http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 4, Pages 894–917 (2024)
DOI:10.31801/cfsuasmas.1357985
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: September 10, 2023; Accepted: June 24, 2024

NONLINEAR SEMILINEAR INTEGRO-DIFFERENTIAL EVOLUTION EQUATIONS WITH IMPULSIVE EFFECTS

Noreddine REZOUG[1], Abdelkrim SALIM[2,3] and Mouffak BENCHOHRA[3]

[1]Department of Mathematics, University of Relizane, Relizane, 48000, ALGERIA
[2]Faculty of Technology, Hassiba Benbouali University of Chlef, P.O. Box 151 Chlef 02000, ALGERIA
[3]Laboratory of Mathematics, Djillali Liabes, University of Sidi Bel-Abbes, PO Box 89, Sidi Bel-Abbes 22000, ALGERIA

Abstract. In this paper, we investigate the existence of a piecewise asymptotically almost automorphic mild solution to some classes of integro-differential equations with impulsive effects in Banach space. The working tools are based on the Mönch's fixed point theorem, the concept of measures of noncompactness theorem and resolvent operator. In order to illustrate our main results, we study the piecewise asymptotically almost automorphic solution of the impulsive differential equations.

1. Introduction

The exploration of impulsive integro-differential equations has witnessed rapid expansion in recent years, finding diverse applications in mathematical models spanning domains such as chemical technology, population dynamics, electrical engineering, medicine, physics, ecology, economics, biology, and beyond. The pioneering work of Milman and Myshkis [36] dates back to 1960 when they first introduced the concept of impulsive differential equations. To delve deeper into the outcomes and practical uses of impulsive integro-differential equations, comprehensive insights can be gleaned from the monographs authored by Bainov and Simeonov [7]. In the books authored by Benchohra *et al.* [9, 10], numerous results concerning differential equations are derived using a range of tools, including the utilization

2020 *Mathematics Subject Classification.* 34G20.
Keywords. Asymptotically almost automorphic, integro-differential equations, mild solution, evolution system, Kuratowski measures of noncompactness, Mönch fixed point.
[1] noreddinerezoug@yahoo.fr; 0000-0003-3504-8736
[2] salim.abdelkrim@yahoo.com, a.salim@univ-chlef.dz-Corresponding author; 0000-0003-2795-6224
[3] benchohra@yahoo.com; 0000-0003-3063-9449.

of measures of noncompactness and fixed point theory, from which we drew motivation. In the papers [11–15], the authors investigated several types of integro-differential equations under different conditions with qualitative and quantitative results. In [6, 33, 52], the authors considered some fractional integro-differential equations with state-dependent delay. See [2–4, 26–28, 48, 49], for some recent results on impulsive equations.

The notion of almost automorphy stands as a significant extension of Bohr's classical concept of almost periodicity, initially introduced by Bochner in [16] in connection with certain aspects of differential geometry. Since its inception, the realm of almost automorphic functions has witnessed substantial advancement and application across diverse fields such as ordinary differential equations, partial differential equations, functional differential equations, integro-differential equations, fractional differential equations, and even stochastic differential equations. A notable array of references, including [5, 17–19, 24, 25, 32, 35, 37, 38, 41, 42, 45, 50, 51, 54], serve to illustrate these developments. Subsequently, this conceptual framework has undergone compelling, natural, and potent generalizations. To exemplify, N'Guérékata [40] introduced the notion of asymptotically almost automorphic functions, which has been fruitfully applied within the realm of differential equations. For a deeper exploration of outcomes in this domain, one can turn to [1, 34, 44, 47, 53] and their associated references. For a comprehensive understanding of the contemporary theory and applications surrounding asymptotically almost automorphic functions, N'Guérekata's monographs [43] offer valuable insights.

In [29], Goldstein and N'Guérékata studied the following semilinear differential equation in a Banach space $\mathbb{X}$,

$$x'(t) = Ax(t) + F(t, x(t)), \quad t \in \mathbb{R},$$

where A generates an exponentially stable C_0-semigroup and $F(t,x) : \mathbb{R}\times\ \mathbb{X} \to \mathbb{X}$ is a function of the form $F(t,x) = P(t)Q(x)$. Under appropriate conditions on P and Q, and using the Schauder fixed point theorem, they proved the existence of an almost automorphic mild solution to the above equation.

José and Claudio [46] investigated the existence and uniqueness of an asymptotically almost automorphic mild solution to the following abstract fractional integro-differential neutral equation with unbounded delay:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} D\left(t, u_t\right) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} A D\left(s, u_s\right) \mathrm{d}s + g\left(t, u_t\right), \quad t \geq 0, \\ &u_0 = \varphi \in B, \end{aligned}$$

where $1 < \alpha < 2, D(t,\varphi) = \varphi(0) + f(t,\varphi), A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a linear densely defined operator of sectorial type on a Banach space $\mathbb{X}$, the history $u_t : (-\infty, 0] \to \mathbb{X}$, defined by $u_t(\theta) = u(t+\theta)$, belongs to an abstract phase space B defined axiomatically, and f, g are functions subject to some additional conditions.

Motivated by the above-mentioned discussions, we are interested in investigating the existence of piecewise asymptotically almost automorphic mild solution for the following integro-differential equations with impulsive differential system

$$\begin{cases} y'(t) = Ay(t) + \displaystyle\int_0^t R(t-s)y(s)ds + f\left(t, y(t), My(t)\right), t \neq t_j, \\ My(t) = \displaystyle\int_0^t H(t,s,y(s))ds, \ t \in \mathbb{R}^+, \\ \Delta y(t_j) = y(t_j^+) - y(t_j^-) = J_j(y(t_j)), \quad j = 1, 2, 3, \dots, \end{cases} \tag{1}$$

where $A : D(A) \subset E \to E$ is the infinitesimal generator of a C_0-semigroup $(S(t))_{t\geq 0} \in E$ and $(E, |\cdot|)$ is a Banach space. Here $R(t)$ is a closed linear operator on E, with domain $D(A) \subset D(R(t))$ which is independent of t. Furthermore, the fixed times t_j satisfy $0 = t_0 < t_1 < t_2 < \dots < t_j < \dots$, t_j^+ and t_j^- denote the right and left limits of y at t_j, $\Delta y(t_j) = y(t_j^+) - y(t_j^-)$ represents the jump in the state y at time t_j, where J_j determines the size of the jump. The functions $f : \mathbb{R}^+ \times E \times E \to E$, and $H : D \times E \to E$, $D = \{(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+ : s \leq t\}$, are appropriate functions satisfying certain assumptions that will be specified later.

We note that the results we have obtained and the problem addressed in this paper are regarded as an extension and a natural continuation from the previously cited works, such as [29, 46].

Let us describe the content of this paper. In Section 2, we recall some facts from resolvent operators and measure of noncompactness. In addition, notations about almost automorphic functions and asymptotically almost automorphic functions are also introduced in this section. In Section 3, we study the existence of a piecewise asymptotically almost automorphic mild solutions for system (1) with their proofs, the results are based on Mönch's fixed point theorem under some appropriate assumptions. In last section, we provide an example to illustrate our obtained results.

2. Preliminaries and Basic Results

In this section, we present some mathematical tools needed to demonstrate the main results. Let E and $\widetilde{E}$ be two Banach spaces. For any Banach space E, the norm of E is defined by $|\cdot|$. The space of all bounded linear operators from E to $\widetilde{E}$ is denoted by $L(E, \widetilde{E})$ and $L(E, E)$ is written as $L(E)$. We denote by $\mathfrak{C}(\mathbb{R}^+, E)$ the Banach space of all continuous E-valued function on $\mathbb{R}^+$. We use $\|f\|_{L^p}$ to denote the $L^p(\mathbb{R}^+, E)$ norm of f whenever $f \in L^p(\mathbb{R}^+, E)$ for some p with $1 \leq p < \infty$. We consider the following spaces:

► $\mathfrak{C}^b(\mathbb{R}^+, E)$: the Banach space of all continuous and bounded functions y mapping $\mathbb{R}^+$ into E equipped with the norm

$$\|y\|_{\mathfrak{C}^b} = \sup\{|y(t)| : t \in \mathbb{R}^+\}.$$

► $P\mathfrak{C}(\mathbb{R}^+, E)$: the space formed by all piecewise continuous functions $f : \mathbb{R}^+ \to E$ such that $f(\cdot)$ is continuous at t for any $t \neq (t_j)_{j\in\mathbb{N}}$, $y(t_j^+)$, $y(t_j^-)$ exist, and $y(t_j^-) = y(t_j)$ for all $j \in \mathbb{N}$.

► $P\mathfrak{C}(\mathbb{R}^+ \times \widetilde{E} \times \widetilde{E}, E)$: the space formed by all piecewise continuous functions $f : \mathbb{R}^+ \times \widetilde{E} \times \widetilde{E} \to \widetilde{E}$ such that for any $(y, \nu) \in \widetilde{E} \times \widetilde{E}$, $f(\cdot, y, \nu) \in P\mathfrak{C}(\mathbb{R}^+, E)$, and for any $t \in \mathbb{R}^+$, $f(t, \cdot, \cdot)$ is continuous at $(y, \nu) \in \widetilde{E} \times \widetilde{E}$.

► $P\mathfrak{C}_0(\mathbb{R}^+, E)$: the space formed by all piecewise continuous functions $\Upsilon : \mathbb{R}^+ \to E$ such that $\lim_{t\to\infty} \Upsilon(t) = 0$.

► $P\mathfrak{C}_0(\mathbb{R}^+ \times \widetilde{E} \times \widetilde{E}, E)$: the space of all piecewise continuous functions $\Upsilon : \mathbb{R}^+ \times \widetilde{E} \times \widetilde{E} \to E$ satisfying $\lim_{t\to\infty} \Upsilon(t, y, \nu) = 0$ in t and uniformly for all $(y, \nu) \in K$, where K is any bounded subset of $\widetilde{E} \times \widetilde{E}$.

► $P\mathfrak{C}^b(\mathbb{R}^+, E)$ the subspace of $P\mathfrak{C}(\mathbb{R}^+, E)$ consisting of all bounded functions.

It is well-known that $P\mathfrak{C}^b(\mathbb{R}^+, E)$ is a Banach space with the norm

$$\|y\|_{P\mathfrak{C}^b} = \sup\{|y(t)|, t \in \mathbb{R}^+\}.$$

First, let's recall some basic defnitions and results on the strong continuous evolution family which will be used later.

We consider the following Cauchy problem

$$\begin{cases} y'(t) = Ay(t) + \displaystyle\int_0^t R(t-s)y(s)ds, \quad t \geq 0, \\ y(t) = y_0. \end{cases} \tag{2}$$

Definition 1. *([23, 30]) A resolvent for Equation (2) is a bounded linear operator valued function $S(t) \in L(E)$ for $t \geq 0$, satisfying the following properties:*

- **(a):** *For any $t \in \mathbb{R}^+$, $S(0) = I$ and $\|S(t)\|_{B(E)} \leq \eta e^{-\lambda(t-s)}$ for some constants η and λ.*
- **(b):** *For each $y \in E$, $S(t)y$ is strongly continuous for $t \geq 0$.*
- **(c):** *For $y \in E$, $S(\cdot)y \in \mathfrak{C}^1([0, +\infty), E) \cap \mathfrak{C}([0, +\infty), \widetilde{E})$ and*

$$\begin{aligned} S'(t)y &= AS(t)y + \int_0^t R(t-s)S(s)\widetilde{E}ds \\ &= S(t)Ay + \int_0^t S(t-s)R(s)\widetilde{E}ds. \end{aligned}$$

We introduce the following assumptions:

- **(T_1):** A is the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on E.

(T_2): For all $t \geq 0$, $B(t)$ is closed linear operator from $D(A)$ to E and $R(t) \in L(\widetilde{E}, E)$. For any $y \in E$, the map $t \to R(t)y$ is bounded, differentiable and its derivative $R'(t)y$ is bounded and uniformly continuous on $\mathbb{R}^+$.

Theorem 1. *([23,30]) Assume that (T_1) and (T_2) hold. Then there exists a unique resolvent operator for the Cauchy problem (2).*

Definition 2 ([16,41,42])**.** *Let u: $\mathbb{N} \to E$ be a bounded sequence. u is called almost automorphic sequence, if for each real sequence $\{j_i'\}$, there exists a subsequence $\{j_i'\} \subset \{j_i\}$ such that*

$$\widehat{u}(j) = \lim_{n\to\infty} u(j + j_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} u(j + j_n) = \widehat{u}(j),$$

for all $j \in \mathbb{N}$. Represent this class of all sequences as $AA(\mathbb{N}, E)$.

Definition 3. *[1] A bounded piecewise continuous function $f \in P\mathfrak{C}(\mathbb{R}^+, E)$ is said to be almost automorphic if*

(A_1)**:** *sequence of impulsive moments $\{t_j\}$ is a almost automorphic sequence,*

(A_2)**:** *for every sequence of real numbers $\{\sigma_n\}$, there exists a subsequence $\{\sigma_{n_j}\}$ such that*

$$\mathbb{G}(t) = \lim_{n\to\infty} f(t + \sigma_{n_j})$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} \mathbb{G}(t - \sigma_{n_j}) = f(t)$$

for each $t \in \mathbb{R}$.

Denote by $AA_{P\mathfrak{C}}(\mathbb{R}, E)$ the set of all such functions.

Lemma 1. *[41] $AA_{P\mathfrak{C}}(\mathbb{R}, E)$ is a Banach space with the norm*

$$\|f\|_{P\mathfrak{C}} = \sup_{t\in\mathbb{R}} |f(t)|.$$

Definition 4. *[1,41] A bounded piecewise continuous function $f \in P\mathfrak{C}(\mathbb{R}^+ \times \widetilde{E}, E)$ is called almost automorphic if*

(A_1)**:** *sequence of impulsive moments $\{t_j\}$ is a almost automorphic sequence*

(A_2)**:** *for every sequence of real numbers $\{\sigma_n\}$, there exists a subsequence $\{\sigma_{n_j}\}$ such that*

$$\lim_{n\to\infty} f(t + \sigma_{n_j}, y) = g(t, y)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t - \sigma_{n_j}, y) = f(t, y)$$

for each $t \in \mathbb{R}$ and each $y \in E$.
Denote by $AA_{PC}(\mathbb{R} \times E, E)$ the set of all such functions.

The following definition, which is the Bi-almost automorphicity, is a crucial ingredient in our approach.

Definition 5 ([43])**.** *A bounded piecewise continuous function $f \in P\mathfrak{C}(\mathbb{R}^+ \times \mathbb{R}^+ \times \widetilde{E}, E))$ is Bi-almost automorphic if*

(A_1)**:** *sequence of impulsive moments $\{t_j\}$ is an almost automorphic sequence*
(A_2)**:** *for every sequence of real numbers $\{\sigma_n\}$, there exists a subsequence $\{\sigma_{n_j}\}$ such that*

$$\lim_{n\to\infty} f(t+\sigma_{n_j}, s+\sigma_{n_j}, y) = \mathcal{G}(t,s,y)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} \mathcal{G}(t-\sigma_{n_j}, s-\sigma_{n_j}, y) = f(t,s,y)$$

for each $t \in \mathbb{R}$ and each $y \in E$.

Definition 6. *[41] A piecewise continuous function $f \in P\mathfrak{C}(\mathbb{R}^+, E)$ is said to be asymptotically almost automorphic if it can be decomposed as*
$f(t) = \mathbb{G}(t) + \Upsilon(t)$,
where
$\mathbb{G}(t,y) \in AA_{P\mathfrak{C}}(\mathbb{R}^+, E), \quad \Upsilon(t,y) \in P\mathfrak{C}_0(\mathbb{R}^+, E)$.
The space of these kinds of functions is denoted by $AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$.

Definition 7. *[41] A piecewise continuous function $f \in P\mathfrak{C}(\mathbb{R}^+ \times \widetilde{E}, E))$ is said to be asymptotically almost automorphic if it can be decomposed as*

$$f(t,y) = \mathbb{G}(t,y) + \Upsilon(t,y),$$

where

$$\mathbb{G}(t,y) \in AA_{P\mathfrak{C}}(\mathbb{R}^+ \times \widetilde{E}, E), \quad \Upsilon(t,y) \in P\mathfrak{C}_0(\mathbb{R}^+ \times \widetilde{E}, E).$$

This class of functions is denoted by $AAA_{P\mathfrak{C}}(\mathbb{R}^+ \times \widetilde{E}, E)$.

We state a lemma inspired by the paper of J. Cao et al. [19] about the composition result.

Lemma 2. *[20] Let $y, \nu \in AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$, $K = \overline{\{\nu(t) : t \in \mathbb{R}^+\}} \times \overline{\{y(t) : t \in \mathbb{R}^+\}}$ and*

$$f \in AAA_{P\mathfrak{C}}(\mathbb{R}^+ \times E \times E, E) \cap C_K(\mathbb{R}^+ \times E \times E, E),$$

then $f(\cdot, y(\cdot), \nu(\cdot)) \in AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$.

The proof of the above lemma is similar to the proof of Lemma 2.5 of [19]. Now, we introduce the Kuratowski measure of noncompactness χ defined by

$$\chi(\Theta) = \inf\{ \Delta > 0 : \Theta \text{ has a finite cover by sets of diameter } \leq \Delta\},$$

for a bounded set Θ in any space E. Some basic properties of $\chi(\cdot)$ are given in the following lemma. For more details, please see [8].

Lemma 3. *([8]) Let E be a Banach space and $\Theta_1, \Theta_2 \subset E$ be bounded, and the following properties are satisfied:*

- (j_1) *Θ is pre-compact if and only if $\chi(\Theta) = 0$,*
- (j_2) *$\chi(\Theta) = \chi(\overline{\Theta}) = \chi(Conv\Theta)$, where $\overline{\Theta}$ and $conv\Theta$ are the closure and the convex hull of Θ, respectively,*
- (j_3) *$\chi(\Theta_1) \leq \chi(\Theta_2)$ when $\Theta_1 \subset \Theta_2$,*
- (j_4) *$\chi(\Theta_1 + \Theta_2) \leq \chi(\Theta_1) + \chi(\Theta_2)$,*
- (j_5) *$\chi(k\Theta) = |k|\chi(\Theta)$ for any $k \in \mathbb{R}$,*
- (j_6) *$\chi(\Theta_2 + \Theta_1) \leq \chi(\Theta_2) + \chi(\Theta_2)$ where $\Theta_2 + \Theta_1 = \{y + \nu : y \in \Theta, \nu \in \Theta_2\}$,*
- (j_7) *$\chi(\Theta_2 \cup \Theta_1) \leq \max\left(\chi(\Theta_2), \chi(\Theta_2)\right)$,*
- (j_8) *if $\Gamma : E \to E$ is a Lipschitz continuous map with constant k, then $\chi(\Gamma(\Theta)) \leq k\chi(\Theta)$ for all bounded subset Θ of E.*

Lemma 4. *([21]) Let E be a Banach space, $\Theta \subset E$ be bounded. Then there exists a countable set $\Theta_0 \subset \Theta$, such that*

$$\chi(\Theta) \leq 2\chi(\Theta_0).$$

Lemma 5. *([31]) Let V be a Banach space, and let $\Theta = \{y_n\} \subset \mathfrak{C}([c,d], E)$ be a bounded and countable set for constants $-\infty < c < d < +\infty$. Then $\Psi(v(t))$ is Lebesgue integral on $[c,d]$, and*

$$\chi\Big(\Big\{ \int_c^d y_n(t)dt : n \in \mathbb{N}\Big\}\Big) \leq 2\int_c^d \chi(\Theta(t))dt.$$

Now, we recall a useful compactness criterion.

Lemma 6. *[22][Corduneanu]*
A set $C \subset P\mathfrak{C}^b(\mathbb{R}^+, E)$ is relatively compact if the following conditions hold

- ***(i)*: C is bounded in $P\mathfrak{C}^b(\mathbb{R}^+, E)$,**
- ***(ii)*:** *C is a locally equicontinuous family of function, i.e., for any constant $d > 0$, the functions in C are equicontinuous in $[0,d]$,*
- ***(iii)*:** *the set $C(t) := \{y(t) : y \in C\}$ is relatively compact on any compact interval of $\mathbb{R}^+$,*
- ***(iv)*:** *the functions from C are equiconvergent, i.e For each $\varepsilon > 0$,there exists $d(\varepsilon) > 0$ such that $|y(t) - y(+\infty)| < \varepsilon$ for all $t \geq d(\varepsilon)$ and for all $y \in C$.*

Finally, we will make use of Mönch's fixed point theorem

Theorem 2. *(Mönch fixed point) [39]. Suppose that Ω is a closed convex subset of X; $0 \in \Omega$. If the map $N : \Omega \to X$ is continuous and of Mönch type, namely, Q satisfies the following property*

$$\Theta \subset \Omega, \Theta \text{ is countable}, \quad \Theta \subset \overline{Conv}\,(N(\Theta) \cup \{0\}) \Longrightarrow \overline{\Theta} \text{ is compact},$$

then, N has a fixed point in Ω.

3. The Main Results

Before starting our main results, we recall the definition of the mild solution of (1).

Definition 8. *A function $y \in P\mathfrak{C}^b(\mathbb{R}^+, E)$ is called a mild solution to the problem (1) if y satisfies the integral equation*

$$y(t) = S(t)y_0 + \sum_{0<t_j<t} S(t-t_j)J_j(y(t_j)) +_0^t S(t-s)f\left(s, y(s), My(s)\right) ds,\ t \in \mathbb{R}^+. \quad (3)$$

The following assumptions are needed to establish our results.

($\mathbb{H}_1$)**:** The resolvent operator given by Theorem 1 satisfies the following condition:

$$\|S(t-s)\|_{L(E)} \leq \eta e^{-\lambda(t-s)} \text{ where } \eta > 0 \text{ and } \lambda > 0.$$

($\mathbb{H}_2$)**:** The function $f : \mathbb{R}^+ \times E \times E \to E$ satisfies:

(i)**:** For a.e. $t \in \mathbb{R}^+$, the function $f(t, \cdot, \cdot) : \mathbb{R}^+ \times E \times E \to E$ is continuous, and for each $(y, \nu) \in E \times E$, the function $f(\cdot, y, \nu) : \mathbb{R}^+ \times E \times E$ is strongly measurable.

(i)**:** The function $f(t, y, \nu)$ asymptotically almost automorphic i.e., $f(t, y, \nu) = \mathbb{G}(t, y, \nu) + \Upsilon(t, y, \nu)$ with

$$\mathbb{G}(t, y, \nu) \in AA_{P\mathfrak{C}}(\mathbb{R} \times E \times E, E), \quad \Upsilon(t, y, \nu) \in \mathfrak{C}_0(\mathbb{R}^+ \times E \times E, E),$$

and $\mathbb{G}(t, y, \nu)$ is uniformly continuous on any bounded subset $K \subset E \times E$ uniformly for $t \in \mathbb{R}$.

(ii)**:** There exists a function $\hbar \in L^{\frac{1}{p_1}}(\mathbb{R}^+, \mathbb{R}^+)$, for a constant $p_1 \in (0, 1)$ such that:

$$|f(t, y, \nu)| \leq \hbar(t)(|y| + |\nu|) \text{ for a.e } t \in \mathbb{R}^+ \text{ and each } y, \nu \in E.$$

(iii)**:** There exists a function $\rho \in L^{\frac{1}{p_2}}(\mathbb{R}^+, \mathbb{R}^+)$, for a constant $p_2 \in (0, 1)$ such that:

$$\chi(f(t, \Omega_1, \Omega_2)) \leq \rho(t)\left(\chi(\Omega_1) + \chi(\Omega_2)\right), t \in \mathbb{R}^+,$$

for any bounded countable subsets $\Omega_1, \Omega_2 \subset P\mathfrak{C}^b(\mathbb{R}^+, E)$.

($\mathbb{H}_3$): The function $H : D \times E \to E$ have the decomposition $H = H^a + H_0^\rho$ in which H^a is Bi-almost automorphic functions which satisfies Bi-almost automorphic in (t, s) uniformly on bounded subsets of E and is ϱ-bounded. Moreover,

$$\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \varrho(t,s)\mathrm{d}s = \varrho^* < +\infty.$$

Also, the Bi-almost automorphic functions H^a is $(\phi, \widehat{\phi})$-Lipschitz (see [20]), with

$$\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \phi(t,s)\mathrm{d}s = \phi^* < +\infty;$$

and for every compact interval $[a, b] \subset \mathbb{R}$, the following limit holds

$$\lim_{t\to+\infty} \int_{a}^{b} \phi(t,s)\mathrm{d}s = 0,$$

we also assume that there exists a function $\pi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ such that $|H(t,s,0)| \leq \pi(t,s)$,

$$\lim_{t\to+\infty} \int_{-\infty}^{0} \pi(t,s)\mathrm{d}s = 0$$

and $H_0^\theta \in \mathfrak{C}_0^\theta(D \times E, E)$, with

$$\int_{0}^{d} \theta(t,s)\mathrm{d}s = 0, \text{ for a.e } d > 0.$$

and

$$\sup_{t\in\mathbb{R}^+} \int_{0}^{t} \theta(t,s)\mathrm{d}s = q < +\infty$$

(i): There exists a positive function $\upsilon(t,s) \in L^1(D, \mathbb{R}^+)$ such that:

$|H(t,s,y)| \leq \upsilon(t,s)(1 + |y|)$, for a.e $t \in \mathbb{R}^+$ and each $y \in E$.

(ii): There exists a positive function $\vartheta(t,s) \in L^1(D, \mathbb{R}^+)$ such that for any bounded countable $\Omega \subset P\mathfrak{C}^b(\mathbb{R}^+, E)$

$$\chi(H(t,s,\Omega) \leq \vartheta(t,s)\chi(\Omega), t \in \mathbb{R}^+.$$

($\mathbb{H}_4$): The impulse functions $J_j : E \to E$ for $j = 1, 2, 3\dots,$ is a sequence of almost asymptotically automorphic function and satisfies:

(i): There exist positive constant numbers σ_j and ς_j such that

$$|J_j(y)| \leq \sigma_j\,|y| + \varsigma_j, \text{ for a.e } t \in \mathbb{R}^+ \text{ and each } y \in E.$$

(ii): There exist $\theta_j > 0$; $j = 1, 2, \dots.$ such that for any bounded countable $\Omega \subset P\mathfrak{C}^b(\mathbb{R}^+, E)$

$$\chi(J_j(\Omega)) \leq \theta_j\chi(\Omega).$$

In the proofs of our results, we need the following auxiliary result.

Lemma 7. *[20] Let* $f = \mathbb{G}+\Upsilon \in AAA_{P\mathfrak{C}}(\mathbb{R}^+\times E\times E, E)$ *with* $\mathbb{G} \in AA_{P\mathfrak{C}}(\mathbb{R}, E), \Upsilon \in P\mathfrak{C}_0(\mathbb{R}^+, E)$*. Then*

$$E_1(t) := \int_0^t S(t-s)f(s)ds \in AAA_{P\mathfrak{C}}(\mathbb{R}^+, E).$$

Lemma 8. *[20] Suppose the functions* $H : \mathbb{R}\times\mathbb{R}\times E \to E$ *satisfies condition* $(\mathbb{H}_3)$*. Then, the integral operators* E_2 *such that*

$$E_2 y(t) = \int_0^t H(t, s, y(s))\mathrm{d}s, \ t \in \mathbb{R}^+,$$

maps $AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$ *into* $AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$*.*

Theorem 3. *Assume that the hypotheses* $(\mathbb{H}_1) - (\mathbb{H}_4)$ *are satisfied. Then the problem (1) has an asymptotically almost automorphic mild solution if*

$$\eta \max\left(\frac{\varsigma_j}{1-e^{-\lambda\varpi}} + \frac{\sigma_j}{1-e^{-\lambda\varpi}} + (1+\upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}}, \frac{\theta_j}{1-e^{-\lambda\varpi}} + 4(1+2\omega^*)\|\rho\|_{L^{\frac{1}{p_2}}}\right) \leq 1. \tag{4}$$

Proof. Let $\overline{\mho}_\kappa = \left\{y \in P\mathfrak{C}^b(\mathbb{R}^+, E) \cap AAA(\mathbb{R}^+, E) : \|y\| \leq \kappa\right\}$. Define an operator Q on $\overline{\mho}_\kappa$ by

$$(Qy)(t) = S(t)y_0 + \sum_{0<t_j<t} S(t-t_j)J_j(y(t_j)) +_0^t S(t-s)f\left(s, y(s), My(s)\right)ds, \ t \in \mathbb{R}^+. \tag{5}$$

We next show that Q has a fixed point in $\overline{\mho_\kappa}$. We divide the proof into several steps.

Step 1. For every $y \in \overline{\mho}_\kappa, \ Qy \in P\mathfrak{C}^b(\mathbb{R}^+, E)$.

For $t \in \mathbb{R}^+$, from the hypotheses $(\mathbb{H}_1)$-$(\mathbb{H}_4)$, we get

$$\begin{aligned}
|(Qy)(t)| \leq{} & \|S(t)\|_{L(E)} |y_0| + \sum_{0<t_j<t} \|S(t-t_j)\|_{L(E)} |J_j(y(t_j))| \\
& + \int_0^t \|S(t-s)\|_{L(E)} \hbar(s)\left(|y(s)| +_0^s \upsilon(s,\tau)(1+|y(\tau)|)d\tau\right) ds \\
& \leq \eta\, |y_0| + \eta \sum_{0<t_j<t} \sigma_j\, |y(t_j)| + \varsigma_j \\
& + \eta \int_0^t e^{-\lambda(t-s)}\hbar(s)\left(|y(s)| +_0^s \upsilon(s,\tau)(1+|y(\tau)|)d\tau\right) ds \\
& \leq \eta|y_0| + \eta \sum_{0<t_j<t} e^{-\lambda(t-t_j)}(\sigma_j\, |y(t_j)| + \varsigma_j) \\
& + \eta \int_0^t e^{-\lambda(t-s)}\hbar(s)\left(\sup_{s\in\mathbb{R}^+} |y(s)| + \upsilon^*(1+\sup_{s\in\mathbb{R}^+} |y(s)|)\right) ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \eta|y_0| + \eta \sum_{0<t_j<t} e^{-\lambda(t-t_j)}(\sigma_j\,|y(t_j)| + \varsigma_j) \\
&+ \eta \int_0^t e^{-\lambda(t-s)} \hbar(s) \left((1+\upsilon^*)(1+\sup_{s\in\mathbb{R}^+}|y(s)|) \right) ds \\
&\leq \eta|y_0| + \eta(\sigma_j\|y\|_{P\mathfrak{C}^b} + \varsigma_j) \sum_{0<t_j<t} e^{-\lambda(t-t_j)} \\
&+ \eta(1+\upsilon^*) \int_0^t e^{-\lambda(t-s)} \hbar(s) ds \|y\|_{P\mathfrak{C}^b} \\
&\leq \eta|y_0| + \frac{\eta(\sigma_j\|y\|_{P\mathfrak{C}^b} + \varsigma_j)}{1-e^{-\lambda\varpi}} \\
&+ \eta(1+\upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}} \left(\int_0^t e^{-\frac{\lambda}{1-p_1}(t-s)} ds \right)^{1-p_1} (1+\|y\|_{P\mathfrak{C}^b}) \\
&\leq \eta|y_0| + \frac{\eta(\sigma_j\|y\|_{P\mathfrak{C}^b} + \varsigma_j)}{1-e^{-\lambda\varpi}} \\
&+ \eta(1+\upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}} \left(1 - e^{-\frac{\lambda t}{1-p_1}} \right) (1+\|y\|_{P\mathfrak{C}^b}) \\
&\leq \eta|y_0| + \frac{\eta\varsigma_j}{1-e^{-\lambda\varpi}} + \eta \left(\frac{\sigma_j}{1-e^{-\lambda\varpi}} + (1+\upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}} \right) (1+\|y\|_{P\mathfrak{C}^b}),
\end{aligned}
$$

which implies that $Qy \in P\mathfrak{C}^b(\mathbb{R}^+, E)$.

Step 2. For every $y \in \overline{\mho}_\kappa$, $Qy \in AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$.
Claim 1. Proving that $(Py)(t)$ belongs to $AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$, where

$$(Py)(t) = S(t)y_0 +_0^t S(t-s) f\left(s, y(s), My(s)\right) ds,\ t \in \mathbb{R}^+.$$

Let

$$E(t) = S(t)y_0,$$

then

$$|E(t)| = |S(t)y_0| \leq |S(t)y_0| \leq \eta e^{-\lambda t}|y_0|.$$

Since $\lambda > 0$, we get $\lim\limits_{t\to+\infty} |(E(t)| = 0$. That is

$$E \in P\mathfrak{C}_0(\mathbb{R}^+, E). \tag{6}$$

Applying Lemma 8 and Lemma 2, we infer that $My(t)$ and $f(\cdot, y(\cdot), My(t)(\cdot))$ belong to $AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$. By Lemma 7 and 6, we obtain that P is $AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$-valued.

Claim 2. Proving that $\sum\limits_{0<t_j<t} S(t-t_j)J_j(y(t_j))$ belongs to $AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$.

From the assumption $(\mathbb{H}_4)$, $J_j(y(t_j)) \in AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$. By definition, it can be expressed as

$$J_j(y(t_j)) = J_{j1}(y(t_j)) + J_{j2}(y(t_j))$$

such that $J_{j1}(y(t_j)) \in AAA(\mathbb{R}^+, E)$,$J_{j2}(y(t_j)) \in P\mathfrak{C}_0(\mathbb{R}^+, E)$ Then:

$$\begin{aligned}\sum_{0<t_j<t} S(t-t_j)J_j(y(t_j)) &= \sum_{0<t_j<t} S(t-t_j)J_{j1}(y(t_j)) + \sum_{0<t_j<t} S(t-t_j)J_{j2}(y(t_j)) \\ &= \wp^a(t) + \wp^0(t).\end{aligned}$$

Since $J_{j1} \in AA(\mathbb{R}^+, E)$, for every real sequence $\{t_j\}$, there exists a subsequence $\{t_{j_n}\}$ such that

$$\lim_{n\to\infty} J_{j1}(y(t_j + t_{j_n})) = \mathbb{J}_{j1}(y(t_j))$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} \mathbb{J}_{j1}(y(t_j - t_{j_n})) = J_{j1}(y(t_j)),$$

Now, we have

$$\wp^a(t + t_{j_n}) = \sum_{0<t_j<t+t_{j_n}} S(t + t_{j_n} - t_j)J_{j1}(y(t_j)) = \sum_{0<t_j<t} S(t-t_j)J_{j1}(y(t_j + t_{j_n})),$$

then

$$\lim_{n\to\infty} \wp^a(t+t_{j_n}) = \lim_{n\to\infty} \sum_{0<t_j<t} S(t-t_j)J_{j1}(y(t_j+t_{j_n})) = \sum_{0<t_j<t} S(t-t_j)\mathbb{J}_{j1}(y(t_j)) = \overline{\wp}^a(t),$$

Similarly

$$\overline{\wp}^a(t-t_{j_n}) = \sum_{0<t_j<t-t_{j_n}} S(t-t_{j_n}-t_j)\mathbb{J}_{j1}(y(t_j))) = \sum_{0<t_j<t} S(t-t_j)\mathbb{J}_{j1}(y(t_j-t_{j_n}))),$$

then

$$\lim_{n\to\infty} \overline{\wp}^a(t-t_{j_n}) = \lim_{n\to\infty} \sum_{0<t_j<t} S(t-t_j)\mathbb{J}_{j1}(y(t_j-t_{j_n}))) = \sum_{0<t_j<t} S(t-t_j)J_{j1}(y(t_j)),$$

then,

$$\wp^a(t) = \sum_{0<t_j<t} S(t-t_j)J_{j1}(y(t_j))$$

belongs to $AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$.

Next, we show that $\wp^0(t) \in \mathfrak{C}_0(\mathbb{R}^+, E)$. Since $J_{j2} \in PC_0(\mathbb{R}^+, E)$, one can choose a $T > 0$ such that

$$|J_{j2}| \le \varepsilon.$$

This enables us to conclude that for all $t > T$,

$$\begin{aligned}\wp^0(t) = \left|\sum_{0<t_j<t} S(t-t_j)J_{j2}(y(t_j))\right| &\le \sum_{0<t_j<t} \|S(t-t_j)\|_{L(E)} |J_{j2}(y(t_j))| \\ &\le \eta \sum_{0<t_j<t} e^{-\lambda(t-t_j)} |J_{j2}(y(t_j))|\end{aligned}$$

$$\begin{aligned} &\leq \eta\,|J_{j2}| \sum_{0<t_j<t} e^{-\lambda(t-t_j)} \\ &\leq \frac{\eta\,|J_{j2}|}{1-e^{-\lambda\varpi}} \\ &\leq \varepsilon \end{aligned}$$

So, $\wp^0(t) \in P\mathfrak{C}_0(\mathbb{R}^+, E)$. Finally by (6), we prove our claim that $Qy \in AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$.

Step 3. We prove that $Q\left(\overline{\mho}_\kappa\right) \subset \overline{\mho}_\kappa$.
If this condition fails, then for every positive constant $\kappa > 0$ and $t \geq 0$, there exists a function $\widehat{y} \in \overline{\mho}_\kappa$ but $Q(\widehat{y}) \notin \overline{\mho}_\kappa$, i.e $|(Q\widehat{y})(t)| > \kappa$. Thus, by the Hölder inequality, the conditions $(\mathbb{H}_1)-(\mathbb{H}_4)$, based on the above estimations, we can easily demonstrate that

$$|(Q\widehat{y})(t)| \leq \eta|y_0| + \frac{\eta\varsigma_j}{1-e^{-\lambda\varpi}} + \eta\left(\frac{\sigma_j}{1-e^{-\lambda\varpi}} + (1+\upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}}\right)(1+\kappa).$$

Thus,

$$\kappa < \eta|y_0| + \frac{\eta\varsigma_j}{1-e^{-\lambda\varpi}} + \eta\left(\frac{\sigma_j}{1-e^{-\lambda\varpi}} + (1+\upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}}\right)(1+\kappa).$$

Dividing on both sides by κ and taking the lower limit as $\kappa \to +\infty$, we can obtain that

$$1 < \frac{\eta\varsigma_j}{1-e^{-\lambda\varpi}} + \eta\left(\frac{\sigma_j}{1-e^{-\lambda\varpi}} + (1+\upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}}\right).$$

This contradicts (4). Hence, for some positive number κ, we must have $Q(\overline{\mho}_\kappa) \subset \overline{\mho}_\kappa$.
Step 4. We show that Q is continuous $\overline{\mho}_\kappa$.
To demonstrate the continuity of Q, we assume that there exists a sequence y_n, y in $\overline{\mho}_\kappa$ and $y_n \to y$ as $n \to +\infty$.
Case 1. If $t \in [0, d]$, $d > 0$, and $y_n \in \overline{\mho}_\kappa$, we have

$$\begin{aligned} &|(Qy_n)(t) - (Qy)(t)| \\ &\leq_{0<t_j<t} S(t-t_j)\,|J_j(y(t_j)) - J_j(y_n(t_j))| \\ &+\eta\int_0^t \left| f\left(s, y_n(s), \int_0^s H(s,\tau,y_n(\tau))d\tau\right) - f\left(s, y(s), \int_0^s H(s,\tau,y(\tau))d\tau\right)\right| ds. \end{aligned}$$

By the Lebesgue dominated convergence theorem accompanying with $(\mathbb{H}_2)(i)$, we get

$$\|Qy_n - Qy\| \to 0 \qquad \text{as} \quad n \to +\infty.$$

Case 2. If $t \in (d, +\infty)$, $d > 0$, by $(\mathbb{H}_2)(i)$, we can see that

$$|J_j(y_n(t_j)) - J_j(y(t_j))| \leq \frac{(1-e^{-\lambda\varpi})\varepsilon_j}{2} \quad \text{for } t \geq d. \tag{7}$$

and

$$\left| f\left(t, y_n(t), \int_0^t H(t,s,y_n(s))ds\right) - f\left(t, y(t), \int_0^t H(t,s,y(s))ds\right) \right| \leq \frac{\lambda\varepsilon}{2\eta} \quad \text{for } t \geq d. \quad (8)$$

Hence, according to the dominated convergence theorem and (8), we obtain that for every $t \geq 0$,

$$\begin{aligned}
&|(Qy_n)(t) - (Qy)(t)| \\
&\leq \sum_{0<t_j<t} S(t-t_j)\,|J_j(y_n(t_j)) - J_j(y(t_j))| \\
&+{}_0^t \|S(t-s)\|_{L(E)} \left| f\left(s, y_n(s), \int_0^s H(s,\tau,y(\tau))d\tau\right) - f\left(s, y(s), \int_0^s H(s,\tau,y(\tau))d\tau\right) \right| ds \\
&\leq \frac{1-e^{-\lambda\varpi}}{2} \sum_{0<t_j<t} \varepsilon_j e^{-\lambda(t-t_j)} + \frac{\lambda\varepsilon}{2\eta} \int_0^t e^{-\lambda(t-s)} ds \\
&\leq \frac{\varepsilon}{2} + \frac{\eta}{\lambda}\frac{\lambda\varepsilon}{2\eta}(1-e^{-\lambda t}) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned} \quad (9)$$

Then the inequality (9) reduces to

$$\|Q(y_n) - Q(y)\|_{P\mathfrak{C}^b} \to 0 \quad \text{as } n \to \infty.$$

This implies that Q is continuous in $\overline{\mho}_\kappa$.

Next, we demonstrate that the operator Q is equi-continuous on every compact interval $[0, d]$ of $[0, +\infty)$, for $d > 0$ and is equi-convergent in $y \in \overline{\mho}_\kappa$.

Step 5. $Q(\overline{\mho}_\kappa)$ is equicontinuous.

Let $0 < d < +\infty$ be an arbitrary constant. Generally, let $0 \leq \tau_1 \leq \tau_2 \leq d$, for any $y \in \overline{\mho}_\kappa$, we know that

$$\begin{aligned}
&|(Qy)(\tau_2) - (Qy)(\tau_1)| \\
&= \Bigg| S(\tau_2)y_0 + \sum_{0<t_j<\tau_2} S(\tau_2 - t_j)J_j(y(t_j)) \\
&+ {}_0^{\tau_2} S(s-\tau_2) f\left(s, y(s), \int_0^s H(s,\tau,y(\tau))d\tau\right) ds \\
&- S(\tau_1)y_0 + \sum_{0<t_j<\tau_1} S(\tau_1 - t_j)J_j(y(t_j)) \\
&+ \int_0^{\tau_1} S(t-s) f\left(s, y(s), \int_0^s H(s,\tau,y(\tau))d\tau\right) ds \Bigg| \\
&\leq |S(\tau_2)y_0 - S(\tau_1)y_0| \\
&+ \left| \sum_{0<t_j<\tau_2} S(\tau_1 - t_j)J_j(y(t_j)) - \sum_{0<t_j<\tau_1} S(\tau_2 - t_j)J_j(y(t_j)) \right|
\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_0^{\tau_1}(S(\tau_2,s)-S(\tau_1,s))f\left(s,y(s),\int_0^s H(s,\tau,y(\tau))d\tau\right)ds\right.\\
&\left.+\int_{\tau_1}^{\tau_2} S(\tau_2,\tau)f\left(s,y(s),\int_0^s H(s,\tau,y(\tau))d\tau\right)ds\right|\\
&\leq |S(\tau_2)y_0-S(\tau_1)y_0|\\
&+\sum_{0<t_j<\tau_1}\|S(\tau_1-t_j)-S(\tau_2-t_j)\|_{L(E)}\,|J_j(y(t_j))|\\
&+\sum_{\tau_1<t_j<\tau_2}\|S(\tau_1-t_j)\|_{L(E)}\,|J_j(y(t_j))|\\
&+_0^{\tau_1}\|S(\tau_2,\tau)-S(\tau_1,\tau)\|_{B(V)}\ \hbar(s)\left(|y(s)|+_0^s \upsilon(s,\tau)(1+y(\tau)|)d\tau\right)ds\\
&+\eta_{\tau_1}^{\tau_2}e^{-\lambda(t-s)}\hbar(s)\left(|y(s)|+_0^s \upsilon(s,\tau)(1+y(\tau)|)d\tau\right)ds.
\end{aligned}
$$

It follows from the Hölder's inequality that

$$
\begin{aligned}
|(Qy)(\tau_2)-(Qy)(\tau_1)| \leq\ &\|S(\tau_2)-S(\tau_1)\|_{L(E)}\,|y_0|\\
&+(\sigma_j\varrho+\varsigma_j)\sum_{0<t_j<\tau_1}\|\mathrm{I}-S(\tau_2-\tau_1)\|_{L(E)}\\
&+\eta(\sigma_j\varrho+\varsigma_j)\sum_{\tau_1<t_j<\tau_2}e^{-\lambda(t-t_j)}\\
&+(1+\upsilon^*)\varrho_0^{\tau_1}\|S(\tau_2-s)-S(\tau_1-s)\|_{B(V)}\ \hbar(s)ds\\
&+\eta\|\hbar\|_{L^{\frac{1}{p_1}}}(1+\upsilon^*)\varrho\|\hbar\|_{L^{\frac{1}{p_1}}}\left(\int_0^t e^{-\frac{\lambda}{1-p_1}(t-s)}ds\right)^{1-p_1}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
|(Qy)(\tau_2)-(Qy)(\tau_1)| \leq\ &\|S(\tau_2)-S(\tau_1\|_{L(E)}\,|v_0|\\
&+(\sigma_j\varrho+\varsigma_j)\sum_{0<t_j<\tau_1}\|S(\tau_1-t_j)-S(\tau_2-t_j)\|_{L(E)}\\
&+\frac{\eta(\sigma_j\varrho+\varsigma_j)(\tau_2-\tau_1)}{\varpi}\\
&+(1+\upsilon^*)\varrho_0^{\tau_1}\|S(\tau_2-s)-S(\tau_1-s)\|_{B(V)}\ \hbar(s)ds\\
&+\frac{\eta\|\hbar\|_{L^{\frac{1}{p_1}}}(1+\upsilon^*)\varrho(1-p_1)^{1-p_1}}{\lambda^{1-p_1}}\left(e^{-\frac{\lambda}{1-p_1}(t-\tau_2)}-e^{-\frac{\lambda}{1-p}(t-\tau_1)}\right)^{1-p_1}.
\end{aligned}
$$

The right-hand side tends to zero as $\tau_2\to\tau_1$. This proves the equicontinuity of $Q(\overline{\mho}_\kappa)$.

Step 6. $\overline{\mho}_\kappa(t)=\left\{(Qy)(t): y\in\overline{\mho}_\kappa\right\}$ is a relatively compact subset of E in each $t\in\mathbb{R}^+$.

Let H be a subset of $\overline{\mho}_\kappa$ such that $H\in\overline{conv}(Q(M)\cup\{0\})$. In addition, by Lemma 4, we know that there is a countable set $\{y\}_{n=0}^{n=+\infty}\subset\Theta$ such that $\chi(Q(\Theta))\leq$

$2\chi(Q(\{y\}_{n=0}^{n=+\infty}))$ for any bounded set Θ. Thus for $\{y_n\}_{n=0}^{+\infty} \subset H$, for the appropriate choice of H, for every $t \in [0, d]$, by utilizing Lemma 5 and the properties of the measure χ, we obtain

$$
\begin{aligned}
& \chi(Q(H(t))) \\
\leq \;& 2\chi(Q(\{y_n(t)\}_{n=0}^{\infty})) \\
\leq \;& 2\chi\left(\left\{S(t)y_0 + \sum_{0<t_j<t} S(t-t_j)J_j(y_n(t_j))\right.\right. \\
+ \;& {}_0^t S(t-s)f\left(s, y_n(s), {}_0^s H(s,\tau,y_n(\tau))d\tau\right) ds\big\}_{n=0}^{\infty}\big) \\
\leq \;& \sum_{0<t_j<t} S(t-t_j)\chi(J_j(y_n(t_j)))) \\
+ \;& 2\chi\left(\left\{{}_0^t S(t-s)f\left(s, y_n(s), {}_0^s H(s,\tau,y_n(\tau)d\tau\right) ds\right\}_{n=0}^{\infty}\right) \\
\leq \;& \sum_{0<t_j<t} S(t-t_j)\theta_j\chi((y_n(t_j)))) \\
+ \;& 2\chi\left({}_0^t S(t-s)f\left(s, \{y_n(s)\}_{n=0}^{\infty}, {}_0^s H(s,\tau,\{y_n(\tau)\}_{n=0}^{\infty})d\tau\right) ds\right) \\
\leq \;& \sum_{0<t_j<t} S(t-t_j)\theta_j \sup_{\tau\in[0,s]} \chi(\{y_n(s)\}_{n=0}^{\infty}) \\
+ \;& 4\eta{}_0^t e^{-\lambda(t-s)}\rho(t)\left(\sup_{s\in[0,t]}\chi(\{y_n(s)\}_{n=0}^{\infty}) + 2{}_0^s\vartheta(s,\tau)\sup_{\tau\in[0,s]}\chi(\{y_n(\tau)\}_{n=0}^{\infty}))d\tau\right) ds \\
\leq \;& \sum_{0<t_j<t} S(t-t_j)\theta_j \sup_{\tau\in[0,s]} \chi(\{y_n(s)\}_{n=0}^{\infty}) \\
+ \;& 4\eta{}_0^t e^{-\lambda(t-s)}\rho(t)\left(\sup_{s\in[0,t]}\chi(\{y_n(s)\}_{n=0}^{\infty}) + 2\sup_{\tau\in[0,s]}\chi(\{y_n(s)\}_{n=0}^{\infty}){}_0^s\vartheta(s,\tau)d\tau\right) ds \\
\leq \;& \eta\sup_{\tau\in[0,s]}\theta_j \sum_{0<t_j<t} e^{-\lambda(t-t_j)} \sup_{\tau\in[0,s]}\chi(\{y_n(s)\}_{n=0}^{\infty}) \\
+ \;& 4\eta(1+2\vartheta^*){}_0^t e^{-\lambda(t-s)}\rho(s)\sup_{s\in[0,t]}\chi(\{y_n(s)\}_{n=0}^{\infty})ds \\
\leq \;& \sum_{0<t_j<t} S(t-t_j)\theta_j \sup_{\tau\in[0,s]} \chi(\{y_n(s)\}_{n=0}^{\infty}) \\
+ \;& 4\eta(1+2\vartheta^*){}_0^t e^{-\lambda(t-s)}\rho(s)ds\chi(\{y_n\}_{n=0}^{\infty}) \\
\leq \;& \frac{\eta\theta_j}{1-e^{-\lambda\varpi}}\sup_{\tau\in[0,s]}\chi(\{y_n(s)\}_{n=0}^{\infty}) \\
+ \;& 4\eta(1+\vartheta^*)\|\rho\|_{L^{\frac{1}{p_2}}}\left(\int_0^t e^{-\frac{\lambda}{1-p_2}(t-s)}ds\right)^{1-p_2}\chi(\{y_n\}_{n=0}^{\infty})
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{\eta\theta_j}{1-e^{-\lambda\varpi}} \sup_{\tau\in[0,s]} \chi(\{y_n(s)\}_{n=0}^{\infty}) \\
&+ 4\eta(1+2\vartheta^*)\|\rho\|_{L^{\frac{1}{p_2}}} \left(1-e^{-\frac{\lambda t}{1-p_2}}\right) \chi(\{y_n\}_{n=0}^{\infty}) \\
&\leq \frac{\eta\theta_j}{1-e^{-\lambda\varpi}} \sup_{\tau\in[0,s]} \chi(\{y_n(s)\}_{n=0}^{\infty}) \\
&+ 4\eta(1+2\vartheta^*)\|\rho\|_{L^{\frac{1}{p_2}}} \chi(\{y_n\}_{n=0}^{\infty}),
\end{aligned}$$

which ensures that

$$\chi((Q(H)(t)) \leq \left(\frac{\eta\theta_j}{1-e^{-\lambda\varpi}} + 4\eta(1+2\omega^*)\|\rho\|_{L^{\frac{1}{p_2}}}\right) \chi(H(t)).$$

Then

$$\chi(H) \leq \chi(Q(\Theta)(t)) \leq \left(\frac{\eta\theta_j}{1-e^{-\lambda\varpi}} + 4\eta(1+2\omega^*)\|\rho\|_{L^{\frac{1}{p_2}}}\right) \chi(H).$$

That is to say

$$\left(\frac{\eta\theta_j}{1-e^{-\lambda\varpi}} + 4\eta(1+2\omega^*)\|\rho\|_{L^{\frac{1}{p_2}}}\right) \chi(H(t)) \leq 0.$$

From (10), we observe that $\chi(H) = 0$.
Step 7. $Q(\overline{\mho}_\kappa)$ is equiconvergent.
Let $y \in \overline{\mho}_\kappa$. For $t \in \mathbb{R}^+$, we have

$$\begin{aligned}
|(Qy)(t)| &\leq |S(t)y_0| + \sum_{0<t_j<t} \|S(t-t_j)\|_{L(E)} |J_j(y(t_j))| \\
&+ {}_0^t S(t-s) f\left(s, y(s), \int_0^s H(s,\tau,y(\tau))d\tau\right) ds \\
&\leq \eta\, |y_0|\, e^{-\lambda t} + \eta \sum_{0<t_j<t} e^{-\lambda(t-t_j)}(\sigma_j\, |y(t_j)| + \varsigma_j) \\
&+ \eta \int_0^t e^{-\lambda(t-s)} \hbar(s) \left(|y(s)| +_0^s \upsilon(s,\tau)|y(\tau)|d\tau\right) ds \\
&\leq \eta|y_0|e^{-\lambda t} + \frac{\eta(\sigma_j(1+\|y\|_{P\mathfrak{C}^b}) + \varsigma_j)}{1-e^{-\lambda\varpi}} \\
&+ \eta(1+\upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}} \left(\int_0^t e^{-\frac{\lambda}{1-p_1}(t-s)} ds\right)^{1-p_1} ds(1+\|y\|_{P\mathfrak{C}^b}) \\
&\leq \eta|y_0|e^{-\lambda t} + \frac{\eta(\sigma_j(1+\|y\|_{P\mathfrak{C}^b}) + \varsigma_j)}{1-e^{-\lambda\varpi}} \\
&+ \eta(1+\upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}} \left(1-e^{-\frac{\lambda t}{1-p_1}}\right) (1+\|y\|_{P\mathfrak{C}^b})
\end{aligned}$$

$$\leq \quad \eta|y_0|e^{-\lambda t} + \frac{\eta\varsigma_j + \kappa\sigma_j}{1 - e^{-\lambda\varpi}} + (1+\upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}}\left(1 - e^{-\frac{\lambda t}{1-p_1}}\right)(1+\kappa).$$

Then, we get

$$|(Qy)(t)| \to \frac{\eta\varsigma_j + \kappa\sigma_j}{1 - e^{-\lambda\varpi}} + (1+\upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}}(1+\kappa) \quad as \quad t \to +\infty.$$

Hence, as $t \to +\infty$, we have $|(Qy)(t) - (Qy)(+\infty)| \to 0$.
Thus, from the above results $\overline{\mho_\kappa}$ is a relatively compact set. By Lemma 2, we know that Q has a fixed point in $\overline{\mho_\kappa}$. The proof is complete.

4. Example

To end this work, we apply our abstract results to the study of an integro-differential equation with impulsive effects. Consider the system

$$\begin{cases} \partial\partial t\psi(t,\xi) = \partial^2\partial\xi^2\psi(t,\xi) + \int_0^t f(t-s)\partial^2\partial s^2\psi(t,\xi)ds \\ +2^{-t}\sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)\left(e^{-\psi(t,\xi)} + \int_0^t a(t)e^{-(t-s)}(1+\psi(t,s))ds\right) \\ +2^{-t}\left(\psi(t,\xi) + \int_0^t a(t)e^{-(t-s)}(1+\psi(t,s))ds\right), t\in\mathbb{R}^+, t\neq t_j, j=1,2,3\ldots,\xi\in[0,1], \\ \Delta\psi(t_j,\xi) = (1-e^{-\lambda\varpi})\ln(1+2^{-j-2})\psi(t_j,\xi) + 2^{-j-2}(1-e^{-\lambda\varpi})\sin(\psi(t_j,\xi)), j=1,2,3\ldots, \\ \psi(t,0)=\psi(t,1)=0, \quad \psi(0,\xi)=\psi_0(\xi), \quad t\in\mathbb{R}^+, \quad \xi\in[0,1], \end{cases} \tag{10}$$

where $t_j = \sin\left(\frac{1}{2+\cos j+\cos\sqrt{2}j}\right)$ and the function $a \in AA_{P\mathfrak{C}}(\mathbb{R})$ such that

$|a| \leq \frac{3-4(\ln 2)^2}{8(\ln 2)^2}$. Here $f : \mathbb{R} \to \mathbb{R}$ is bounded uniformly continuous and continuously differentiable. Set $E = L^2(0,1)$ and let A be be the Laplace operator

$$(A\psi)(\xi) = \partial^2\partial s^2\psi(\xi),$$

then $A : D(A) = H^2(0,1)\cap H_0^1(0,1) \to L^2(0,1)$. Note that, the operator A has eigenvalues $\{-n^2\pi^2\}_1^{+\infty}$ and generates a C_0-semigroup $(S(t))_{t\geq 0}$ on E such that

$$\|S(t)\|_{L(E)} \leq \eta e^{-\lambda t},$$

with $\eta = 1$, $\lambda = \pi^2$ for all $t \geq 0$.
We define the operator $B(t) : B : E \to E$ as follows:

$$B(t)\psi = f(t)A\psi \quad \text{for } t\geq 0 \quad \text{and } \psi\in D(A).$$

Furthermore we set

$$\psi(t)(\xi) = \psi(t,\xi) \ \text{ for } t\in\mathbb{R}^+ \text{ and } \xi\in[0,1].$$

$$\psi(0) = \psi(0,\xi) \ \text{ for } t\in\mathbb{R}^+ \text{ and } \xi\in[0,1].$$

Then the system (10) takes the following abstract form

$$\begin{cases} \psi'(t) = A\psi(t) + \int_0^t B(t-s)\psi(s)ds + f\left(t, \psi(t), \int_0^t H(t,s,\psi(s))ds\right), \ t \geq 0, \\ \psi(0) = \psi_0, \end{cases} \tag{11}$$

where the nonlinear function $f : \mathbb{R}^+ \times E \times E \to E$ given by

$$\begin{aligned} f\left(t, \psi(t), \int_0^t H(t,s,\psi(s))ds\right) &= 2^{-t} \sin\left(\frac{1}{2+\cos t + \cos\sqrt{2}t}\right) \\ &\times \left(e^{-\psi} + \int_0^t a(t)e^{-(t-s)}(1+\psi(t,s))ds\right) \\ &+ 2^{-t}\left(\psi(t) + \int_0^t a(t)e^{-(t-s)}(1+\psi(t,s))ds\right). \end{aligned}$$

Let

$$\mathbb{G}\left(t, \psi(t), \varphi(t)\right) = 2^{-t} \sin\left(\frac{1}{2+\cos t + \cos\sqrt{2}t}\right)\left(\sin\psi(t) + \varphi(t)\right),$$

$$\Upsilon(t, \psi(t), \varphi(t)) = 2^{-t}(\psi(t) + \varphi(t)),$$

and

$$H(t,s,\psi(s)) = a(t)e^{-(t-s)}(1+\psi(t,s)).$$

Then it is easy to verify that $\mathbb{G}, \Upsilon : \mathbb{R} \times E \times E \to E$ are continuous and $\mathbb{G}(t, \psi(t), \varphi(t)) \in AA(\mathbb{R} \times E \times E \to E)$ and

$$|\Upsilon(t, \psi(t), \varphi(t))| \leq 2^{-t}(|\psi| + |\varphi|),$$

which implies $\Upsilon(t, \psi(t), \varphi(t)) \in C_0(\mathbb{R}^+ \times E \times E \to E)$ and

$$f(t, \psi(t), \varphi(t)) = \mathbb{G}(t, \psi(t), \varphi(t)) + \Upsilon(t, \omega(t), \vartheta(t)) \in AAA_{P\mathfrak{C}}(\mathbb{R}^+ \times E \times E, E).$$

Observe that

$$|f(t, \psi(t), \varphi(t))| \leq 2^{-t}(|\psi(t)| + |\varphi(t)|).$$

Moreover, for a bounded subset Ω_1, Ω_2 of E, and from properties of measure of noncompactness χ, we have

$$\chi(f(t, \Omega_1, \Omega_2)) \leq 2^{-t}\left(\chi(\Omega_1) + \chi(\Omega_2)\right).$$

Moreover, let $p_1 = p_2 = \frac{1}{2}$, then, the assumptions $(\mathbb{H}_2)$ hold with

$$\hbar(t) = \rho(t) = 2^{-t}.$$

Similarly, H clearly satisfies the following:

$$|H(t,s,\psi_2) - H(t,s,\psi_1)| \leq |a(t)|\, e^{-(t-s)}\, |\psi_2 - \psi_1|.$$

Now, by the property of measure of noncompactness for bounded subset Ω of E, we have

$$\chi(H(t,s,\Omega)) \leq |a(t)|\, e^{-(t-s)}\chi(\Omega).$$

In addition

$$\|a\|_{P\mathfrak{C}} \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-(t-s)} \mathrm{d}s = \|a\|_{P\mathfrak{C}} < +\infty.$$

and for every compact interval $[c,d] \subset \mathbb{R}$, we have

$$\lim_{t\in+\infty} \int_{a}^{b} a(t)e^{-(t-s)}\mathrm{d}s = \lim_{t\in+\infty} \|a\|_{P\mathfrak{C}}(e^{-(t-d)} - e^{-(t-c)}) = 0,$$

and

$$H(t,s,0) = a(t)e^{-(t-s)}.$$

Then the assumptions $(\mathbb{H}_1)$ hold with

$$\varrho(t,s) = \phi(t,s) = \theta(t,s) = \pi(t,s) = a(t)e^{-(t-s)} \text{ and } \widehat{\phi}(t,s) = b(t)e^{-(t-s)},$$

b the limit functions given in Definition 3 with $f = a,\ \mathbb{G} = b$.
Moreover,

$$|J_j(\psi)| \leq (1 - e^{-\lambda\varpi}) \ln(1 + 2^{-j})|\psi(t)| + 2^{-j-2}(1 - e^{-\lambda\varpi}).$$

Now, by the property of measure of noncompactness for bounded subset Ω of E, we have

$$\chi(J_j(\Omega)) \leq 2^{-j-2}(1 - e^{-\lambda\varpi})\chi(\Omega).$$

Furthermore, from Theorem 3, we obtain

$$\begin{aligned}
\Delta &= \eta \max\left(\frac{\varsigma_j}{1 - e^{-\lambda\varpi}} + \frac{\sigma_j}{1 - e^{-\lambda\varpi}} + (1 + \upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}}, \frac{\theta_j}{1 - e^{-\lambda\varpi}} + 4(1 + 2\vartheta^*)\|\rho\|_{L^{\frac{1}{p_2}}}\right) \\
&\leq \max\left(\frac{1}{2} + \frac{4(1 + |a|)}{(\ln 2)^2}, \frac{1}{4} + \frac{16(1 + 2|a|)}{(\ln 2)^2}\right) \\
&\leq \frac{1}{4} + \frac{4}{(\ln 2)^2} \max\left(1 + |a|, 4(1 + 2|a|)\right) \\
&\leq \frac{1}{4} + \frac{16(1 + 2|a|)}{(\ln 2)^2} \\
&\leq 1.
\end{aligned}$$

So, all the conditions of Theorem 3 are satisfied. Hence by the conclusion of Theorems 3, it follows that the problem (1) has at least one an asymptotically almost automorphic mild solution $\psi \in \overline{\mho_\kappa}$.

5. Conclusions

In the present research, we have investigated existence for the piecewise asymptotically almost automorphic mild solutions of impulsive integro-differential equations with instantaneous impulses in Banach space. To achieve the desired results for the given problems, the fixed-point approach was used, namely Mönch's fixed point theorem, combined with resolvent operators from the Grimmer perspective and the concept of measures of non-compactnes. An example is provided to demonstrate how the major results can be applied. Our results in the given configuration are novel and substantially contribute to the literature on this field of study. We feel that there are multiple potential study avenues such as coupled systems, problems with infinite delays, problems with inclusions and many more due to the limited number of publications on integro-differential equations and inclusions, particularly with impulses. We hope that this article will serve as a starting point for such an undertaking.

Author Contribution Statements The study was carried out in collaboration of all authors. All authors read and approved the final manuscript.

Declaration of Competing Interests It is declared that authors has no competing interests.

References

[1] Abbas, S., Mahto, L., Hafayed, M., Alimi, A. M., Asymptotic almost automorphic solutions of impulsive neural network with almost automorphic coefficients, *Neurocomputing*, 142 (2014), 326-334. https://doi.org/10.1016/j.neucom.2014.04.028

[2] Akgöl, S. D., Asymptotic equivalence of impulsive dynamic equations on time scales, *Hacet. J. Math. Stat.*, 52(2) (2023), 277-291. https://doi.org/10.15672/hujms.1103384

[3] Akgöl, S. D., Existence of solutions for impulsive boundary value problems on infinite intervals, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 72(3) (2023), 721-736. https://doi.org/10.31801/cfsuasmas.1186785

[4] Akgöl, S. D., Oscillation of impulsive linear differential equations with discontinuous solutions, *Bull. Aust. Math. Soc.*, 107(1) (2023), 112-124. https://doi.org/10.1017/s0004972722000429

[5] Araya, D., Lizama, C., Almost automorphic mild solutions to fractional differential equations, *Nonlinear Anal.*, 69(11) (2008), 3692-3705. http://dx.doi.org/10.1016/j.na.2007.10.004

[6] Arjunan, M. M., Mlaiki, N., Kavitha, V., Abdeljawad, T., On fractional state-dependent delay integro-differential systems under the Mittag-Leffler kernel in Banach space, *AIMS Math.*, 8(1) (2023), 1384-1409. https://doi.org/10.3934/math.2023070

[7] Bainov, D. D., Simeonov, P. S., Impulsive Differential Equations: Periodic Solutions and Applications, In: Pitman Monographs and Surveys in Pure and Applied Mathematics Vol. 66. Harlow, Longman Scientific Technical, New York, 1993.

[8] Banaŝ, J., Goebel, K., Measures of Noncompactness in Banach Spaces, Lecture Note in Pure App. Math., New York, 1980.

[9] Benchohra, M., Karapınar, E., Lazreg, J. E., Salim, A., Advanced Topics in Fractional Differential Equations: A Fixed Point Approach, Springer, Cham, 2023. https://doi.org/10.1007/978-3-031-26928-8

[10] Benchohra, M., Karapınar, E., Lazreg, J. E., Salim, A., Fractional Differential Equations: New Advancements for Generalized Fractional Derivatives, Springer, Cham, 2023. https://doi.org/10.1007/978-3-031-34877-8

[11] Benkhettou, N., Aissani, K., Salim, A., Benchohra, M., Tunc, C., Controllability of fractional integro-differential equations with infinite delay and non-instantaneous impulses, *Appl. Anal. Optim.*, 6 (2022), 79-94.

[12] Benkhettou, N., Salim, A., Aissani, K., Benchohra, M., Karapınar, E., Non-instantaneous impulsive fractional integro-differential equations with state-dependent delay, *Sahand Commun. Math. Anal.*, 19 (2022), 93-109. https://doi.org/10.22130/scma.2022.542200.1014

[13] Bensalem, A., Salim, A., Ahmad, B., Benchohra, M., Existence and controllability of integrodifferential equations with non-instantaneous impulses in Fréchet spaces, *CUBO.*, 25(2) (2023), 231–250. https://doi.org/10.56754/0719-0646.2502.231

[14] Bensalem, A., Salim, A., Benchohra, M., Ulam-Hyers-Rassias stability of neutral functional integrodifferential evolution equations with non-instantaneous impulses on an unbounded interval, *Qual. Theory Dyn. Syst.*, 22 (2023), 29 pages. https://doi.org/10.1007/s12346-023-00787-y

[15] Bensalem, A., Salim, A., Benchohra, M., Fečkan, M., Approximate controllability of neutral functional integro-differential equations with state-dependent delay and non-instantaneous impulses, *Mathematics*, 11 (2023), 1-17. https://doi.org/10.3390/math11071667

[16] Bochner, S., Continuous mappings of almost automorphic and almost periodic functions, *Proc. Natl. Acad. Sci.*, 52 (1964), 907-910.

[17] Caraballo, T., Cheban, D., Almost periodic and almost automorphic solutions of linear differential/difference equations without Favard's separation condition, *J. Differential Equations*, 246(1) (2009), 108-128. http://dx.doi.org/10.1016/j.jde.2008.04.001

[18] Cao, J., Yang, Q., Huang, Z., Existence and exponential stability of almost automorphic mild solutions for stochastic functional differential equations, *Stoch.: An Int. J. Probab. Stoch. Processes*, 83 (2011), 259-275. http://dx.doi.org/10.1080/17442508.2010.533375

[19] Cao, J., Huang, Z., N'Guérékata, G. M., Existence of asymptotically almost automorphic mild solutions for nonautonomous semilinear evolution equations, *Elec. J. Differential Equations*, 2018(37) (2018), 16 pp.

[20] Chavez, A., Pinto, M., Zavaleta, U., On almost automorphic type solutions of abstract integral equations, a Bohr-Neugebauer type property and some applications, *J. Math. Anal. Appl.*, 494(1) (2021), 38 pp. http://dx.doi.org/10.1016/j.jmaa.2020.124395

[21] Chen, P., Li, Y., Monotone iterative technique for a class of semilinear evolution equations with nonlocal conditions, *Results Math.*, 63 (2013), 731-744. http://dx.doi.org/10.1007/s00025-012-0230-5

[22] Corduneanu, C., Integral Equations and Stability of Feedback Systems, Acadimic Press, New York, 1973.

[23] Desch, W., Grimmer, R., Schappacher, W., Some considerations for linear integro-differential equations, *J. Math. Anal. Appl.*, 104 (1984), 219-234.

[24] Dianaga, T., N'Guérékata, G. M., Almost automorphic solutions to some classes of partial evolution equations, *Appl. Math. Lett.*, 20 (2007), 462-466. http://dx.doi.org/10.1016/j.aml.2006.05.015

[25] Ezzinbi, K., N'Guérékata, G. M., Almost automorphic solutions for some partial functional differential equations, *J. Math. Anal. Appl.*, 328 (1) (2007), 344-358. https://doi.org/10.1016/j.jmaa.2006.05.036

[26] Fen, M. O., Fen, F. T., Homoclinic and heteroclinic motions in hybrid systems with impacts, *Mathematica Slovaca.*, 67(5) (2017), 1179-1188. https://doi.org/10.1515/ms-2017-0041

[27] Fen, M. O., Fen, F. T., Replication of period-doubling route to chaos in impulsive systems, *Electron. J. Qual. Theory Differ. Equ.*, 2019(58) (2019), 1-20. https://doi.org/10.14232/ejqtde.2019.1.58

[28] Fen, M. O., Fen, F. T., Unpredictability in quasilinear non-autonomous systems with regular moments of impulses, *Mediterr. J. Math.*, 20(4) (2023), 191. https://doi.org/10.1007/s00009-023-02401-6

[29] Goldstein, J. A., N'Guérékata, G. M., Almost automorphic solutions of semilinear evolution equations, *Proc. Amer. Math. Soc.*, 133 (2005), 2401-2408. http://dx.doi.org/10.2307/4097881

[30] Grimmer, R. C., Resolvent operators for integral equations in a Banach space, *Trans. Amer. Math. Soc.,* 273 (1982), 333-349.

[31] Heinz, H. P., On the behaviour of measure of noncompactness with respect to differentiation and integration of rector-valued functions, *Nonlinear Anal.*, 7 (1983), 1351-1371.

[32] Kavitha, V., Baleanu, D., Grayna, J., Measure pseudo almost automorphic solution to second order fractional impulsive neutral differential equation, *AIMS Math.*, 6(8) (2021), 8352-8366. http://dx.doi.org/10.3934/math.2021484

[33] Kavitha, V., Arjunan, M., Baleanu, D., Grayna, J., Weighted pseudo almost automorphic functions with applications to impulsive fractional integro-differential equation, *An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat.*, 31(1) (2023), 143-166. https://doi.org/10.2478/auom-2023-0007

[34] Liang, J., Zhang, J., Xiao, T., Composition of pseudo almost automorphic and asymptotically almost automorphic functions, *J. Math. Anal. Appl.*, 340(2) (2008), 1493-1499. https://doi.org/10.1016/j.jmaa.2007.09.065

[35] Mahto, L., Abbas, S., PC-almost automorphic solution of impulsive fractional diferential equations, *Mediter. J. Math.*, 12(3) (2015), 771-790. http://dx.doi.org/10.1007/s00009-014-0449-3

[36] Milman, V. D., Myshkis, A. D., On the stability of motion in presence of impulses, *Sib. Math. J.*, 1 (1960), 233-237.

[37] Mishra, I., Bahuguna, D., Abbas, S., Existence of almost automorphic solutions of neutral functional differential equation, *Nonlinear Dyn. Syst. Theory.,* 11(2) (2011), 165-172.

[38] Mophoua, G., N'Guérékata, G. M., On some classes of almost automorphic functions and applications to fractional differential equations, *Compu. Math. Appl.*, 59 (2010), 1310-1317. http://dx.doi.org/10.1016/j.camwa.2009.05.008

[39] Mönch, H., Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal.*, 4 (1980), 985-999.

[40] N'Guérékata, G. M., Sur les solutions presqu'Automorphes d'équations différentielles abstraites [On almost automorphic solutions of abstract differential equations], *Ann. Sci. Math. Québec.*, 5 (1981), 69-79.

[41] N'Guérékata, G. M., Almost Automorphic and Almost Periodic Functions in Abstract Spaces, Kluwer Academic, New York, 2001.

[42] N'Guérékata, G. M., Topics in Almost Automorphy, Springer, New York, Boston, Dordrecht, London, Moscow, 2005.

[43] N'Guérékata, G. M., Spectral Teory for Bounded Functions and Applications to Evolution Equations, Nova Science Pub. NY, 2017.

[44] Rezoug, N., Benchohra, M., Ezzinbi, K., Asymptotically automorphic solutions of abstract fractional evolution equations with non-instantaneous impulses, *Surv. Math. Appl.*, 17 (2022), 113-138.

[45] Rezoug, N., Salim, A., Benchohra, M., Asymptotically almost automorphy for impulsive integrodifferential evolution equations with infinite time delay via Mönch fixed point, *Evol. Equ. Control Theory*, 13(4) (2024), 989-1014. http://dx.doi.org/10.3934/eect.2024014

[46] Santos, J. P. C., Cuevas, C., Asymptotically almost automorphic solutions of abstract fractional integro-differential neutral equations, *Appl. Math. Lett.*, 23(9) (2010), 960-965. https://doi.org/10.1016/j.aml.2010.04.016

[47] Singh, V., Pandey, D., Doubly weighted pseudo almost automorphic solutions for two-term fractional order differential equations, *J. Nonlinear Evol. Equ. Appl.*, (4) (2018), 39-56.

[48] Svetlin, G. G., Akgöl, S. D., Kuş, M. E., Existence of solutions for first order impulsive periodic boundary value problems on time scales, *Filomat*, 37(10) (2023), 3029-3042. https://doi.org/10.2298/FIL2310029G

[49] Tokmak Fen, F., Fen, M. O., Modulo periodic Poisson stable solutions of dynamic equations on a time scale, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 72(4) (2023), 907-920. https://doi.org/10.31801/cfsuasmas.1220565

[50] Veech, W., Almost automorphic functions, *Proc. Natl. Acad. Sci.*, 49 (1963), 462-464.

[51] Xia, Z., Piecewise asymptotically almost periodic solution of neutral Volterra integro-differential equations with impulsive effects, *Turkish J. Math.*, 41(6) (2017), 23. https://doi.org/10.3906/mat-1408-11

[52] Yan, Z., Zhang, H., Asymptotic stability of fractional impulsive neutral stochastic partial integro-differential equations with state-dependent delay, *Electron. J. Differential Equations,* (206) (2013), 29 pp.

[53] Zheng, X. J., Ye, C. Z., Ding, H. S., Asymptotically almost automorphic solutions to nonautonomous semilinear evolution equations, *Afr. Diaspora J. Math.*, 12(2) (2011), 104-112.

[54] Zhao, Z., Chang, Y., Nieto, J., Almost automorphic and pseudo-almost automorphic mild solutions to an abstract differential equation in Banach spaces, *Nonlinear Anal. Theo. Meth. Appl.*, 72 (2010), 1886-1894. http://dx.doi.org/10.1016/j.na.2009.09.028