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Research Article

$\star$-Ricci–Yamabe solitons on almost coKähler manifolds

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Abstract

The aim of the present article is to analyze $\star$-Ricci–Yamabe solitons on almost coKähler manifolds and to characterize them when the potential vector field is pointwise collinear with the Reeb vector field. It is proved that a compact almost coKähler manifold admitting a $\star$-Ricci–Yamabe soliton under certain restriction on $\star$-scalar curvature is coKähler and $\star$-Ricci flat; in addition, that the soliton is steady. (κ, μ)-almost coKähler manifolds admitting such solitons are also considered and finally, the obtained results are completed by non-trivial examples.

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1. Introduction

Geometric flows, like Ricci flow and Yamabe flow, enabled mathematical physicists to explain certain relativistic and cosmological phenomenon [18] in a sophisticated way; in addition, to analyze the mathematical context of geometric quantization ([1,2]). It is known that techniques in Ricci flow are deeply associated with general relativity, particularly for static metric [40].

Investigations on Hamilton's Ricci flow ([21,22]) acquired acceleration after its successful application by Perelman to solve the famous Poincaré conjecture which was unsolved for a long time. Since then, a good number of articles have turned up in the arena of existing literature with variations of the original concept in different kinds of manifolds from different perspectives. Though the seed of Yamabe flow was implied within the ground breaking work of Hamilton, later Yamabe individually created its distinct foundation. A self similar solution of the Ricci flow is a Ricci soliton and a similar definition applies for a Yamabe soliton. It is to be mentioned that solitons play a fundamental role in formation of singularities of the flow.

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In [37], Wang studied $\star$-Ricci solitons on contact 3-manifolds using $\star$-Ricci curvature. A combination of Ricci solitons and Yamabe solitons, namely, Ricci–Yamabe solitons on (κ, μ)-almost coKähler manifolds have been studied by Mandal in the paper [24]. For further details about Ricci–Yamabe solitons we refer to [19]. The theory of Ricci solitons has been enriched by a good number of geometers. Through them are ([3,4,8–11,14–17,33–38]). Relativistic perfect fluid spacetimes with Ricci–Yamabe solitons have been studied by several authors. After going through the above works, we feel the necessity regarding the analysis of the combination of $\star$-Ricci soliton and Yamabe soliton as $\star$-Ricci–Yamabe soliton. Its study is the main goal of the present article.

In a Riemannian manifold $(\mathcal{M}^n, g)(n \geq 3)$ of dimension n, the equations of the flows named after Ricci and Yamabe are depicted by

$$\frac{\partial g}{\partial t}(t) = -2S(t) \tag{1.1}$$

and

$$\frac{\partial g}{\partial t}(t) = -r(t)g(t), \tag{1.2}$$

obeying certain initial conditions. Here S represents the Ricci curvature tensor and r is the scalar curvature corresponding to g.

A soliton of a geometric flow is a fixed solution, up to diffeomorphisms and scaling. Thus, for a geometric flow, the soliton is obviously the Riemannian metric which provides maximum symmetry of the space, up to diffeomorphisms and scaling.

A smooth manifold $(\mathcal{M}^n, g)$ with Riemannian metric g is named a Ricci soliton if it agrees with the equation

$$\frac{1}{2}\mathcal{L}_V g + S = \gamma g,$$

where V is a vector field called the potential vector field, $\mathcal{L}_V$ stands for the Lie derivative operator in the direction of V and γ is a real scalar. Ricci solitons are generalizations of Einstein metrics and specific solutions of the flow (1.1). For shrinking, expanding or steady cases of Ricci solitons, γ is positive, negative or zero, respectively.

A smooth manifold $(\mathcal{M}^n, g)(n \geq 3)$ with Riemannian metric g is named a Yamabe soliton if it agrees with the equation

$$\frac{1}{2}\mathcal{L}_V g = (\gamma - r)g.$$

Like a Ricci soliton, a Yamabe soliton is a specific solution of the flow (1.2). The soliton is shrinking, expanding or steady according as γ is positive, negative or zero, respectively. For further details, see ([25,27,29,30,33,36–39]).

A Ricci–Yamabe flow of type (l, m) is a linear combination of the Ricci flow and Yamabe flow described by

$$\frac{\partial g}{\partial t}(t) = -2lS(t) + mr(t)g(t),$$

agreeing with certain initial conditions, where l and m are real scalars.

The self-similar solution of the Ricci–Yamabe flow of type (l, m) is known as a Ricci–Yamabe soliton of type (l, m) if it varies as a group of diffeomorphisms with one parameter and changes by scale factor. A Ricci–Yamabe soliton of type (l, m) is characterized by

$$\mathcal{L}_V g + 2lS = (2\gamma - mr)g. \tag{1.3}$$

The soliton is shrinking, expanding or steady according as γ is positive, negative or zero, respectively. A Ricci–Yamabe soliton of type (l, m) becomes a Ricci soliton if $l = 1$ and $m = 0$ and a Yamabe soliton if $l = 0$ and $m = 1$.

In [31], Tachibana initiated the study of $\star$-Ricci curvatures. $\star$-Ricci curvatures have been analyzed from different perspectives in the articles ([12,20,23,26,28]). A $\star$-Ricci curvature is a generalization of the usual Ricci curvature. Hence, it carries more information and it demands a separate study.

If the Ricci curvature and the scalar curvature are substituted by $\star$-Ricci curvatures and $\star$-scalar curvatures in (1.3), respectively, then the soliton is called $\star$-Ricci–Yamabe soliton of type (l, m). Thus, a $\star$-Ricci–Yamabe soliton is represented by

$$\mathcal{L}_V g + 2lS^\star = (2\gamma - mr^\star)g, \tag{1.4}$$

where $S^\star$ indicates the $\star$-Ricci curvature defined by

$$S^\star(V_1, V_2) = \frac{1}{2}\mathrm{tr}\{V_3 \to R(V_1, \tau V_2)\tau V_3\},$$

for any vector fields V_1, V_2 and V_3 on the manifold, R being the Riemann curvature tensor field, $r^\star$ the $\star$-scalar curvature and τ a $(1,1)$-tensor field on the manifold. When the potential vector field V is equal to the gradient of a certain smooth function ψ (which is known as potential function) on $\mathcal{M}$, a $\star$-Ricci–Yamabe soliton is called a gradient $\star$-Ricci–Yamabe soliton and it is described by

$$\nabla\nabla\psi + lS^\star = \Big(\gamma - \frac{1}{2}mr^\star\Big)g, \tag{1.5}$$

where $\nabla\nabla\psi$ stands for the Hessian of ψ. In [33], Wang studied Ricci solitons on compact almost coKähler manifolds and proved that such a manifold is coKähler and Ricci flat admitting steady soliton. One of our purposes is to extend the result for $\star$-Ricci–Yamabe solitons on almost coKähler manifolds.

In the present article, after the formal literature review, we assemble, in Section 2, the known results regarding almost coKähler manifolds. Section 3 contains some results associated with $\star$-Ricci–Yamabe solitons on almost coKähler manifolds. In this section we also study such compact manifolds and extend a result of Wang [33]. We deduce some characteristics of $\star$-Ricci–Yamabe solitons on (κ, μ)-almost coKähler manifolds in Section 4, whereas Section 5 is allotted to analyze gradient $\star$-Ricci–Yamabe solitons. The concluding section strengthens the obtained results by providing illustrative examples that will establish the transparency of the deduced results.

2. Preliminaries

Let $\mathcal{M}$ be a differentiable manifold of dimension $(2n+1)$ equipped with an almost contact metric structure $(\tau, \theta, \omega, g)$, where τ is a tensor field of type $(1,1)$, θ is a vector field, ω is a 1-form and g is a Riemannian metric on $\mathcal{M}$ such that ([13, 15]):

$$\tau^2(V_1) = -V_1 + \omega(V_1)\theta, \quad \omega(\theta) = 1, \tag{2.1}$$

for all $V_1 \in \chi(\mathcal{M})$. As a consequence, we get the following:

$$\tau\theta = 0, \quad g(V_1, \theta) = \omega(V_1), \quad \omega(\tau V_1) = 0,$$
$$g(\tau V_1, \tau V_2) = g(V_1, V_2) - \omega(V_1)\omega(V_2),$$
$$g(\tau V_1, V_2) = -g(V_1, \tau V_2), \quad g(\tau V_1, V_1) = 0,$$

for all V_1, $V_2 \in \chi(\mathcal{M})$. A smooth manifold $\mathcal{M}$ of dimension $(2n+1)$ with an almost contact metric structure is known as an almost contact metric manifold.

Consider the 2-form $\tilde{\tau}$ satisfying

$$\tilde{\tau}(V_1, V_2) = g(V_1, \tau V_2),$$

for all V_1, $V_2 \in \chi(\mathcal{M})$. If $d\omega = \tilde{\tau}$, then an almost contact metric manifold is known as a contact metric manifold. An almost contact metric manifold is an almost coKähler manifold if both ω and $\tilde{\tau}$ are closed, that is, $d\tilde{\tau} = 0$ and $d\omega = 0$. According to Blair [6], an (almost) coKähler manifold and an (almost) cosymplectic manifold coincide.

Let $\mathcal{M}$ be an almost coKähler manifold of dimension $(2n+1)$. We consider the operators h, h' and L defined by $h = \frac{1}{2}\mathcal{L}_\theta\tau$, $h' = h\tau$ and $L = R(\cdot, \theta)\theta$, where R is the curvature tensor and $\mathcal{L}$ is the Lie differentiation operator. These operators agree with the following ([13, 15]):

$$h\theta = 0, \quad \mathrm{tr}(h) = 0, \quad \mathrm{tr}(h') = 0, \quad h\tau = -\tau h, \quad h^2 = h'^2, \quad \mathrm{div}\theta = 0,$$
$$\nabla\theta = h', \quad \nabla_\theta \tau = 0, \tag{2.2}$$
$$\tau L \tau - L = 2h^2.$$

In an almost coKähler manifold, the 1-form ω is closed, so

$$(\nabla_{V_1}\omega)V_2 = (\nabla_{V_2}\omega)V_1,$$

for all $V_1,\, V_2 \in \chi(\mathcal{M})$.

The almost coKähler structure is integrable if and only if

$$(\nabla_{V_1}\tau)V_2 = g(hV_1, V_2)\theta - \omega(V_2)hV_1, \tag{2.3}$$

for all $V_1,\, V_2 \in \chi(\mathcal{M})$.

The idea of (κ,μ)-nullity distribution on contact metric manifolds was coined by Blair *et al.* [7]. The contact metric manifold $\mathcal{M}$ whose curvature follows the relation

$$R(V_1, V_2)\theta = \kappa[\omega(V_2)V_1 - \omega(V_1)V_2] + \mu[\omega(V_2)hV_1 - \omega(V_1)hV_2], \tag{2.4}$$

for all $V_1,\, V_2 \in \chi(\mathcal{M})$ and for some real scalars κ, μ, is known as a (κ,μ)-contact metric manifold and it is said that θ belongs to the (κ,μ)-nullity distribution. The manifold is called a generalized (κ,μ)-contact metric manifold if $\kappa,\ \mu$ are differentiable functions of any order and $\mathcal{M}$ is said to be a (κ,μ)-almost coKähler manifold whenever $\kappa,\ \mu$ are real numbers.

A (κ,μ)-almost coKähler manifold of dimension $(2n+1)$ has the following curvature restrictions ([13, 15]):

$$h^2 V_1 = \kappa\tau^2 V_1, \tag{2.5}$$
$$S(V_1, \theta) = 2n\kappa\omega(V_1),$$
$$Q\theta = 2n\kappa\theta,$$

for all $V_1 \in \chi(\mathcal{M})$, where Q is the Ricci operator defined by $g(QV_1, V_2) = S(V_1, V_2)$.

Definition 2.1 ([5])**.** An almost coKähler manifold is known as an ω-Einstein manifold if the Ricci curvature agrees with the following

$$S(V_1, V_2) = ag(V_1, V_2) + b\omega(V_1)\omega(V_2), \tag{2.6}$$

for all $V_1,\, V_2 \in \chi(\mathcal{M})$, where a and b are smooth functions on the manifold.

Tracing V_1 and V_2 in the above equation, we infer

$$r = (2n+1)a + b, \tag{2.7}$$

where r is the scalar curvature.

Lemma 2.2 ([10])**.** *In a (κ,μ)-almost coKähler manifold of dimension $(2n+1)$ with $\kappa < 0$, the following relations hold*

$$QV_1 = \mu hV_1 + 2n\kappa\omega(V_1)\theta, \tag{2.8}$$

$$\begin{aligned}(\nabla_{V_1}h)V_2 - (\nabla_{V_2}h)V_1 =& \kappa[\omega(V_2)\tau V_1 - \omega(V_1)\tau V_2 + 2g(\tau V_1, V_2)\theta] \\ &+ \mu[\omega(V_2)\tau hV_1 - \omega(V_1)\tau hV_2],\end{aligned} \tag{2.9}$$

$$(\nabla_{V_1}h\tau)V_2 - (\nabla_{V_2}h\tau)V_1 = \kappa[\omega(V_2)V_1 - \omega(V_1)V_2] + \mu[\omega(V_2)hV_1 - \omega(V_1)hV_2], \tag{2.10}$$

for all $V_1,\, V_2 \in \chi(\mathcal{M})$.

3. ⋆-Ricci–Yamabe solitons on almost coKähler manifolds

In any 3-dimensional Riemannian manifold, the Weyl conformal curvature tensor vanishes identically. So, the curvature tensor of a 3-dimensional ω-Einstein almost coKähler manifold is given by

$$\begin{aligned} R(V_1,V_2)V_3 =&S(V_2,V_3)V_1 - S(V_1,V_3)V_2 + g(V_2,V_3)QV_1 \\ &-g(V_1,V_3)QV_2 - \frac{r}{2}[g(V_2,V_3)V_1 - g(V_1,V_3)V_2], \end{aligned} \tag{3.1}$$

for all V_1, V_2, $V_3 \in \chi(\mathcal{M})$. Using (2.6) and (2.7) in (3.1), we infer

$$\begin{aligned} R(V_1,V_2)V_3 =&\frac{a-b}{2}[g(V_2,V_3)V_1 - g(V_1,V_3)V_2] \\ &+b[g(V_2,V_3)\omega(V_1)\theta - g(V_1,V_3)\omega(V_2)\theta \\ &+\omega(V_2)\omega(V_3)V_1 - \omega(V_1)\omega(V_3)V_2]. \end{aligned}$$

Taking $V_1 = \theta$ in the above equation, we get

$$R(\theta,V_2)V_3 = \frac{a+b}{2}[g(V_2,V_3)\theta - \omega(V_3)V_2]. \tag{3.2}$$

Now we state and prove the following two lemmas.

Lemma 3.1. *For a 3-dimensional ω-Einstein almost coKähler manifold, we have*

$$(\nabla_{V_3}h)V_2 - (\nabla_{V_2}h)V_3 = \frac{a+b}{2}[\omega(V_2)\tau V_3 - \omega(V_3)\tau V_2] + \omega(V_3)\tau hV_2 - \omega(V_2)\tau hV_3, \tag{3.3}$$

for all V_2, $V_3 \in \chi(\mathcal{M})$.

Proof. We have

$$(\nabla_{V_3}\tau h)V_2 = g(hV_2,hV_3)\theta + \tau((\nabla_{V_3}h)V_2),$$

where we applied (2.3). Thus we can write the following

$$(\nabla_{V_3}\tau h)V_2 - (\nabla_{V_2}\tau h)V_3 = \tau((\nabla_{V_3}h)V_2 - (\nabla_{V_2}h)V_3). \tag{3.4}$$

Due to Blair *et al.* [7], we infer

$$g(R(\theta,V_1)V_2,V_3) = g((\nabla_{V_1}\tau)V_2,V_3) + g((\nabla_{V_3}\tau h)V_2 - (\nabla_{V_2}\tau h)V_3, V_1). \tag{3.5}$$

Inserting (2.3), (3.2) and (3.4) in (3.5), we obtain

$$\tau((\nabla_{V_3}h)V_2 - (\nabla_{V_2}h)V_3) = \frac{a+b}{2}[\omega(V_3)V_2 - \omega(V_2)V_3] - \omega(V_3)hV_2 + \omega(V_2)hV_3.$$

Applying τ to the above equation, we infer

$$\begin{aligned} &-[(\nabla_{V_3}h)V_2 - (\nabla_{V_2}h)V_3] + \omega((\nabla_{V_3}h)V_2 - (\nabla_{V_2}h)V_3)\theta \\ &= \frac{a+b}{2}[\omega(V_3)\tau V_2 - \omega(V_2)\tau V_3] - \omega(V_3)\tau hV_2 + \omega(V_2)\tau hV_3. \end{aligned} \tag{3.6}$$

By a straightforward computation, we get

$$\omega((\nabla_{V_3}h)V_2 - (\nabla_{V_2}h)V_3) = 0. \tag{3.7}$$

Using (3.7) in (3.6), we obtain the desired result. □

In contrast with the usual Ricci tensor, the ⋆-Ricci tensor is non-symmetric, in general. But, in the following, we show that, in particular, for a 3-dimensional almost coKähler manifold, the ⋆-Ricci tensor is symmetric.

Lemma 3.2. *The ⋆-Ricci curvature and ⋆-scalar curvature of a 3-dimensional ω-Einstein almost coKähler manifold are, respectively, given by*

$$S^\star = \frac{a-b}{2}(g - \omega\otimes\omega), \tag{3.8}$$

$$r^\star = a-b.$$

Proof. By virtue of (2.3), we infer

$$\nabla_{V_2}\tau V_3 = g(hV_2, V_3)\theta - \omega(V_3)hV_2 + \tau\nabla_{V_2}V_3, \tag{3.9}$$

which implies

$$\begin{aligned}\nabla_{V_1}\nabla_{V_2}\tau V_3 =& \nabla_{V_1}g(hV_2, V_3)\theta + g(hV_2, V_3)\nabla_{V_1}\theta\\ &-\nabla_{V_1}\omega(V_3)hV_2 - \omega(V_3)\nabla_{V_1}hV_2 + \nabla_{V_1}\tau\nabla_{V_2}V_3.\end{aligned} \tag{3.10}$$

Replacing V_2 by V_1 and V_3 by $\nabla_{V_2}V_3$ in (3.9), we obtain

$$\nabla_{V_1}\tau\nabla_{V_2}V_3 = g(hV_1, \nabla_{V_2}V_3)\theta - \omega(\nabla_{V_2}V_3)hV_1 + \tau\nabla_{V_1}\nabla_{V_2}V_3. \tag{3.11}$$

Using (2.2) and (3.11) in (3.10), we infer

$$\begin{aligned}\nabla_{V_1}\nabla_{V_2}\tau V_3 =& \nabla_{V_1}g(hV_2, V_3)\theta + g(hV_2, V_3)h\tau V_1\\ &-\nabla_{V_1}\omega(V_3)hV_2 - \omega(V_3)\nabla_{V_1}hV_2 + g(hV_1, \nabla_{V_2}V_3)\theta\\ &-\omega(\nabla_{V_2}V_3)hV_1 + \tau\nabla_{V_1}\nabla_{V_2}V_3.\end{aligned} \tag{3.12}$$

Interchanging V_1 and V_2 in the above equation, we get

$$\begin{aligned}\nabla_{V_2}\nabla_{V_1}\tau V_3 =& \nabla_{V_2}g(hV_1, V_3)\theta + g(hV_1, V_3)h\tau V_2\\ &-\nabla_{V_2}\omega(V_3)hV_1 - \omega(V_3)\nabla_{V_2}hV_1 + g(hV_2, \nabla_{V_1}V_3)\theta\\ &-\omega(\nabla_{V_1}V_3)hV_2 + \tau\nabla_{V_2}\nabla_{V_1}V_3.\end{aligned}$$

Also, from (3.9), we have

$$\nabla_{[V_1,V_2]}\tau V_3 = g(h[V_1, V_2], V_3)\theta - \omega(V_3)h[V_1, V_2] + \tau\nabla_{[V_1,V_2]}V_3. \tag{3.13}$$

From (3.12) and (3.13), we obtain

$$\begin{aligned}R(V_1, V_2)\tau V_3 =& [g((\nabla_{V_1}h)V_2, V_3) - g((\nabla_{V_2}h)V_1, V_3)]\theta + g(hV_2, V_3)h\tau V_1\\ &-g(hV_1, V_3)h\tau V_2 - (\nabla_{V_1}\omega)(V_3)hV_2 + (\nabla_{V_2}\omega)(V_3)hV_1\\ &-\omega(V_3)[(\nabla_{V_1}h)V_2 - (\nabla_{V_2}h)V_1] + \tau R(V_1, V_2)V_3.\end{aligned} \tag{3.14}$$

Using (3.3) in (3.14), we get

$$\begin{aligned}R(V_1, V_2)\tau V_3 =& \frac{a+b}{2}[g(\tau V_1, V_3)\omega(V_2) - g(\tau V_2, V_3)\omega(V_1)]\theta\\ &+g(\tau hV_2, V_3)\omega(V_1)\theta - g(\tau hV_1, V_3)\omega(V_2)\theta + g(hV_2, V_3)h\tau V_1\\ &-g(hV_1, V_3)h\tau V_2 - g(h\tau V_1, V_3)hV_2 + g(h\tau V_2, V_3)hV_1\\ &-\omega(V_3)\Big[\frac{a+b}{2}[\omega(V_2)\tau V_1 - \omega(V_1)\tau V_2] + \omega(V_1)\tau hV_2\\ &-\omega(V_2)\tau hV_1\Big] + \tau R(V_1, V_2)V_3.\end{aligned}$$

Taking the inner product with τW in the above equation, we have

$$\begin{aligned}g(R(V_1, V_2)\tau V_3, \tau W) =& - g(hV_2, V_3)g(hV_1, W) + g(hV_1, V_3)g(hV_2, W)\\ &-g(h\tau V_1, V_3)g(h\tau V_2, W) + g(h\tau V_2, V_3)g(h\tau V_1, W)\\ &-\omega(V_3)\Big[\frac{a+b}{2}[g(V_1, W)\omega(V_2) - g(V_2, W)\omega(V_1)]\\ &+g(hV_2, W)\omega(V_1) - g(hV_1, W)\omega(V_2)\Big]\\ &+g(R(V_1, V_2)V_3, W) - \omega(R(V_1, V_2)V_3)\omega(W).\end{aligned} \tag{3.15}$$

Contracting the above equation, we obtain

$$S^\star(V_2, V_3) = \frac{a-b}{2}[g(V_2, V_3) - \omega(V_2)\omega(V_3)].$$

Again, contracting the previous relation, we infer

$$r^\star = a - b. \qquad \square$$

Theorem 3.3. *If a 3-dimensional ω-Einstein almost coKähler manifold is a $\star$-Ricci–Yamabe soliton with the potential vector field of gradient type, then the potential vector field is pointwise collinear with the Reeb vector field.*

Proof. Let $V = D\psi$, where D denotes the gradient operator. We denote the star Ricci operator by $Q^\star$ given by $g(Q^\star V_1, V_2) = S^\star(V_1, V_2)$. Then, from equation (1.5), we have

$$\nabla_{V_1} D\psi = \Big(\gamma - \frac{1}{2} m r^\star\Big) V_1 - l Q^\star V_1. \tag{3.16}$$

The above equation yields

$$R(V_1, V_2)D\psi = (\nabla_{V_2} Q^\star)V_1 - (\nabla_{V_1} Q^\star)V_2. \tag{3.17}$$

In view of Lemma 3.2, it follows

$$(\nabla_{V_1} Q^\star)V_2 = -\frac{a-b}{2}[g(h\tau V_1, V_2)\theta + \omega(V_2)h\tau V_1]. \tag{3.18}$$

Applying (3.18) in (3.17), we obtain

$$R(V_1, V_2)D\psi = \frac{a-b}{2}[\omega(V_2)h\tau V_1 - \omega(V_1)h\tau V_2].$$

Putting $V_1 = \theta$ in the above equation, we infer

$$R(\theta, V_2)D\psi = -\frac{a-b}{2} h\tau V_2.$$

Taking the inner product in both sides with V_1, we have

$$g(R(\theta, V_2)D\psi, V_1) = -\frac{a-b}{2} g(V_1, h\tau V_2), \tag{3.19}$$

which yields

$$g(R(\theta, V_2)V_1, D\psi) = \frac{a-b}{2} g(V_1, h\tau V_2).$$

In view of (3.2), the above equation gives

$$(a+b)[g(\theta, D\psi)V_2 - g(V_2, D\psi)\theta] = (a-b)h\tau V_2. \tag{3.20}$$

Taking the inner product in both sides with θ, we have

$$D\psi = \omega(D\psi)\theta,$$

provided that $a + b \neq 0$. Thus, V is pointwise collinear with θ. If $a + b = 0$, (3.20) implies $a - b = 0$, so $a = b = 0$, which is discarded because in that case, the Ricci tensor is identically zero. $\square$

In the following theorem, we will see what happens for the converse case.

Theorem 3.4. *If a 3-dimensional ω-Einstein almost coKähler manifold admits a $\star$-Ricci–Yamabe soliton with the potential vector field V pointwise collinear with the Reeb vector field, $V = f\theta$ with f nowhere zero along the integral curves of θ, then the potential vector field is pointwise collinear with the gradient of f.*

Proof. In view of (2.2), we get

$$\nabla_{V_1} V = (V_1 f)\theta + f(h\tau V_1). \tag{3.21}$$

From (1.4), we have

$$\begin{aligned}&(\mathcal{L}_V g)(V_1, V_2) - g(\nabla_V V_1, V_2) + g((V_1 f)\theta + f(h\tau V_1), V_2)\\ &- g(V_1, \nabla_V V_2) + g(V_1, (V_2 f)\theta + f(h\tau V_2)) + 2lS^\star(V_1, V_2)\\ &= (2\gamma - m r^\star) g(V_1, V_2).\end{aligned} \tag{3.22}$$

Putting $V_2 = \tau V_1$, we have from above

$$g(Df, \tau V_1) = 0.$$

Replacing V_1 by τV_1, in the above equation, we obtain

$$Df = (\theta f)\theta.$$

If f is nowhere zero along the integral curves of θ, then

$$V = \frac{f}{(\theta f)} Df. \quad \square$$

So far, in this section, we have considered 3-dimensional almost coKähler manifolds with $\star$-Ricci–Yamabe solitons. In the following we shall establish a result for $(2n+1)$-dimensional compact manifolds. Let us now prove the following theorem which is a kind of an extension of a result given by Wang [33].

Theorem 3.5. *If a compact and connected almost coKähler manifold is a $\star$-Ricci–Yamabe soliton with the potential vector field pointwise collinear with the Reeb vector field θ, then the manifold is coKähler and $\star$-Ricci flat; in addition, the corresponding soliton is steady, provided that $r^\star$ is constant along the integral curves of θ.*

Proof. If the potential vector field V is pointwise collinear with the Reeb vector field θ, $V = f\theta$, for a smooth function f defined on the manifold, we have

$$\nabla_{V_1} V = (V_1 f)\theta + f(h' V_1).$$

Using the above condition in (1.4), we infer

$$V_1(f)\omega(V_2) + V_2(f)\omega(V_1) + 2fg(h'V_1, V_2) + 2lS^\star(V_1, V_2) = 2\Big(\gamma - \frac{mr^\star}{2}\Big)g(V_1, V_2). \quad (3.23)$$

The above equation gives

$$V_1(f)\theta + \omega(V_1)Df + 2fh'V_1 + 2lQ^\star V_1 = (2\gamma - mr^\star)V_1. \quad (3.24)$$

Contracting V_1 and V_2 in (3.23) with respect to a τ-basis, we obtain

$$\theta f = (2n+1)\Big(\gamma - \frac{mr^\star}{2}\Big) - lr^\star. \quad (3.25)$$

By covariant differentiation of (3.24) with respect to V_2 and then contracting V_2 in the resultant equation, we infer

$$\begin{aligned} &\theta(V_1(f)) + Df(\omega(V_1)) + \omega(V_1)\Delta f + 2(h'V_1)(f) + 2f\mathrm{div}(h'V_1) \\ &+ 2l(\mathrm{div}Q^\star)(V_1) + 2l\sum_{i=1}^{2n+1} g(Q^\star \nabla_{e_i} V_1, e_i) = (2\gamma - mr^\star)\mathrm{div}V_1, \end{aligned} \quad (3.26)$$

where Δ is the Laplacian operator. Replacing V_1 by θ and using the facts that $\mathrm{div}Q^\star = \frac{1}{2}dr^\star$ and $\mathrm{div}\theta = 0$, we get that

$$\theta(\theta(f)) + \Delta f + l\theta r^\star + 2l\mathrm{tr}(Q^\star h') = 0.$$

By virtue of (3.25), the above equation gives

$$\Delta f + 2l\mathrm{tr}(Q^\star h') = \frac{2n+1}{2}\theta(mr^\star). \quad (3.27)$$

In (3.24), replacing V_1 by hV_1 and then contracting V_1 in the resulting equation, we infer

$$l\mathrm{tr}(Q^\star h') + f\mathrm{tr}(h^2) = 0. \quad (3.28)$$

Combining (3.27) and (3.28), we get

$$\Delta f^2 = 2\|Df\|^2 + 2f\Big(\frac{2n+1}{2}\theta(mr^\star) + 2f\mathrm{tr}(h^2)\Big).$$

Assuming $\theta(r^\star)=0$, we have

$$\Delta f^2 = 2\|Df\|^2 + 4f^2 \mathrm{tr}(h^2).$$

Considering the manifold compact and using the divergence theorem, we obtain

$$\int_{\mathcal{M}} (\|Df\|^2 + 2f^2 \mathrm{tr}(h^2)) d\mathcal{M} = 0.$$

As a consequence of the above equation, we get f is a non-zero constant and $h=0$. Hence by following the same arguments of the paper [33], it can be easily concluded that the manifold is coKähler and ⋆-Ricci flat. The corresponding soliton is steady. This completes the proof. □

4. ⋆-Ricci–Yamabe solitons on (κ,μ)-almost coKähler manifolds

Let us first prove the following.

Lemma 4.1. *The ⋆-Ricci curvature and ⋆-scalar curvature of a $(2n+1)$-dimensional (κ,μ)-almost coKähler manifold are, respectively, given by*

$$S^\star = -\kappa(g - \omega\otimes\omega), \tag{4.1}$$

$$r^\star = -2n\kappa. \tag{4.2}$$

Proof. Applying (2.9) in (3.14), we obtain

$$\begin{aligned} R(V_1,V_2)\tau V_3 =& \kappa[g(\tau V_1,V_3)\omega(V_2) - g(\tau V_2,V_3)\omega(V_1)\\ &+2g(\tau V_1,V_2)\omega(V_3)]\theta + \mu[g(\tau h V_1,V_3)\omega(V_2)\\ &-g(\tau h V_2,V_3)\omega(V_1)]\theta + g(hV_2,V_3)h\tau V_1 - g(hV_1,V_3)h\tau V_2\\ &-g(h\tau V_1,V_3)hV_2 + g(h\tau V_2,V_3)hV_1\\ &-\omega(V_3)[\kappa(\omega(V_2)\tau V_1 - \omega(V_1)\tau V_2 + 2g(\tau V_1,V_2)\theta)\\ &+\mu(\omega(V_2)\tau h V_1 - \omega(V_1)\tau h V_2)] + \tau R(V_1,V_2)V_3. \end{aligned}$$

Therefore,

$$\begin{aligned} g(R(V_1,V_2)\tau V_3,\tau W) =& -g(hV_2,V_3)g(hV_1,W) + g(hV_1,V_3)g(hV_2,W)\\ &-g(h\tau V_1,V_3)g(h\tau V_2,W) + g(h\tau V_2,V_3)g(h\tau V_1,W)\\ &-\kappa[g(V_1,W)\omega(V_2)\omega(V_3) - g(V_2,W)\omega(V_1)\omega(V_3)]\\ &-\mu[g(hV_1,W)\omega(V_2)\omega(V_3) - g(hV_2,W)\omega(V_1)\omega(V_3)]\\ &+g(R(V_1,V_2)V_3,W) - \omega(R(V_1,V_2)V_3)\omega(W). \end{aligned}$$

Contracting V_1 and W in the above equation, we obtain

$$\begin{aligned} S^\star(V_2,V_3) =& S(V_2,V_3) - \kappa[g(V_2,V_3) - \omega(V_2)\omega(V_3)]\\ &-2n\kappa\omega(V_2)\omega(V_3) - \mu g(hV_2,V_3). \end{aligned} \tag{4.3}$$

Using (2.8) in (4.3), we have

$$S^\star(V_2,V_3) = -\kappa[g(V_2,V_3) - \omega(V_2)\omega(V_3)].$$

Tracing V_2 and V_3 in the above equation, we obtain

$$r^\star = -2n\kappa.$$ □

Theorem 4.2. *If a (κ,μ)-almost coKähler manifold with $\kappa<0$ is a ⋆-Ricci–Yamabe soliton, then $\gamma = -nm\kappa$.*

Proof. From (1.4), (4.1) and (4.2), we get
$$(\mathcal{L}_V g)(V_2, V_3) = (2\gamma + 2l\kappa + 2nm\kappa)g(V_2, V_3) - 2l\kappa\omega(V_2)\omega(V_3). \tag{4.4}$$
By covariant derivative, we infer
$$(\nabla_{V_1}(\mathcal{L}_V g))(V_2, V_3) = -2l\kappa[(\nabla_{V_1}\omega)(V_2)\omega(V_3) + \omega(V_2)(\nabla_{V_1}\omega)(V_3)]. \tag{4.5}$$
From (2.2), we have
$$(\nabla_{V_1}\omega)(V_2) = g(h\tau V_1, V_2). \tag{4.6}$$
From (4.5) and (4.6), we obtain
$$(\nabla_{V_1}(\mathcal{L}_V g))(V_2, V_3) = -2l\kappa[g(h\tau V_1, V_2)\omega(V_3) + g(h\tau V_1, V_3)\omega(V_2)]. \tag{4.7}$$
Using formula for commutativity of Lie derivative and covariant derivative (for details see Yano [41], p.23), we have
$$\begin{aligned}&(\mathcal{L}_V(\nabla_{V_1} g) - \nabla_{V_1}(\mathcal{L}_V g) - \nabla_{[V,V_1]} g)(V_2, V_3)\\ &= -g((\mathcal{L}_V\nabla)(V_1, V_2), V_3) - g((\mathcal{L}_V\nabla)(V_1, V_3), V_2).\end{aligned}$$
Because of the parallelism of the metric tensor g, the above equation reduces to
$$(\nabla_{V_1}(\mathcal{L}_V g))(V_2, V_3) = g((\mathcal{L}_V\nabla)(V_1, V_2), V_3) + g((\mathcal{L}_V\nabla)(V_1, V_3), V_2).$$
From the above equation, we have
$$\begin{aligned}2g((\mathcal{L}_V\nabla)(V_1, V_2), V_3) =&(\nabla_{V_1}(\mathcal{L}_V g))(V_2, V_3) + (\nabla_{V_2}(\mathcal{L}_V g))(V_3, V_1)\\ &-(\nabla_{V_3}(\mathcal{L}_V g))(V_1, V_2).\end{aligned} \tag{4.8}$$
Using (4.7) in (4.8), we obtain
$$g((\mathcal{L}_V\nabla)(V_1, V_2), V_3) = -2l\kappa g(h\tau V_1, V_2)\omega(V_3),$$
from which we get
$$(\mathcal{L}_V\nabla)(V_1, V_2) = -2l\kappa g(h\tau V_1, V_2)\theta.$$
Taking the covariant derivative of the above equation with respect to the vector field V_1 and using (2.2), we infer
$$(\nabla_{V_1}(\mathcal{L}_V\nabla))(V_2, V_3) = -2l\kappa g((\nabla_{V_1} h\tau)V_2, V_3)\theta - 2l\kappa g(h\tau V_2, V_3)h\tau V_1. \tag{4.9}$$
According to Yano ([41, p. 23]), we have
$$(\mathcal{L}_V R)(V_1, V_2)V_3 = (\nabla_{V_1}(\mathcal{L}_V\nabla))(V_2, V_3) - (\nabla_{V_2}(\mathcal{L}_V\nabla))(V_1, V_3), \tag{4.10}$$
for any vector fields V_1, V_2, V_3.
Substituting (4.9) in (4.10), we obtain
$$\begin{aligned}(\mathcal{L}_V R)(V_1, V_2)V_3 =& - 2l\kappa[g((\nabla_{V_1} h\tau)V_2 - (\nabla_{V_2} h\tau)V_1, V_3)\theta\\ &+g(h\tau V_2, V_3)h\tau V_1 - g(h\tau V_1, V_3)h\tau V_2].\end{aligned}$$
Using (2.10) in the above equation, we get
$$\begin{aligned}(\mathcal{L}_V R)(V_1, V_2)V_3 =& - 2l\kappa\big[\kappa(g(V_1, V_3)\omega(V_2)\theta - g(V_2, V_3)\omega(V_1)\theta)\\ &+\mu[g(hV_1, V_3)\omega(V_2)\theta - g(hV_2, V_3)\omega(V_1)\theta]\\ &+g(h\tau V_2, V_3)h\tau V_1 - g(h\tau V_1, V_3)h\tau V_2\big].\end{aligned}$$
Contracting the above equation over V_1, we obtain
$$(\mathcal{L}_V S)(V_2, V_3) = 2l\kappa\mu g(hV_2, V_3). \tag{4.11}$$
From (2.8), we infer
$$S(V_2, V_3) = \mu g(hV_2, V_3) + 2n\kappa\omega(V_2)\omega(V_3). \tag{4.12}$$

Taking Lie derivative of (4.12) with respect to V and using (4.4), we obtain

$$\begin{aligned}(\mathcal{L}_V S)(V_2, V_3) =&\mu[(2\gamma + 2l\kappa + 2nm\kappa)g(hV_2, V_3)\\ &+g((\mathcal{L}_V h)V_2, V_3)] + 2n\kappa[(\mathcal{L}_V \omega)(V_2)\omega(V_3)\\ &+\omega(V_2)(\mathcal{L}_V \omega)(V_3)].\end{aligned} \tag{4.13}$$

Now,

$$\begin{aligned}(\mathcal{L}_V \omega)(V_2) =&(\mathcal{L}_V g)(V_2, \theta) + g(V_2, \mathcal{L}_V \theta)\\ =&(2\gamma + 2nm\kappa)\omega(V_2) + g(V_2, \mathcal{L}_V \theta),\end{aligned} \tag{4.14}$$

where we have used (4.4).
Using (4.14) in (4.13), we have

$$\begin{aligned}(\mathcal{L}_V S)(V_2, V_3) =&\mu[(2\gamma + 2l\kappa + 2nm\kappa)g(hV_2, V_3)\\ &+g((\mathcal{L}_V h)V_2, V_3)] + 2n\kappa[4(\gamma + nm\kappa)\omega(V_2)\omega(V_3)\\ &+g(V_2, \mathcal{L}_V \theta)\omega(V_3) + g(V_3, \mathcal{L}_V \theta)\omega(V_2)].\end{aligned} \tag{4.15}$$

Comparing (4.11) and (4.15), we obtain

$$\begin{aligned}&\mu[(2\gamma + 2l\kappa + 2nm\kappa)g(hV_2, V_3)\\ &+ g((\mathcal{L}_V h)V_2, V_3)] + 2n\kappa[4(\gamma + nm\kappa)\omega(V_2)\omega(V_3)\\ &+ g(V_2, \mathcal{L}_V \theta)\omega(V_3) + g(V_3, \mathcal{L}_V \theta)\omega(V_2)] = 2l\kappa\mu g(hV_2, V_3).\end{aligned}$$

Let $\{e_1, \ldots, e_{2n+1}\}$ denote an orthonormal τ-basis, with $e_{2n+1} = \theta$, of the tangent space at each point of the manifold, where $he_i = \sqrt{-\kappa}e_i$. Contracting V_2 and V_3 with respect to the above basis, we get

$$\omega(\mathcal{L}_V \theta) = -2(\gamma + nm\kappa). \tag{4.16}$$

Again, putting $V_2 = V_3 = \theta$ in (4.4), we obtain

$$\omega(\mathcal{L}_V \theta) = -(\gamma + nm\kappa). \tag{4.17}$$

Comparing (4.16) and (4.17), we get

$$\gamma = -nm\kappa.$$ □

As $\kappa < 0$, we state

Corollary 4.3. *Under the hypotheses of Theorem 4.2, the soliton is expanding or steady or shrinking according as the value of m is negative or zero or positive.*

Remark 4.4. If $m = 0$, we see from Theorem 4.2 that $\gamma = 0$ and the soliton is l-almost ⋆-Ricci soliton. In that case, the soliton is steady. If $l = 1$ and $m = 0$, the soliton is a ⋆-Ricci soliton. Such solitons are also steady in (κ, μ)-almost coKähler manifolds.

Theorem 4.5. *If a (κ, μ)-almost coKähler manifold with $\kappa < 0$ is a ⋆-Ricci–Yamabe soliton and the potential vector field V is pointwise collinear with the Reeb vector field θ, then V is a constant multiple of θ.*

Proof. If the potential vector field V is pointwise collinear with θ, that is, $V = \rho\theta$, where ρ is a smooth function, from (1.4), we get

$$\begin{aligned}&\rho g(\nabla_{V_1}\theta, V_2) + (V_1\rho)\omega(V_2) + \rho g(\nabla_{V_2}\theta, V_1)\\ &+ (V_2\rho)\omega(V_1) + 2lS^\star(V_1, V_2) = (2\gamma - mr^\star)g(V_1, V_2).\end{aligned} \tag{4.18}$$

Using (2.2), (4.1) and (4.2) in (4.18), we obtain

$$\begin{aligned}&2\rho g(h\tau V_1, V_2) + (V_1\rho)\omega(V_2) + (V_2\rho)\omega(V_1)\\ &= (2\gamma + 2nm\kappa + 2l\kappa)g(V_1, V_2) - 2l\kappa\omega(V_1)\omega(V_2).\end{aligned} \tag{4.19}$$

Putting $V_2 = \theta$ in (4.19), we get

$$(V_1\rho) + (\theta\rho)\omega(V_1) = (2\gamma + 2nm\kappa)\omega(V_1). \tag{4.20}$$

Again, putting $V_1 = \theta$ in (4.20), we obtain

$$(\theta\rho) = \gamma + nm\kappa. \tag{4.21}$$

From (4.20) and (4.21), we get

$$(V_1\rho) = (\gamma + nm\kappa)\omega(V_1). \tag{4.22}$$

Using (4.22) in (4.19), we obtain

$$\rho g(h\tau V_1, V_2) = (\gamma + l\kappa + nm\kappa)[g(V_1, V_2) - \omega(V_1)\omega(V_2)]. \tag{4.23}$$

Replacing V_1 by τV_1 in (4.23), we get

$$\rho g(hV_1, V_2) = -(\gamma + l\kappa + nm\kappa)g(\tau V_1, V_2).$$

Again, replacing V_1 by hV_1, we infer

$$g(h\tau V_1, V_2) = -\frac{\rho\kappa}{(\gamma + l\kappa + nm\kappa)}[g(V_1, V_2) - \omega(V_1)\omega(V_2)]. \tag{4.24}$$

Using (4.24) in (4.23), we obtain

$$\Big[\frac{\rho^2\kappa}{(\gamma + l\kappa + nm\kappa)} + (\gamma + l\kappa + nm\kappa)\Big][g(V_1, V_2) - \omega(V_1)\omega(V_2)] = 0,$$

which is true for any vector fields V_1, V_2. Thus, from above, we get

$$\rho^2 = -\frac{(\gamma + l\kappa + nm\kappa)^2}{\kappa},$$

from which we conclude that ρ is a constant. □

Theorem 4.6. *A (κ, μ)-almost coKähler manifold can not be a $\star$-Ricci–Yamabe soliton if the potential vector field is the Reeb vector field θ.*

Proof. If the potential vector field V is the Reeb vector field θ, then from (4.19), we get

$$g(h\tau V_1, V_2) = (\gamma + nm\kappa + l\kappa)[g(V_1, V_2) - \omega(V_1)\omega(V_2)],$$

which implies

$$h\tau V_1 = (\gamma + l\kappa + nm\kappa)V_1 - \omega(V_1)\theta. \tag{4.25}$$

Operating both sides of (4.25) by τ and using (2.1), we obtain

$$hV_1 = (\gamma + l\kappa + nm\kappa)\tau V_1. \tag{4.26}$$

Again, operating both sides of (4.26) by h and using (2.5), we get

$$\kappa\tau^2 V_2 = (\gamma + l\kappa + nm\kappa)h\tau V_1. \tag{4.27}$$

Tracing the above equation and using $\mathrm{tr}(h\tau) = 0$, we infer $\kappa = 0$, which is a contradiction. Thus, a $(2n+1)$-dimensional (κ, μ)-almost coKähler manifold can not be a $\star$-Ricci–Yamabe soliton if the potential vector is the Reeb vector field θ. □

Due to Blair ([5, p. 72]) and Tanno [32], we give the following definition

Definition 4.7. A vector field V on an almost contact metric manifold is called an infinitesimal contact transformation if it satisfies

$$\mathcal{L}_V\omega = f\omega,$$

for f a smooth function on $\mathcal{M}$. If $f = 0$, then the vector field V is called a strict infinitesimal contact transformation.

Theorem 4.8. *If a (κ, μ)-almost coKähler manifold with $\kappa < 0$ is a $\star$-Ricci–Yamabe soliton, then the potential vector field is an infinitesimal contact transformation.*

Proof. Applying $V_3=\theta$ in (4.4), we infer

$$(\mathcal{L}_V g)(V_2,\theta)=2(\gamma+nm\kappa)\omega(V_2). \tag{4.28}$$

Again, replacing V_2 by θ

$$g(\mathcal{L}_V\theta,\theta)=-(\gamma+nm\kappa),$$

which implies

$$\mathcal{L}_V\theta=-(\gamma+nm\kappa)\theta. \tag{4.29}$$

Taking Lie derivative of $\omega(V_2)=g(V_2,\theta)$ with respect to V, we have

$$(\mathcal{L}_V\omega)(V_2)=(\mathcal{L}_V g)(V_2,\theta)+g(V_2,\mathcal{L}_V\theta). \tag{4.30}$$

Using (4.28) and (4.29) in (4.30), we obtain

$$(\mathcal{L}_V\omega)(V_2)=(\gamma+nm\kappa)\omega(V_2),$$

it follows that V is an infinitesimal contact transformation. □

As a consequence of the above theorem, we can state the following

Corollary 4.9. *If the potential vector field of a ⋆-Ricci–Yamabe soliton in a (κ,μ)-almost coKähler manifold is a strict infinitesimal contact transformation, then $\gamma=-nm\kappa$.*

5. Gradient ⋆-Ricci–Yamabe solitons on (κ,μ)-almost coKähler manifolds

In this section we study gradient ⋆-Ricci–Yamabe solitons on (κ,μ)-almost coKähler manifold.

Theorem 5.1. *If a (κ,μ)-almost coKähler manifold with $\kappa<0$ is a gradient ⋆-Ricci–Yamabe soliton, then either $\mu^2=-\kappa$ or the soliton is trivial.*

Proof. With the help of (1.5), we have

$$\nabla_{V_1}D\psi=\Big(\gamma-\frac{1}{2}mr^\star\Big)V_1-lQ^\star V_1, \tag{5.1}$$

where ψ is a smooth function on the manifold.
Using (4.1) and (4.2) in (5.1), we obtain

$$\nabla_{V_1}D\psi=(\gamma+l\kappa+nm\kappa)V_1-l\kappa\omega(V_1)\theta. \tag{5.2}$$

Differentiating (5.2) covariantly with respect to V_2, we get

$$\begin{aligned}\nabla_{V_2}\nabla_{V_1}D\psi=&(\gamma+l\kappa+nm\kappa)\nabla_{V_2}V_1\\&-l\kappa[\nabla_{V_2}\omega(V_1)\theta+\omega(V_1)\nabla_{V_2}\theta].\end{aligned} \tag{5.3}$$

Interchanging V_1 and V_2 in (5.3), we obtain

$$\begin{aligned}\nabla_{V_1}\nabla_{V_2}D\psi=&(\gamma+l\kappa+nm\kappa)\nabla_{V_1}V_2\\&-l\kappa[\nabla_{V_1}\omega(V_2)\theta+\omega(V_2)\nabla_{V_1}\theta].\end{aligned} \tag{5.4}$$

Also, from (5.2), we get

$$\nabla_{[V_1,V_2]}D\psi=(\gamma+l\kappa+nm\kappa)[V_1,V_2]-l\kappa\omega([V_1,V_2])\theta. \tag{5.5}$$

Using (2.2), (5.3)-(5.5), we obtain

$$R(V_1,V_2)D\psi=-l\kappa[\omega(V_2)h\tau V_1-\omega(V_1)h\tau V_2]. \tag{5.6}$$

By (5.6), we get

$$g(R(V_1,V_2)D\psi,\theta)=0. \tag{5.7}$$

Also from (2.4), we have

$$\begin{aligned}g(R(V_1,V_2)\theta,D\psi)=&\kappa[(V_1\psi)\omega(V_2)-(V_2\psi)\omega(V_1)]\\&+\mu[(hV_1\psi)\omega(V_2)-(hV_2\psi)\omega(V_1)].\end{aligned} \tag{5.8}$$

As $g(R(V_1,V_2)V_3,W)=-g(R(V_1,V_2)W,V_3)$, from (5.7) and (5.8), we get

$$\kappa[(V_1\psi)\omega(V_2)-(V_2\psi)\omega(V_1)]+\mu[(hV_1\psi)\omega(V_2)-(hV_2\psi)\omega(V_1)]=0.$$

Putting $V_2=\theta$ in the above equation, we obtain

$$\kappa[(V_1\psi)-(\theta\psi)\omega(V_1)]+\mu(hV_1\psi)=0. \tag{5.9}$$

Replacing V_1 by hV_1 in (5.9), we get

$$(hV_1\psi)=\mu[(V_1\psi)-(\theta\psi)\omega(V_1)]. \tag{5.10}$$

Therefore, from (5.9) and (5.10), we obtain

$$(\kappa+\mu^2)[(V_1\psi)-(\theta\psi)\omega(V_1)]=0,$$

thus we get either $\kappa=-\mu^2$ or $D\psi=(\theta\psi)\theta$.
When $D\psi=(\theta\psi)\theta$, from (5.2), we infer

$$V_1(\theta\psi)\theta+(\theta\psi)h\tau V_1=(\gamma+l\kappa+nm\kappa)V_1-l\kappa\omega(V_1)\theta. \tag{5.11}$$

Taking the inner product of (5.11) with V_2, we get

$$V_1(\theta\psi)\omega(V_2)+(\theta\psi)g(h\tau V_1,V_2)=(\gamma+l\kappa+nm\kappa)g(V_1,V_2)-l\kappa\omega(V_1)\omega(V_2). \tag{5.12}$$

Putting $V_2=\theta$ in the above equation, we obtain

$$V_1(\theta\psi)=(\gamma+nm\kappa)\omega(V_1). \tag{5.13}$$

From (5.12) and (5.13), we get

$$(\theta\psi)g(h\tau V_1,V_2)=(\gamma+l\kappa+nm\kappa)[g(V_1,V_2)-\omega(V_1)\omega(V_2)]. \tag{5.14}$$

Contracting V_1 and V_2 and using $\mathrm{tr}(\tau h)=0$, we obtain

$$(\gamma+l\kappa+nm\kappa)=0. \tag{5.15}$$

Using (5.15) in (5.14), we obtain

$$(\theta\psi)g(h\tau V_1,V_2)=0,$$

which gives $(\theta\psi)=0$. Thus, from the relation $D\psi=(\theta\psi)\theta$, we get $D\psi=0$, i.e, $V=0$, which shows that the soliton is trivial. □

Corollary 5.2. *If a (κ,μ)-almost coKähler manifold is a gradient $\star$-Ricci–Yamabe soliton, then $l=0$.*

Proof. Applying $D\psi=0$ in (5.2), we obtain

$$(\gamma+l\kappa+nm\kappa)g(V_1,V_2)-l\kappa\omega(V_1)\omega(V_2)=0. \tag{5.16}$$

Contracting the above equation, we have

$$\gamma=-\frac{2nl\kappa+n(2n+1)m\kappa}{2n+1}. \tag{5.17}$$

Again, putting $V_1=V_2=\theta$ in (5.16), we infer

$$\gamma=-nm\kappa. \tag{5.18}$$

Comparing (5.17) and (5.18), we get $l\kappa=0$. As $\kappa<0$, we get that $l=0$. □

6. Examples

Example 6.1. Let $\mathcal{M} = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. We consider the linearly independent vector fields

$$u_1 = \frac{\partial}{\partial x}, \quad u_2 = \frac{\partial}{\partial y}, \quad u_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

By a direct computation, we get

$$[u_1, u_2] = 0, \quad [u_1, u_3] = u_1, \quad [u_2, u_3] = u_2.$$

Let g be the Riemannian metric defined by $g(u_i, u_j) = \delta_{ij}$, $i, j = 1, 2, 3$.
We define the 1-form ω by $\omega(V_1) = g(V_1, u_3)$ for all vector fields V_1 on $\mathcal{M}$.
Let τ be the $(1, 1)$-tensor field represented by

$$\tau u_1 = u_2, \quad \tau u_2 = -u_1, \quad \tau u_3 = 0.$$

It is easy to see that

$$\omega(u_3) = 1, \quad \tau^2 V_1 = -V_1 + \omega(V_1)u_3,$$
$$g(\tau V_1, \tau V_2) = g(V_1, V_2) - \omega(V_1)\omega(V_2),$$

for any vector fields V_1, V_2 on $\mathcal{M}$. Thus (τ, u_3, ω, g) defines an almost contact metric structure.
By the formula due to Koszul, we have the following

$$\nabla_{u_1}u_1 = -u_3, \quad \nabla_{u_1}u_2 = 0, \quad \nabla_{u_1}u_3 = u_1,$$
$$\nabla_{u_2}u_2 = -u_3, \quad \nabla_{u_2}u_1 = 0, \quad \nabla_{u_2}u_3 = u_2,$$
$$\nabla_{u_3}u_1 = 0, \quad \nabla_{u_3}u_2 = 0, \quad \nabla_{u_3}u_3 = 0.$$

Clearly $\mathcal{M}$ is an almost coKähler manifold with $hu_1 = -u_2$, $hu_2 = u_1$ and $hu_3 = 0$.
The components of the curvature tensor are

$$R(u_1, u_2)u_1 = u_2, \quad R(u_1, u_2)u_2 = -u_1, \quad R(u_2, u_3)u_3 = -u_2,$$

$$R(u_3, u_2)u_2 = -u_3, \quad R(u_1, u_3)u_3 = -u_1, \quad R(u_3, u_1)u_1 = -u_3,$$

$$R(u_1, u_2)u_3 = 0, \quad R(u_2, u_3)u_1 = 0, \quad R(u_3, u_1)u_2 = 0.$$

The only non-vanishing components of the Ricci curvature tensor are

$$S(u_1, u_1) = -2, \quad S(u_2, u_2) = -2, \quad S(u_3, u_3) = -2,$$

thus

$$S(V_1, V_2) = -2g(V_1, V_2)$$

for any vector fields V_1, V_2 on $\mathcal{M}$ and the scalar curvature r is -6. Thus we can write the curvature tensor as

$$\begin{aligned} R(V_1, V_2)V_3 =& \frac{a-b}{2}[g(V_2, V_3)V_1 - g(V_1, V_3)V_2] \\ &+ b[g(V_2, V_3)\omega(V_1)\theta - g(V_1, V_3)\omega(V_2)\theta \\ &+ \omega(Y)\omega(V_3)V_2 - \omega(V_1)\omega(V_3)V_2], \end{aligned}$$

for any vector fields V_1, V_2 and V_3 on $\mathcal{M}$, where $a = -2$, $b = 0$.
The non-zero components of the ⋆-Ricci curvature are

$$S^\star(u_1, u_1) = -1, \quad S^\star(u_2, u_2) = -1.$$

Therefore the ⋆-scalar curvature $r^\star$ is -2.
Let $V = c_1u_1 + c_2u_2 + c_3u_3$ be the potential vector field, where $c_1, c_2, c_3 \in \mathbb{R}$. Then

$$(\mathcal{L}_V g)(u_1, u_1) = 2c_3, \quad (\mathcal{L}_V g)(u_2, u_2) = 2c_3, \quad (\mathcal{L}_V g)(u_3, u_3) = 0.$$

Thus, from (1.4), we obtain the following equations

$$\gamma - l - m = c_3,$$
$$\gamma - m = 0.$$

From the above two equations, we get $l = -c_3$. Thus $\mathcal{M}$ is a $\star$-Ricci–Yamabe soliton with $l = -c_3$, $\gamma = m$, where m can be any real number.
Now, let $V = fu_3$ where f is a smooth function on $\mathcal{M}$. Then

$$(\mathcal{L}_V g)(u_1, u_1) = 2f, \quad (\mathcal{L}_V g)(u_2, u_2) = 2f, \quad (\mathcal{L}_V g)(u_3, u_3) = 2(u_3 f).$$

Thus, from (1.4), we obtain the following equations

$$f - \gamma - l - m = 0, \tag{6.1}$$
$$(u_3 f) - \gamma - m = 0. \tag{6.2}$$

Equation (6.1) implies $f = \gamma + l + m$, which is a constant. Thus $(u_3 f) = 0$, so, from (6.2) we have $\gamma = -m$ and finally, from (6.1), we get $f = l$. The above data verifies Theorem 3.4.

Example 6.2. Let $\mathcal{M} = \{(x, y, z, u, v) \in \mathbb{R}^5 : v \neq 0\}$, where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$. We consider the linearly independent vector fields

$$u_1 = e^{2v}\frac{\partial}{\partial x}, \quad u_2 = e^{2v}\frac{\partial}{\partial y}, \quad u_3 = e^{-2v}\frac{\partial}{\partial z}, \quad u_4 = e^{-2v}\frac{\partial}{\partial u}, \quad u_5 = \frac{\partial}{\partial v}.$$

By a direct computation, we get

$$[u_1, u_5] = -2u_1, \quad [u_2, u_5] = -2u_2, \quad [u_3, u_5] = 2u_3, \quad [u_4, u_5] = 2u_4$$

and all the remaining brackets $[u_i, u_j] = 0$ for $i, j = 1, 2, 3, 4, 5$.
Let g be the Riemannian metric defined by $g(u_i, u_j) = \delta_{ij}$, $i, j = 1, 2, 3, 4, 5$.
We define the 1-form ω by $\omega(V_1) = g(V_1, u_5)$ for all vector fields V_1 on $\mathcal{M}$.
Let τ be the $(1, 1)$-tensor field represented by

$$\tau u_1 = u_3, \quad \tau u_2 = u_4, \quad \tau u_3 = -u_1, \quad \tau u_4 = -u_2, \quad \tau u_5 = 0.$$

Then it is easy to see that

$$\omega(u_5) = 1, \quad \tau^2 V_1 = -V_1 + \omega(V_1)u_5,$$
$$g(\tau V_1, \tau V_2) = g(V_1, V_2) - \omega(V_1)\omega(V_2),$$

for any vector fields V_1, V_2 on $\mathcal{M}$. Thus (τ, u_5, ω, g) defines an almost contact metric structure.
By the formula due to Koszul, we have the following

$$\nabla_{u_1} u_1 = 2u_5, \quad \nabla_{u_1} u_5 = -2u_1, \quad \nabla_{u_2} u_2 = 2u_5,$$
$$\nabla_{u_2} u_5 = -2u_2, \quad \nabla_{u_3} u_3 = -2u_5, \quad \nabla_{u_3} u_5 = 2u_3,$$
$$\nabla_{u_4} u_4 = -2u_5, \quad \nabla_{u_4} u_5 = 2u_4$$

and all the rest $\nabla_{u_i} u_j = 0$ for $i, j = 1, 2, 3, 4, 5$. Clearly $\mathcal{M}$ is an almost coKähler manifold with $hu_1 = -2u_3$, $hu_2 = -2u_4$, $hu_3 = -2u_1$, $hu_4 = -2u_2$ and $hu_5 = 0$.
The components of the curvature tensor are

$$R(u_1, u_2)u_1 = 4u_2, \quad R(u_1, u_2)u_2 = -4u_1, \quad R(u_1, u_3)u_1 = -4u_3,$$

$$R(u_1, u_3)u_3 = 4u_1, \quad R(u_1, u_4)u_1 = -4u_4, \quad R(u_1, u_4)u_4 = 4u_1,$$

$$R(u_1, u_5)u_1 = 4u_5, \quad R(u_1, u_5)u_5 = -4u_1, \quad R(u_2, u_3)u_2 = -4u_3.$$

$$R(u_2, u_3)u_3 = 4u_2, \quad R(u_2, u_4)u_2 = -4u_4, \quad R(u_2, u_4)u_4 = 4u_2,$$

$$R(u_2,u_5)u_2 = 4u_5, \quad R(u_2,u_5)u_5 = -4u_2, \quad R(u_3,u_4)u_3 = 4u_4,$$

$$R(u_3,u_4)u_4 = -4u_3, \quad R(u_3,u_5)u_3 = 4u_5, \quad R(u_3,u_5)u_5 = -4u_3,$$

$$R(u_4,u_5)u_4 = 4u_5, \quad R(u_4,u_5)u_5 = -4u_4$$

and all the other $R(u_i,u_j)e_k = 0$ for $i,j,k = 1,2,3,4,5$. From above, we conclude that $\mathcal{M}$ is a (κ,μ)-almost coKähler manifold with $\kappa = -4$ and $\mu = 0$.
The only non-vanishing component of the Ricci curvature tensor is $S(u_5,u_5) = -16$ and the scalar curvature r is -16.
The non-zero components of the ⋆-Ricci curvature are

$$S^\star(u_1,u_1) = S^\star(u_2,u_2) = S^\star(u_3,u_3) = S^\star(u_4,u_4) = 4.$$

Therefore, the ⋆-scalar curvature $r^\star$ is 16.
Let us consider the potential vector field V as the Reeb vector field θ. Then

$$(\mathcal{L}_V g)(u_1,u_1) = -4, \quad (\mathcal{L}_V g)(u_2,u_2) = -4,$$
$$(\mathcal{L}_V g)(u_3,u_3) = 4, \quad (\mathcal{L}_V g)(u_4,u_4) = 4, \quad (\mathcal{L}_V g)(u_5,u_5) = 0.$$

Thus, from (1.4), we obtain the following equations

$$\gamma - 4l - 8m + 2 = 0, \tag{6.3}$$
$$\gamma - 4l - 8m - 2 = 0, \tag{6.4}$$
$$\gamma = 8m.$$

Equations (6.3) and (6.4) are inconsistent. Thus $\mathcal{M}$ does not admit a ⋆-Ricci–Yamabe soliton if the potential vector field is the Reeb vector field θ, which verifies Theorem 4.5.
Now, let the potential vector field V be the gradient of a smooth function ψ on $\mathcal{M}$. Then, from (1.5), we obtain

$$\nabla_{V_1} D\psi = \Big(\gamma - \frac{1}{2}mr^\star\Big)V_1 - lQ^\star V_1.$$

Also, $D\psi$ can be written as

$$D\psi = (u_1\psi)u_1 + (u_2\psi)u_2 + (u_3\psi)u_3 + (u_4\psi)u_4 + (u_5\psi)u_5.$$

Thus,

$$\begin{aligned}
\nabla_{u_1} D\psi =& (u_1(u_1\psi) - 2(u_5\psi))u_1 + u_1(u_2\psi)u_2 + u_1(u_3\psi)u_3 \\
&+ u_1(u_4\psi)u_4 + (u_1(u_5\psi) + 2(u_1\psi))u_5, \\
\nabla_{u_2} D\psi =& u_2(u_1\psi)u_1 + (u_2(u_2\psi) - 2(u_5\psi))u_2 + u_2(u_3\psi)u_3 \\
&+ u_2(u_4\psi)u_4 + (u_2(u_5\psi) + 2(u_2\psi))u_5, \\
\nabla_{u_3} D\psi =& u_3(u_1\psi)u_1 + u_3(u_2\psi)u_2 + (u_3(u_3\psi) + 2(u_5\psi))u_3 \\
&+ u_3(u_4\psi)u_4 + (u_3(u_5\psi) - 2(u_3\psi))u_5, \\
\nabla_{u_4} D\psi =& u_4(u_1\psi)u_1 + u_4(u_2\psi)u_2 + u_4(u_3\psi)u_3 \\
&+ (u_4(u_4\psi) + 2(u_5\psi))u_4 + (u_4(u_5\psi) - 2(u_4\psi))u_5, \\
\nabla_{u_5} D\psi =& u_5(u_1\psi)u_1 + u_5(u_2\psi)u_2 + u_5(u_3\psi)u_3 \\
&+ u_5(u_4\psi)u_4 + u_5(u_5\psi)u_5.
\end{aligned}$$

The last five equations imply

$$\begin{aligned}
u_1(u_1\psi) - 2(u_5\psi) &= \gamma - 4l - 8m, \\
u_2(u_2\psi) - 2(u_5\psi) &= \gamma - 4l - 8m, \\
u_3(u_3\psi) + 2(u_5\psi) &= \gamma - 4l - 8m, \\
u_4(u_4\psi) + 2(u_5\psi) &= \gamma - 4l - 8m,
\end{aligned}$$

$$u_5(u_5\psi) = \gamma - 8m,$$
$$u_1(u_5\psi) + 2(u_1\psi) = 0,$$
$$u_2(u_5\psi) + 2(u_2\psi) = 0,$$
$$u_3(u_5\psi) - 2(u_3\psi) = 0,$$
$$u_4(u_5\psi) - 2(u_4\psi) = 0,$$
$$u_1(u_2\psi) = 0, \quad u_1(u_3\psi) = 0,$$
$$u_1(u_4\psi) = 0, \quad u_2(u_3\psi) = 0,$$
$$u_2(u_4\psi) = 0, \quad u_3(u_4\psi) = 0,$$
$$u_5(u_1\psi) = 0, \quad u_5(u_2\psi) = 0,$$
$$u_5(u_3\psi) = 0, \quad u_5(u_4\psi) = 0.$$

Thus we get the following partial differential equations

$$e^{4v}\frac{\partial^2\psi}{\partial x^2} - 2\frac{\partial\psi}{\partial v} = \gamma - 4l - 8m, \tag{6.5}$$
$$e^{4v}\frac{\partial^2\psi}{\partial y^2} - 2\frac{\partial\psi}{\partial v} = \gamma - 4l - 8m,$$
$$e^{-4v}\frac{\partial^2\psi}{\partial z^2} + 2\frac{\partial\psi}{\partial v} = \gamma - 4l - 8m,$$
$$e^{-4v}\frac{\partial^2\psi}{\partial u^2} + 2\frac{\partial\psi}{\partial v} = \gamma - 4l - 8m,$$
$$\frac{\partial^2\psi}{\partial v^2} = \gamma - 8m, \tag{6.6}$$
$$e^{2v}\frac{\partial^2\psi}{\partial x\partial v} + 2e^{2v}\frac{\partial\psi}{\partial x} = 0,$$
$$e^{2v}\frac{\partial^2\psi}{\partial y\partial v} + 2e^{2v}\frac{\partial\psi}{\partial y} = 0,$$
$$e^{-2v}\frac{\partial^2\psi}{\partial z\partial v} - 2e^{-2v}\frac{\partial\psi}{\partial z} = 0,$$
$$e^{-2v}\frac{\partial^2\psi}{\partial u\partial v} - 2e^{-2v}\frac{\partial\psi}{\partial u} = 0,$$
$$e^{4v}\frac{\partial^2\psi}{\partial x\partial y} = 0, \quad \frac{\partial^2\psi}{\partial x\partial z} = 0,$$
$$\frac{\partial^2\psi}{\partial x\partial u} = 0, \quad \frac{\partial^2\psi}{\partial y\partial z} = 0,$$
$$\frac{\partial^2\psi}{\partial y\partial u} = 0, \quad e^{-4v}\frac{\partial^2\psi}{\partial z\partial u} = 0,$$

which show that ψ is a constant. Thus, Theorem 5.1 is verified. Hence, from (6.5)-(6.6), we have

$$\gamma - 4l - 8m = 0,$$
$$\gamma - 8m = 0.$$

The above two equations indicate that $l = 0$ which verifies Corollary 5.2. The value of γ is $8m$, where m is a real number. In particular, if we take $m = 1$, then the soliton becomes a gradient $\star$-Yamabe soliton and it is shrinking.

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References

[1] J.E. Andersen, *Geometric quantization of symplectic manifolds with respect to reducible non-negative polarization*, Commun. Math. Phys., **183**, 401–421, 1997.

[2] R.J. Berman, *Relative Kähler Ricci flow and their quantization*, Anal. PDE, **6**, 131–180, 2013.

[3] A.M. Blaga, *Geometric solitons in a D-homothetically deformed Kenmotsu manifold*, Filomat, **36**, 175–186, 2022.

[4] A.M. Blaga and H.M. Taştan, *Some results on almost η-Ricci–Bourguignon solitons*, J. Geom. Phys. **168**, 104316, 2021.

[5] D.E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, **203**, Birkhäuser, New York, 2010.

[6] D.E. Blair, *The theory of quasi-Sasakian structures*, J. Differential Geom. **1**, 331–345, 1967.

[7] D.E. Blair, T. Koufogiorgos, and B.J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel J. Math. **91**, 189–214, 1995.

[8] X. Chen, *Almost quasi-Yamabe solitons on almost cosymplectic manifolds*, Int. J. Geom. Methods Mod. Phys. **17**, 2050070, 2020.

[9] X. Chen, *Cotton solitons on almost coKähler 3-manifolds*, Quaest. Math. **44**, 1055–1075, 2021.

[10] X. Chen, *The k-almost Yamabe solitons and coKähler manifolds*, Int. J. Geom. Methods Mod. Phys. **18**, 2150179, 2021.

[11] X. Chen, *Three-dimensional contact metric manifolds with cotton solitons*, Hiroshima J. Math. **51**, 275–299, 2021.

[12] X. Dai, Y. Zhao, and U.C. De, *⋆-Ricci curvature on (κ,μ)-almost Kenmotsu manifolds*, Open Math. **17**, 874–882, 2019.

[13] U.C. De, S.K. Chaubey, and Y.J. Suh, *A note on almost coKähler manifolds*, Int. J. Geom. Methods Mod. Phys. **17** (10), 2050153, 2020.

[14] U.C. De, S.K. Chaubey, and Y.J. Suh, *Gradient Yamabe and gradient m-quasi-Einstein metrics on three-dimensional cosymplectic manifolds,* Mediterr. J. Math. **18**, Art. No. 80, 2021.

[15] U.C. De and A. Sardar, *Classification of (κ,μ)-almost coKähler manifolds with vanishing Bach tensor and divergence free Cotton tensor*, Commun. Korean Math. Soc. **35**, 1245–1254, 2020.

[16] U.C. De and Y.J. Suh, *Yamabe and quasi-Yamabe solitons in para contact manifolds*, Int. J. Geom. Methods Mod. Phys. **18**, 2150196, 2021.

[17] U.C. De, Y.J. Suh, and S.K. Chaubey, *Conformal vector fields on almost coKähler manifolds*, Math. Slovaca **71**, 1545–4552, 2021.

[18] A.E. Fischer, *An introduction to conformal Ricci flow*, Class. Quantum Grav. **21**, 171–218, 2004.

[19] S. Güler and M. Crasmareanu, *Ricci–Yamabe maps for Riemannian flow and their volume variation and volume entropy*, Turk. J. Math. **43**, 2631–2641, 2019.

[20] T. Hamada, *Real hypersurfaces of complex space forms in terms of Ricci $\star$-tensor*, Tokyo J. Math. **25**, 473–483, 2002.

[21] R.S. Hamilton, Lectures on geometric flows, 1989, (Unpublished).

[22] R.S. Hamilton, *The Ricci flow on surfaces*, Contemp. Math. **71**, 237–261, 1988.

[23] G. Kaimakamis and K. Panagiotidou, *$\star$-Ricci solitons of real hypersurfaces in non-flat complex space forms*, J. Geom. Phys. **86**, 408–413, 2014.

[24] T. Mandal, *Ricci–Yamabe solitons on (κ, μ)-almost coKähler manifolds*, Afr. Mat. **33**, Art. No. 38, 10pp, 2022.

[25] C. Özgür, *On Ricci solitons with a semisymmetric metric connection*, Filomat, **35**, 3635–3641, 2021.

[26] A. Sarkar and G.G. Biswas, *$\star$-Ricci solitons on three dimensional trans-Sasakian manifolds*, The Mathematics Student, **88**, 153–164, 2019.

[27] A. Sarkar and G.G. Biswas, Ricci solitons on generalized Sasakian space forms with quasi-Sasakian metric, Afr. Mat. **31**, 455–463, 2020.

[28] A. Sardar, M.N.I. Khan, and U.C. De, *η-$\star$-Ricci solitons and almost coKähler manifolds*, Mathematics **9**(24), 3200, 2021.

[29] R. Sharma, *A 3-dimensional Sasakian metric as a Yamabe soliton*, Int. J. Geom. Methods Mod. Phys, **9**(4), 1220003, 2012.

[30] Y.J. Suh and U.C. De, *Yamabe solitons and Ricci Yamabe solitons on almost coKähler manifolds*, Canad. Math. Bull. **62**, 653–661, 2019.

[31] S. Tachibana, *On almost-analytic vectors in almost Kählerian manifolds*, Tohoku Math. J. **11**, 247–265, 1959.

[32] S. Tanno, *Some transformations on manifolds with almost contact and contact metric structures II*, Tohoku Math. J. **15**, 322–331, 1963.

[33] Y. Wang, *A generalization of the Goldberg conjecture for coKähler manifolds*, Mediterr. J. Math. **13**, 2679–2690, 2016.

[34] Y. Wang, *Yamabe solitons on three-dimensional Kenmotsu manifolds*, Bull. Belg. Math. Soc. Simon Stevin **23**, 345–355, 2016.

[35] Y. Wang, *Ricci solitons on 3-dimensional cosymplectic manifolds*, Math. Slovaca, **67**, 979–984, 2017.

[36] Y. Wang, *Ricci solitons on almost coKähler manifolds*, Canad. Math. Bull. **62**, 912–922, 2019.

[37] Y. Wang, *Contact 3-manifolds and $\star$-Ricci solitons*, Kodai Math. J. **43**, 256–267, 2020.

[38] Y. Wang, *Almost Kenmotsu (κ, μ)-manifolds with Yamabe solitons*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. **115**, Art. No. 14, 2021.

[39] W. Wang, *Almost cosymplectic (κ, μ)-metrics as η-Ricci solitons*, J. Nonlinear Math. Phys. **29**, 58–72, 2022.

[40] E. Woolgar, *Some applications of Ricci flow in Physics,* Canadian J. Phys. **86**, 2008.

[41] K. Yano, *Integral formulas in Riemannian geometry*, Marcel Dekker, New York, 1970.