae. By solving this vector differential equation, we obtain the position vector of the slant helix in parametric form. Before constructing the vector differential equation, it will be more useful to use new derivative formulae obtained by the parameter transformation $\theta = \int f(s)ds$ instead of the derivative formulae given in Equality 3 where $f(s)$ is the first alternative curvature. According to this new parameter θ, the derivative formulae of the alternative moving frame become the following form [19]:

$$\begin{bmatrix} \boldsymbol{N}'(\theta) \\ \boldsymbol{C}'(\theta) \\ \boldsymbol{W}'(\theta) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \sigma(\theta) \\ 0 & -\sigma(\theta) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{N}(\theta) \\ \boldsymbol{C}(\theta) \\ \boldsymbol{W}(\theta) \end{bmatrix} \tag{9}$$

such that $\sigma(\theta) = \frac{g(\theta)}{f(\theta)}$.

Theorem 3.1. Let $\alpha = \alpha(s)$ be a unit speed slant helix and $\alpha = \alpha(\theta)$ be another parametric representation of this curve where $\theta = \int f(s)ds$. The principal normal vector $\boldsymbol{N}(\theta)$ satisfies the following vector differential equation

$$\boldsymbol{N}'''(\theta) + (1 + \sigma^2)\boldsymbol{N}'(\theta) = 0 \tag{10}$$

where $\sigma(\theta) = \frac{g(\theta)}{f(\theta)}$.

Proof.

From the second equation of Equality 9, the vector $\boldsymbol{W}(\theta)$ can be written as

$$\boldsymbol{W}(\theta) = \frac{1}{\sigma(\theta)}\left(\boldsymbol{C}'(\theta) + \boldsymbol{N}(\theta)\right) \tag{11}$$

Differentiating the first equation of Equality 9,

$$\boldsymbol{N}''(\theta) = \boldsymbol{C}'(\theta) \tag{12}$$

Substituting Equality 12 into Equality 11,

$$\boldsymbol{W}(\theta) = \frac{1}{\sigma(\theta)}\left(\boldsymbol{N}''(\theta) + \boldsymbol{N}(\theta)\right) \tag{13}$$

By differentiating Equality 13 and by using the first and third equations of Equality 9,

$$\left(\frac{1}{\sigma(\theta)}\left(\boldsymbol{N}''(\theta) + \boldsymbol{N}(\theta)\right)\right)' + \sigma(\theta)\boldsymbol{N}'(\theta) = 0 \tag{14}$$

According to Theorem 2.1, $\sigma(\theta)$ is a constant. Thus, Equality 14 becomes Equality 10 which completes the proof. □

Equality 10 is a third-order vector differential equation with constant coefficients. The principal normal vector $\boldsymbol{N}$ can be found by solving this equation. The following theorem provides a solution for the problem of determining the position vector of a slant helix given the curvature and torsion functions.

Theorem 3.2. Let $\kappa(s) > 0$ and $\tau(s)$ be two differentiable functions on an interval I. On this interval, the position vector $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ of a slant helix for which s, $\kappa(s)$ and $\tau(s)$ become the arc-length, the curvature and the torsion, respectively, can be determined up to rigid motions in Euclidean 3-space as follows:

$$\begin{cases} \alpha_1(s) = \frac{1}{\sqrt{1+\sigma^2}}\int\left(\int f(s)\cos\left(\sigma\int f(s)ds\right)\cos\left(\sqrt{1+\sigma^2}\int f(s)ds\right)ds\right)ds \\ \alpha_2(s) = \frac{1}{\sqrt{1+\sigma^2}}\int\left(\int f(s)\cos\left(\sigma\int f(s)ds\right)\sin\left(\sqrt{1+\sigma^2}\int f(s)ds\right)ds\right)ds \\ \alpha_3(s) = \frac{\sigma}{\sqrt{1+\sigma^2}}\int\left(\int f(s)\cos\left(\sigma\int f(s)ds\right)ds\right)ds \end{cases} \tag{15}$$

where $\sigma = \frac{g}{f}$ and f and g are the alternative curvatures defined in Equalities 4 and 5, respectively.

Proof.
Let $\alpha = \alpha(s)$ be a unit speed slant helix in E^3 and $\alpha = \alpha(\theta)$ be another parametrization of the same where $\theta = \int f(s)ds$. We can write the principal normal vector $\boldsymbol{N}(\theta)$ of the curve $\alpha(\theta)$ with the standard basis of E^3 as $\boldsymbol{N}(\theta) = N_1(\theta)\boldsymbol{e_1} + N_2(\theta)\boldsymbol{e_2} + N_3(\theta)\boldsymbol{e_3}$. We can choose the axis of the slant helix to be parallel to $\boldsymbol{e_3}$ without losing generality. Note that selections of different axes will produce the same slant helix up to rigid motion in Euclidean 3-space. Since the vector $\boldsymbol{N}(\theta)$ makes a constant angle with the axis, then

$$\langle \boldsymbol{N}(\theta), \boldsymbol{e_3}\rangle = N_3(\theta) = n \tag{16}$$

where n is a constant real number. It is clear that $N_3(\theta) = n$ satisfies Equality 10. Additionally, since $\boldsymbol{N}(\theta)$ is a unit vector,

$$N_1^2(\theta) + N_2^2(\theta) = 1 - n^2 \tag{17}$$

From the general solution of Equality 17, the components N_1 and N_2 can be written as

$$N_1(\theta) = \sqrt{1-n^2}\cos(t(\theta)) \quad \text{and} \quad N_2(\theta) = \sqrt{1-n^2}\sin(t(\theta))$$

where t is a function of θ. The components N_1 and N_2 should satisfy Equality 10. Substituting $N_1(\theta)$ and $N_2(\theta)$ into Equality 10, we obtain the following equalities:

$$(3t't'')\cos t + \left(-(t')^3 + t''' + (1+(\sigma)^2)t'\right)\sin t = 0 \tag{18}$$

$$(3t't'')\sin t - \left(-(t')^3 + t''' + (1 + (\sigma)^2)t'\right)\cos t = 0 \tag{19}$$

From Equalities 18 and 19, we have the following equalities:

$$3t''t' = 0 \tag{20}$$

$$t'' - (t')^3 + (1 + \sigma^2)t' = 0 \tag{21}$$

From Equality 20,

$$t(\theta) = c_1\theta + c_2 \tag{22}$$

where c_1 and c_2 are constants. If we change the parameter $t \to t + c_2$, then

$$t(\theta) = c_1\theta \tag{23}$$

Substituting Equality 23 into Equality 21,

$$c_1(-{c_1}^2 + 1 + \sigma^2) = 0 \tag{24}$$

Solving Equality 24,

$$c_1 = \pm\sqrt{1 + \sigma^2} \tag{25}$$

Substituting Equality 25 into Equality 23, the function $t(\theta)$ can be obtained as

$$t = \pm\sqrt{1 + \sigma^2}\theta$$

Therefore, the vector $\boldsymbol{N}(\theta)$ can be found as follows:

$$\boldsymbol{N}(\theta) = \left(\sqrt{1 - n^2}\cos(\sqrt{1 + \sigma^2}\theta), \pm\sqrt{1 - n^2}\sin\left(\sqrt{1 + \sigma^2}\theta\right), n\right) \tag{26}$$

Moreover, the constant n in Equality 26 can be written in terms of σ. Differentiating Equality 16 and using the first equation of Equality 9,

$$\langle \boldsymbol{C}(\theta), \boldsymbol{e_3} \rangle = 0 \tag{27}$$

Differentiating Equality 27 and using the second equation of Equality 9,

$$\langle \boldsymbol{W}(\theta), \boldsymbol{e_3} \rangle = \frac{n}{\sigma}$$

Since the axis of the slant helix is a unit vector,

$$n = \pm\frac{\sigma}{\sqrt{1 + \sigma^2}} \tag{28}$$

Substituting Equality 28 into Equality 26, the principal normal vector $\boldsymbol{N}(\theta)$ of the slant helix can be written in terms of the parameter θ as

$$\boldsymbol{N}(\theta) = \frac{1}{\sqrt{1 + \sigma^2}}\left(\cos\left(\sqrt{1 + \sigma^2}\theta\right), \pm\sin\left(\sqrt{1 + \sigma^2}\theta\right), \pm\sigma\right)$$

or in terms of the arc-length parameter s as

$$\boldsymbol{N}(s) = \frac{1}{\sqrt{1 + \sigma^2}}\left(\cos\left(\sqrt{1 + \sigma^2}\int f(s)ds\right), \pm\sin\left(\sqrt{1 + \sigma^2}\int f(s)ds\right), \pm\sigma\right) \tag{29}$$

Moreover, from Equalities 1 and 2, the curve α can be written as

$$\alpha(s) = \int \left(\int \kappa(s)\boldsymbol{N}(s)ds\right)ds$$

By using Equality 4 and the parameter transformation $\theta = \int f(s)ds$,

$$\alpha(\theta) = \int \frac{1}{f(\theta)}\left(\int \cos(\sigma\theta)\boldsymbol{N}(\theta)d\theta\right)d\theta$$

If the curve α is written again in terms of the arc length parameter s,

$$\alpha(s) = \int \left(\int f(s) \cos \left(\sigma \int f(s) ds \right) \boldsymbol{N}(s) ds \right) ds \tag{30}$$

By choosing the positive sign in Equality 29 and substituting it into Equality 30, the position vector $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ of the slant helix can be found as

$$\begin{cases} \alpha_1(s) = \frac{1}{\sqrt{1+\sigma^2}} \int \left(\int f(s) \cos \left(\sigma \int f(s) ds \right) \cos \left(\sqrt{1+\sigma^2} \int f(s) ds \right) ds \right) ds \\ \alpha_2(s) = \frac{1}{\sqrt{1+\sigma^2}} \int \left(\int f(s) \cos \left(\sigma \int f(s) ds \right) \sin \left(\sqrt{1+\sigma^2} \int f(s) ds \right) ds \right) ds \\ \alpha_3(s) = \frac{\sigma}{\sqrt{1+\sigma^2}} \int \left(\int f(s) \cos \left(\sigma \int f(s) ds \right) ds \right) ds \end{cases} \tag{31}$$

Given the curvature and torsion functions, the functions f, g, and σ can be determined using Equalities 4 and 5 and Theorem 2.1. Substituting f and σ into Equality 31, the position vector of the slant helix can be obtained. □

4. Illustrative Examples

In this section, we obtain parametric representations of some examples of slant helices for chosen some special curvature and torsion functions.

Example 4.1. If $\kappa = \cos(s)$ and $\tau = \sin(s)$, then $f = 1$, $g = 1$, and $\sigma = 1$ from Equalities 4, 5, and 8. From Equality 15, the position vector $\alpha(s)$ of the slant helix can be found as

$$\alpha(s) = \left(-\frac{3\sqrt{2}}{2} \cos(\sqrt{2}s) \cos(s) - 2 \sin(\sqrt{2}s) \sin(s), -\frac{3\sqrt{2}}{2} \sin(\sqrt{2}s) \cos(s) + 2 \cos(\sqrt{2}s) \sin(s), -\frac{\sqrt{2}}{2} \cos(s) \right)$$

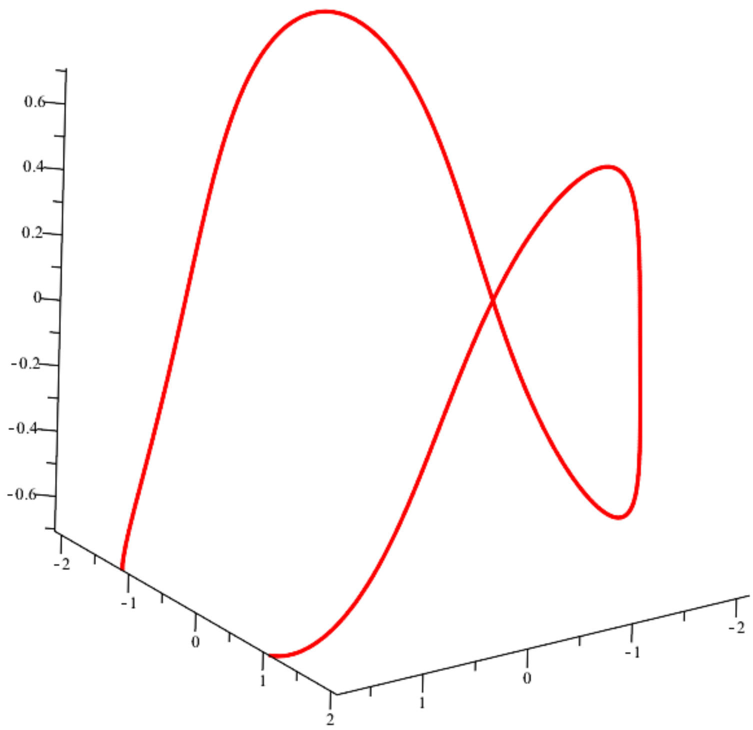

Figure 1. Slant helix with $\kappa = \cos(s)$ and $\tau = \sin(s)$

Example 4.2. Let curvature and torsion functions be given as $\kappa = \frac{1}{(1+s^2)^{3/2}}$ and $\tau = \frac{s}{(1+s^2)^{3/2}}$, respectively. From Equalities 4, 5, and 8, $f = \frac{1}{1+s^2}$, $g = \frac{1}{1+s^2}$, and $\sigma = 1$. Using Equality 15, the position vector of the slant helix can be obtained in the parametric representation as

$$\alpha(s) = \left(\frac{\sqrt{2}}{2}\sqrt{1+s^2}\cos\left(\sqrt{2}\arctan(s)\right), -\frac{\sqrt{2}}{2}\sqrt{1+s^2}\sin\left(\sqrt{2}\arctan(s)\right), \frac{\sqrt{2}}{2}\sqrt{1+s^2}\right)$$

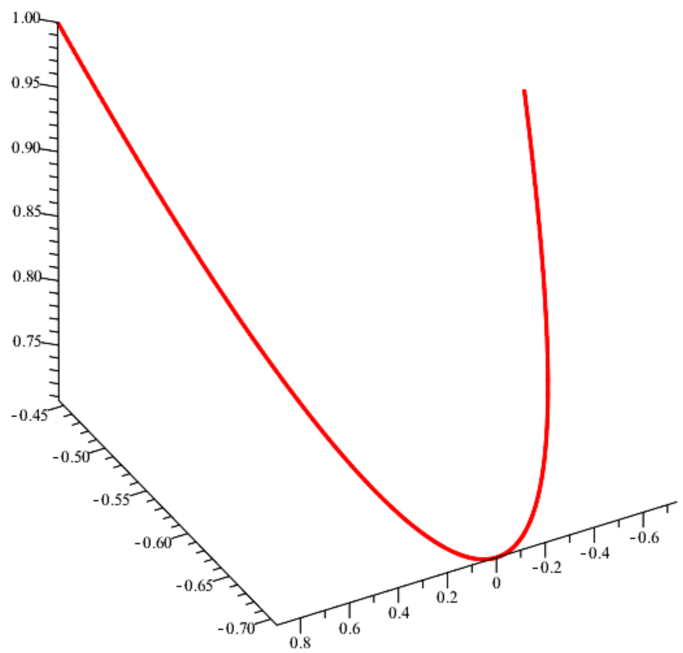

Figure 2. Slant helix with $\kappa = \frac{1}{(1+s^2)^{3/2}}$ and $\tau = \frac{s}{(1+s^2)^{3/2}}$

Example 4.3. Let curvature and torsion functions be given as $\kappa = \frac{\cos(\sqrt{s})}{2\sqrt{s}}$ and $\tau = \frac{\sin(\sqrt{s})}{2\sqrt{s}}$, respectively. The functions f, g, and σ can be found as $f = \frac{1}{2\sqrt{s}}$, $g = \frac{1}{2\sqrt{s}}$, and $\sigma = 1$, respectively. Using Equality 15, the position vector $\alpha(s)$ of the slant helix can be expressed in the parametric representation as

$$\alpha(s) = \left(-\tfrac{\sqrt{2}}{2}\cos\left(\sqrt{2s}\right)\sin\left(\sqrt{s}\right) + \sin\left(\sqrt{2s}\right)\cos\left(\sqrt{s}\right), -\tfrac{\sqrt{2}}{2}\sin\left(\sqrt{2s}\right)\sin\left(\sqrt{s}\right) - \cos\left(\sqrt{2s}\right)\cos\left(\sqrt{s}\right), \tfrac{\sqrt{2}}{2}\sin\left(\sqrt{s}\right)\right)$$

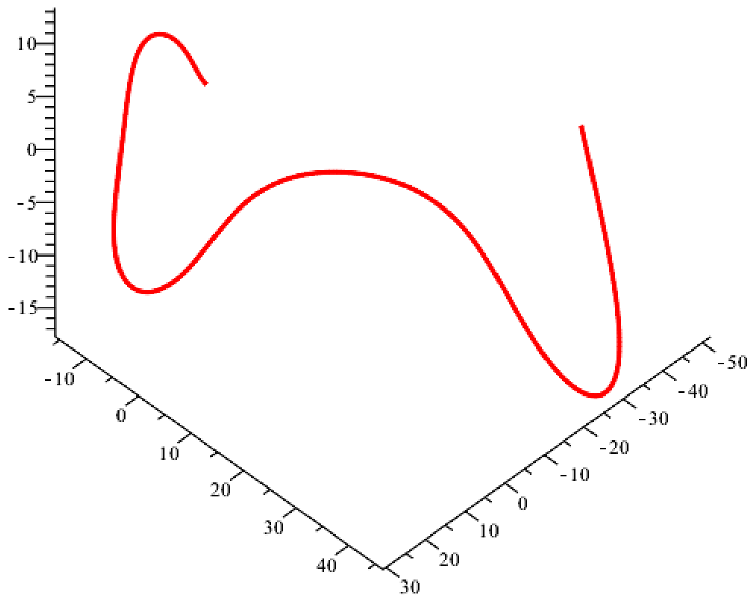

Figure 3. Slant helix with $\kappa = \frac{\cos(\sqrt{s})}{2\sqrt{s}}$ and $\tau = \frac{\sin(\sqrt{s})}{2\sqrt{s}}$

5. Conclusion

In this paper, we give a method for the determination of the position vector of a slant helix. This method basically includes two important steps. The first step is constructing a vector differential equation in terms of the principal normal vector of the curve with the help of the alternative moving frame. The second step is determining the position vector of the slant helix by solving the equation in the first step. Thanks to this method, the problem of determining the position vector of a curve given the curvature and torsion functions is solved for a slant helix case. As an application of the method, some examples of slant helices in parametric form are obtained for given special curvature and torsion functions. It is thought that the obtained examples of slant helices will contribute to the variety of examples of slant helices in the literature.

The problem of determining the position vector of a slant helix, solved in this paper, is also discussed in [11] and [19]. In [11], a method based on solving a vector differential equation in terms of the principal normal vector constructed with the help of Frenet formulae is presented. Since the characterization of a slant helix cannot be used directly in this vector differential equation, this method for determining the position vector of a slant helix involves much more complicated mathematical operations than the method in the present paper. In [19], the authors give a method based on the alternative moving frame to solve the same problem. Unlike the method in the present paper, the vector differential equation is constructed in terms of the vector $\boldsymbol{C}$ of the alternative moving frame in the method given in [19]. Moreover, this method has been developed to determine a slant helix's position vector in Minkowski 3-space, not Euclidean 3-space. The necessity of finding all the vectors of the alternative moving frame to determine the position vector of the slant helix is an important disadvantage of this method. In light of all these comparisons, it is hoped that the method in the present paper will be an important alternative to the methods in the literature for the problem of determining the position vector of a slant helix.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the first author's master's thesis supervised by the second author. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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