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Pseudostarlikeness and Pseudoconvexity of Multiple Dirichlet Series

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Abstract

Let $p \in \mathbb{N}$, $s = (s_1, \ldots, s_p) \in \mathbb{C}^p$, $h = (h_1, \ldots, h_p) \in \mathbb{R}^p_+$, $(n) = (n_1, \ldots, n_p) \in \mathbb{N}^p$ and the sequences $\lambda_{(n)} = (\lambda^{(1)}_{n_1}, \ldots, \lambda^{(p)}_{n_p})$ are such that $0 < \lambda^{(j)}_1 < \lambda^{(j)}_k < \lambda^{(j)}_{k+1} \uparrow +\infty$ as $k \to \infty$ for every $j = 1, \ldots, p$. For $a = (a_1, \ldots, a_p)$ and $c = (c_1, \ldots, c_p)$ let $(a, c) = a_1c_1 + \ldots + a_pc_p$, and we say that $a > c$ if $a_j > c_j$ for all $1 \le j \le p$. For a multiple Dirichlet series

$$F(s) = e^{(s,h)} + \sum_{\lambda_{(n)} > h} f_{(n)} \exp\{(\lambda_{(n)}, s)\}$$

absolutely converges in $\Pi^p_0 = \{s : \mathrm{Re}\, s < 0\}$, concepts of pseudostarlikeness and pseudoconvexity are introduced and criteria for pseudostarlikeness and the pseudoconvexity are proved. Using the obtained results, we investigated neighborhoods of multiple Dirichlet series, Hadamard compositions, and properties of solutions of some differential equations.

1. Introduction

Let S be a class of functions $f(z) = z + \sum_{n=2}^{\infty} f_n z^n$ analytic univalent in $\mathbb{D} = \{z : |z| < 1\}$. The function $f \in S$ is said to be starlike if $f(\mathbb{D})$ is a starlike domain concerning the origin. It is well-known [1] (see p. 202) that the condition

$$\mathrm{Re}\,\{zf'(z)/f(z)\} > 0 \ (z \in \mathbb{D})$$

is necessary and sufficient for the starlikeness of f. A. W. Goodman [2] (see also [3] p. 9) proved that if $\sum_{n=2}^{\infty} n|f_n| \le 1$ then function $f \in S$ is starlike.
The concept of the starlikeness of function $f \in S$ got the series of generalizations. I. S. Jack [4] studied starlike functions of order $\alpha \in [0, 1)$, i. e. such functions $f \in S$, for which

$$\mathrm{Re}\,\{zf'(z)/f(z)\} > \alpha \ (z \in \mathbb{D}).$$

It is proved [4], ([3], p. 13) that if $\sum_{n=2}^{\infty} (n - \alpha)|f_n| \le 1 - \alpha$ then function $f \in S$ is starlike function of order α. V. P. Gupta [5] introduced the concept of starlike function of order $\alpha \in [0, 1)$ and type $\beta \in (0, 1]$. A function $f \in S$ is so named for that

$$|zf'(z)/f(z) - 1| < \beta |zf'(z)/f(z) + 1 - 2\alpha| \quad \text{for all} \quad z \in \mathbb{D}.$$

It is proved [5] that if

$$\sum_{n=2}^{\infty} \{(1 + \beta)n - \beta(2\alpha - 1) - 1)\}|f_n| \le 2\beta(1 - \alpha)$$

then function $f \in S$ is starlike function of the order α and the type β.

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For $f \in S$, following A. W. Goodman [6] and S. Ruscheweyh [7], its neighborhood is called a set

$$N_{\delta}(f)=\left\{g(z)=z+\sum_{k=2}^{\infty}g_k z^k\in S:\sum_{k=2}^{\infty}k|g_k-f_k|\le\delta\right\}.$$

The neighborhoods of various classes of analytical in $\mathbb{D}$ functions were studied by many authors (we indicate here only in articles [8–14]). For power series

$$f_j(z)=\sum_{k=0}^{\infty}f_{k,j}z^k \quad (j=1,2)$$

the series

$$(f_1*f_2)(z)=\sum_{k=0}^{\infty}f_{k,1}f_{k,2}z^k$$

is called the Hadamard composition (product) [15]. Obtained by J. Hadamard properties of this composition find the applications [16–18] in the theory of the analytic continuation of the functions represented by power series. Many authors (see for example [7, 19–22]) have studied Hadamard compositions of univalent, starlike, meromorphically starlike functions.
Let $h\ge 1$, $\Lambda=(\lambda_k)$ be an increasing to $+\infty$ sequence of positive numbers ($\lambda_1>h$) and $S(\Lambda)$ be a class of Dirichlet series

$$F(s)=e^{sh}+\sum_{k=1}^{\infty}f_k\exp\{s\lambda_k\}\ (s=\sigma+it)$$

absolutely convergent in half-plane $\Pi_0=\{s:\mathrm{Re}\,s<0\}$. It is known [24], ([3], p. 135) that each function $F\in S(\Lambda)$ is non-univalent in Π_0, but there exist conformal in Π_0 functions $F\in S(\Lambda)$, and if

$$\sum_{k=2}^{\infty}\lambda_k|f_k|\le h$$

then function F is conformal in Π_0. A conformal function F in Π_0 is said to be pseudostarlike if

$$\mathrm{Re}\{F'(s)/F(s)\}>0\,(s\in\Pi_0).$$

In [24] (see also [3], p. 139) it is proved that if

$$\sum_{k=2}^{\infty}\lambda_k|f_k|\le h$$

then function F is pseudostarlike. A conformal function $F\in S(\Lambda)$ is said to be pseudostarlike of the order α if

$$\mathrm{Re}\{F'(s)/F(s)\}>\alpha\in[0,\,1) \quad \text{for all} \quad s\in\Pi_0.$$

Since the inequality $|w-h|<|w-(2\alpha-h)|$ holds if and only if $\mathrm{Re}\,w>\alpha$, function $F\in S(\Lambda)$ is pseudostarlike of the order α if and only if

$$|F'(s)/F(s)-h|<|F'(s)/F(s)-(2\alpha-h)|fors\in\Pi_0.$$

Therefore, as in [25], we call a conformal function $F\in S(\Lambda)$ in Π_0 pseudostarlike of the order $\alpha\in[0,\,1)$ and the type $\beta\in(0,\,1]$ if

$$|F'(s)/F(s)-h|<\beta|F'(s)/F(s)-(2\alpha-h)| \quad \text{for} \quad s\in\Pi_0.$$

In [25], it is proved that if

$$\sum_{k=1}^{\infty}\{(1+\beta)\lambda_k-2\beta\alpha-h(1-\beta)\}|f_k|\le 2\beta(h-\alpha)$$

then F is pseudostarlike of the order α and the type β. If in the definition of the pseudostarlikeness instead of F'/F to put F''/F' then we will get the definition of the pseudoconvexity.
S. M. Shah [26] indicated conditions on real parameters γ_0, γ_1, γ_2 of the differential equation

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+(\gamma_0 z^2+\gamma_1 z+\gamma_2)w=0,$$

under which there exists an entire transcendental solution $f(z)=z+\sum_{n=2}^{\infty}f_n z^n$ such that f and all its derivatives are close-to-convex in $\mathbb{D}$. The convexity of solutions of the Shah equation has been studied in [27, 28]. Substituting $z=e^s$ we obtain the differential equation

$$\frac{d^2w}{ds^2}+(\gamma_0 e^{2hs}+\gamma_1 e^{hs}+\gamma_2)w=0$$

with $h=1$. The pseudoconvexity and pseudostarlikeness of solutions of the last equation have been studied in [3] (see p. 147-153).
In the proposed article we will get similar results for multiple Dirichlet series. The theorems proved here complement the results of the papers [29–31].

2. Pseudostarlikeness and Pseudoconvexity

Let $p\in\mathbb{N}$, $h=(h_1,\dots,h_p)\in\mathbb{R}_+^p$, $(n)=(n_1,\dots,n_p)\in\mathbb{N}^p$ and the sequences $\lambda_{(n)}=\left(\lambda_{n_1}^{(1)},\dots,\lambda_{n_p}^{(p)}\right)$ are such that

$$0<\lambda_1^{(j)}<\lambda_k^{(j)}<\lambda_{k+1}^{(j)}\uparrow+\infty \quad \text{as} \quad k\to\infty \quad \text{for every} \quad j=1,\dots,p.$$

We denote $\mathbf{H}=h_1\cdot\ldots\cdot h_p$, $\Lambda_{(n)}=\left(\lambda_{n_1}^{(1)}\cdot\ldots\cdot\ldots\lambda_{n_p}^{(p)}\right)$. Also let $s=(s_1,\dots,s_p)\in\mathbb{C}^p$, $s_j=\sigma_j+it_j$, $\sigma=(\sigma_1,\dots,\sigma_p)$, and for $a=(a_1,\dots,a_p)$ and $c=(c_1,\dots,c_p)$ let $(a,c)=a_1c_1+\ldots+a_pc_p$. We say that $a>c$ if $a_j>c_j$ for all $1\le j\le p$.
Suppose that the multiple Dirichlet series

$$F(s)=e^{(s,h)}+\sum_{\lambda_{(n)}>h} f_{(n)}\exp\{(\lambda_{(n)},s)\} \tag{2.1}$$

absolutely converges in $\Pi_0^p=\{s:\operatorname{Re}s<0\}$, where $\operatorname{Re}s<0\Longleftrightarrow(\operatorname{Re}s_1<0,\dots,\operatorname{Re}s_p<0)$.
For the definition of the pseudostarlikeness of the function (2.1) can be used either by one variable or in joint variables or in the direction. Here we will look at the pseudostarlikeness in joint variables.
We denote

$$F^{(p)}(s)=\frac{\partial^pF(s)}{\partial s_1\cdot\ldots\cdot\partial s_p} \quad \text{and} \quad F^{(2p)}(s)=\frac{\partial^pF^{(p)}(s)}{\partial s_1\cdot\ldots\cdot\partial s_p}=\frac{\partial^{2p}F(s)}{\partial^2 s_1\cdot\ldots\cdot\partial^2 s_p}.$$

We say that function (2.1) is pseudostarlike of the order $\alpha\in[0,\mathbf{H})$ and the type $\beta>0$ in joint variables if

$$\left|\frac{F^{(p)}(s)}{F(s)}-\mathbf{H}\right|<\beta\left|\frac{F^{(p)}(s)}{F(s)}-(2\alpha-\mathbf{H})\right|, \quad s\in\Pi_0^p. \tag{2.2}$$

and function (2.1) is pseudoconvex of the order $\alpha\in[0,\mathbf{H})$ and the type $\beta>0$ in joint variables if

$$\left|\frac{F^{(2p)}(s)}{F^{(p)}(s)}-\mathbf{H}\right|<\beta\left|\frac{F^{(2p)}(s)}{F^{(p)}(s)}-(2\alpha-\mathbf{H})\right|, \quad s\in\Pi_0^p.$$

Theorem 2.1. *Let $\alpha\in[0,\mathbf{H})$ and $\beta>0$. If*

$$\sum_{\lambda_{(n)}>h}\{(1+\beta)\Lambda_{(n)}-2\beta\alpha-\mathbf{H}(1-\beta)\}|f_{(n)}|\le 2\beta(\mathbf{H}-\alpha) \tag{2.3}$$

then function (2.1) is pseudostarlike of the order α and the type β in joint variables. If

$$\sum_{\lambda_{(n)}>h}\Lambda_{(n)}\{(1+\beta)\Lambda_{(n)}-2\beta\alpha-\mathbf{H}(1-\beta)\}|f_{(n)}|\le 2\mathbf{H}\beta(\mathbf{H}-\alpha) \tag{2.4}$$

then function (2.1) is pseudoconvex of the order α and the type β in joint variables.

Proof. Since

$$F^{(p)}(s)=\mathbf{H}e^{(s,h)}+\sum_{\lambda_{(n)}>h}\Lambda(n)f_{(n)}\exp\{(\lambda_{(n)},s)\},$$

we have

$$\begin{aligned}\left|F^{(p)}(s)-\mathbf{H}F(s)\right|-\beta\left|F^{(p)}(s)-(2\alpha-\mathbf{H})F(s)\right|&=\left|\sum_{\lambda_{(n)}>h}(\Lambda(n)-\mathbf{H})f_{(n)}\exp\{(\lambda_{(n)},s)\}\right|\\&\quad-\beta\left|2(\mathbf{H}-\alpha)e^{(s,h)}+\sum_{\lambda_{(n)}>h}(\Lambda(n)+\mathbf{H}-2\alpha)f_{(n)}\exp\{(\lambda_{(n)},s)\}\right|.\end{aligned}$$

Suppose that $\alpha<\mathbf{H}$. Since $-|a+b|\le-|a|+|b|$ and $\sigma<0$, hence in view of (2.3) we get

$$\begin{aligned}&\left|F^{(p)}(s)-\mathbf{H}F(s)\right|-\beta\left|F^{(p)}(s)-(2\alpha-\mathbf{H})F(s)\right|\\&\le\left|\sum_{\lambda_{(n)}>h}(\Lambda(n)-\mathbf{H})f_{(n)}\exp\{(\lambda_{(n)},s)\}\right|-\left|2\beta(\mathbf{H}-\alpha)e^{(s,h)}\right|+\left|\beta\sum_{\lambda_{(n)}>h}(\Lambda_{(n)}+\mathbf{H}-2\alpha)f_{(n)}\exp\{(\lambda_{(n)},s)\}\right|\\&\le\sum_{\lambda_{(n)}>h}(\Lambda(n)-\mathbf{H})|f_{(n)}|\exp\{(\lambda_{(n)},\sigma)\}-2\beta(\mathbf{H}-\alpha)e^{(\sigma,h)}+\beta\sum_{\lambda_{(n)}>h}(\Lambda_{(n)}+\mathbf{H}-2\alpha)|f_{(n)}|\exp\{(\lambda_{(n)},\sigma)\}\\&=e^{(\sigma,h)}\left(\sum_{\lambda_{(n)}>h}\{(1+\beta)\Lambda_{(n)}-2\beta\alpha-\mathbf{H}(1-\beta)\}|f_{(n)}|\exp\{(\lambda_{(n)}-h,\sigma))\}-2\beta(\mathbf{H}-\alpha)\right)\\&<\sum_{\lambda_{(n)}>h}\{(1+\beta)\Lambda_{(n)}-2\beta\alpha-\mathbf{H}(1-\beta)\}|f_{(n)}|-2\beta(\mathbf{H}-\alpha)\le 0,\end{aligned}$$

i. e.

$$\left|F^{(p)}(s)-\mathbf{H}F(s)\right|-\beta\left|F^{(p)}(s)-(2\alpha-\mathbf{H})F(s)\right|<0,\quad s\in\Pi_0^p. \tag{2.5}$$

Since conditions (2.2) and (2.5) are equivalent, function (2.1) is pseudostarlike of the order α and the type β in joint variables.
Since $F^{(2p)}(s)/F^{(p)}(s)=G^{(p)}(s)/G(s)$, where

$$G(s)=e^{sh}+\sum_{\lambda_{(n)}>h}g_{(n)}\exp\{(\lambda_{(n)},s)\},\quad g_{(n)}=\frac{\Lambda_{(n)}}{\mathbf{H}}f_{(n)},$$

the function F is pseudoconvex of the order $\alpha\in[0,\mathbf{H})$ and the type $\beta>0$ in joint variables if and only if the function G is pseudostarlike of the order $\alpha\in[0,\mathbf{H})$ and the type $\beta>0$ in joint variables. Therefore, if (2.4) holds then the function F is pseudoconvex of the order $\alpha\in[0,\mathbf{H})$ and the type $\beta>0$ in joint variables. The proof of Theorem 2.1 is complete. □

The following theorem complements the statement of Theorem 2.1.

Theorem 2.2. *Let $\alpha\in[0,\mathbf{H})$ and $\beta>0$. If function (2.1) is pseudostarlike of the order α and the type β in joint variables and all $f_{(m)}\le 0$ then (2.3) holds. If function (2.1) is pseudoconvex of the order α and the type β in joint variables and all $f_{(m)}\le 0$ then (2.4) holds.*

Proof. If function (2.1) is pseudostarlike of the order α and the type β in joint variables and $f_{(m)}=-|f_{(m)}|$ then in view of (2.5) as above we have for all $s\in\Pi_0^p$

$$\left|\frac{-\sum\limits_{\lambda_{(n)}>h}(\Lambda(n)-\mathbf{H})|f_{(n)}|\exp\{(\lambda_{(n)},s)\}}{2(\mathbf{H}-\alpha)e^{(s,h)}-\sum\limits_{\lambda_{(n)}>h}(\Lambda(n)+\mathbf{H}-2\alpha)|f_{(n)}|\exp\{(\lambda_{(n)},s)\}}\right|=\left|\frac{F^{(p)}(s)-\mathbf{H}F(s)}{F^{(p)}(s)-(2\alpha-\mathbf{H})F(s)}\right|<\beta$$

and therefore,

$$\mathrm{Re}\,\frac{\sum\limits_{\lambda_{(n)}>h}(\Lambda(n)-\mathbf{H})|f_{(n)}|\exp\{(\lambda_{(n)},s)\}}{2(\mathbf{H}-\alpha)e^{(s,h)}-\sum\limits_{\lambda_{(n)}>h}(\Lambda(n)+\mathbf{H}-2\alpha)|f_{(n)}|\exp\{(\lambda_{(n)},s)\}}<\beta$$

whence for all $\sigma<0$ we obtain

$$\frac{\sum\limits_{\lambda_{(n)}>h}(\Lambda(n)-\mathbf{H})|f_{(n)}|\exp\{(\lambda_{(n)},\sigma)\}}{2(\mathbf{H}-\alpha)e^{(\sigma,h)}-\sum\limits_{\lambda_{(n)}>h}(\Lambda(n)+\mathbf{H}-2\alpha)|f_{(n)}|\exp\{(\lambda_{(n)},\sigma)\}}<\beta$$

Letting $\sigma\to 0$ from here we get

$$\frac{\sum\limits_{\lambda_{(n)}>h}(\Lambda(n)-\mathbf{H})|f_{(n)}|}{2(\mathbf{H}-\alpha)-\sum\limits_{\lambda_{(n)}>h}(\Lambda(n)+\mathbf{H}-2\alpha),|f_{(n)}|}<\beta$$

whence (2.3) follows. The first part of Theorem 2.2 is proved. The second part is proved similarly. □

Theorems 2.1 and 2.2 imply the following statements.

Corollary 2.3. *In order that the function (2.1) is pseudostarlike of the order $\alpha\in[0,\mathbf{H})$ in joint variables, it is sufficient and in the case, when all $f_{(n)}\le 0$, it is necessary that*

$$\sum_{\lambda_{(n)}>h}\{\Lambda_{(n)}-\alpha)\}|f_{(n)}|\le\mathbf{H}-\alpha. \tag{2.6}$$

In order that the function (2.1) is pseudoconvex of the order $\alpha\in[0,\mathbf{H})$ in joint variables, it is sufficient and in the case, when all $f_{(n)}\le 0$, it is necessary that

$$\sum_{\lambda_{(n)}>h}\Lambda_{(n)}\{\Lambda_{(n)}-\alpha\}|f_{(n)}|\le\mathbf{H}(\mathbf{H}-\alpha).$$

Corollary 2.4. *In order that the function (2.1) is pseudostarlike in joint variables, it is sufficient and in the case, when all $f_{(n)}\le 0$, it is necessary that*

$$\sum_{\lambda_{(n)}>h}\Lambda_{(n)}|f_{(n)}|\le\mathbf{H}.$$

In order that the function (2.1) is pseudoconvex in joint variables, it is sufficient and in the case, when all $f_{(n)}\le 0$, it is necessary that $\sum\limits_{\lambda_{(n)}>h}(\Lambda_{(n)})^2|f_{(n)}|\le\mathbf{H}^2$.

3. Neighborhoods of Multiple Dirichlet Series

Here the class of series (2.1) absolutely convergent in Π_0^p we denote by D and we say that $F \in D^*$ if $F \in D$ and all $f_{(n)} \leq 0$. By $PSD_{\alpha,\beta}$ we denote a class of pseudostarlike functions (2.1) of the order α and the type β in joint variables, and by $PCD_{\alpha,\beta}$ we denote a class of pseudoconvex functions (2.1) of the order α and the type β in joint variables.
For $j > 0$ and $\delta > 0$ we define the neighborhood of $F \in D$ in joint variables as follows

$$O_{j,\delta}(F) = \left\{ G(s) = e^{(s,h)} + \sum_{\lambda_{(n)} > h} g_{(n)} \exp\{(\lambda_{(n)}, s)\} \in D : \sum_{\lambda_{(n)} > h} \Lambda_{(n)}^j |g_{(n)} - f_{(n)}| \leq \delta \right\}.$$

Similarly, for $F \in D^*$

$$O_{j,\delta}^*(F) = \left\{ G(s) = e^{(s,h)} + \sum_{\lambda_{(n)} > h} g_{(n)} \exp\{(\lambda_{(n)}, s)\} \in D^* : \sum_{\lambda_{(n)} > h} \Lambda_{(n)}^j |g_{(n)} - f_{(n)}| \leq \delta \right\}.$$

Here we will establish a connection between classes $PSD_{\alpha,\beta}$, $PCD_{\alpha,\beta}$ and $O_{j,\delta}(F)$, $O_{j,\delta}^*(F)$. We need the following lemma.

Lemma 3.1. *Let $F \in D$. Then $G \in O_{2,\mathbf{H}\delta}(F)$ if and only if $\dfrac{G^{(p)}}{\mathbf{H}} \in O_{1,\delta}\left(\dfrac{F^{(p)}}{\mathbf{H}}\right)$.*

Indeed,

$$\frac{F^{(p)}}{\mathbf{H}} = e^{sh} + \sum_{\lambda_{(n)} > h} \frac{\Lambda_{(n)}}{\mathbf{H}} f_{(n)} \exp\{(\lambda_{(n)}, s)\} \in D.$$

Therefore, $\dfrac{G^{(p)}}{\mathbf{H}} \in O_{1,\delta}\left(\dfrac{F^{(p)}}{\mathbf{H}}\right)$ if and only if $\sum\limits_{\lambda_{(n)} > h} \Lambda_{(n)} \left| \dfrac{\Lambda_{(n)}}{\mathbf{H}} f_{(n)} - \dfrac{\Lambda_{(n)}}{\mathbf{H}} g_{(n)} \right| \leq \delta$, i. e. $G \in O_{2,\mathbf{H}\delta}(F)$.
At first, we consider the case when $F(s) = E(s) := e^{(h,s)}$ and we prove such theorem.

Theorem 3.2. *For the function $E(s) = e^{(h,s)}$ the following correlations are correct: $O_{1,\mathbf{H}}(E) \subset PSD_{0,1}$, $O_{1,\mathbf{H}}^*(E) = PSD_{0,1} \bigcap D^*$, $O_{2,\mathbf{H}^2}(E) \subset PCD_{0,1}$ and $O_{2,\mathbf{H}^2}^*(E) = PCD_{0,1} \bigcap D^*$.*

Proof. If $G \in O_{1,\mathbf{H}}(E)$ then $G \in D$ and $\sum\limits_{\lambda_{(n)} > h} \Lambda_{(n)} |g_{(n)}| \leq \mathbf{H}$. Since

$$G^{(p)}(s) = \mathbf{H} e^{(s,h)} + \sum_{\lambda_{(n)} > h} \Lambda(n) g_{(n)} \exp\{(\lambda_{(n)}, s)\},$$

we have

$$\begin{aligned}
\left| \frac{G^{(p)}(s)}{\mathbf{H}} - G(s) \right| &= \left| \sum_{\lambda_{(n)} > h} \frac{\Lambda_{(n)}}{\mathbf{H}} g_{(n)} \exp\{(\lambda_{(n)}, s)\} - \sum_{\lambda_{(n)} > h} g_{(n)} \exp\{(\lambda_{(n)}, s)\} \right| \\
&= \left| \sum_{\lambda_{(n)} > h} \left(\frac{\Lambda_{(n)}}{\mathbf{H}} - 1 \right) g_{(n)} \exp\{(\lambda_{(n)}, s)\} \right| \\
&\leq \sum_{\lambda_{(n)} > h} \left(\frac{\Lambda_{(n)}}{\mathbf{H}} - 1 \right) |g_{(n)}| \exp\{\lambda_{(n)}, \sigma)\} \\
&= \sum_{\lambda_{(n)} > h} \frac{\Lambda_{(n)}}{\mathbf{H}} |g_{(n)}| \exp\{\lambda_{(n)}, \sigma)\} - \sum_{\lambda_{(n)} > h} |g_{(n)}| \exp\{\lambda_{(n)}, \sigma)\} \\
&\leq \exp\{h, \sigma)\} \sum_{\lambda_{(n)} > h} \frac{\Lambda_{(n)}}{\mathbf{H}} |g_{(n)}| - \sum_{\lambda_{(n)} > h} |g_{(n)}| \exp\{\lambda_{(n)}, \sigma)\} \\
&\leq \exp\{h, \sigma)\} - \sum_{\lambda_{(n)} > h} |g_{(n)}| \exp\{\lambda_{(n)}, \sigma)\}.
\end{aligned}$$

On the other hand,

$$|G(s)| = \left| e^{(h,s)} + \sum_{\lambda_{(n)} > h} |g_{(n)}| \exp\{\lambda_{(n)}, \sigma)\} \right| \geq e^{(h,\sigma)} - \sum_{\lambda_{(n)} > h} |g_{(n)}| e^{(\lambda_{(n)}, \sigma)}$$

and thus, $\left| \dfrac{G^{(p)}(s)}{\mathbf{H}} - G(s) \right| \leq |G(s)|$, i. e. $\left| \dfrac{G^{(p)}(s)}{\mathbf{H}G(s)} - 1 \right| \leq 1$ for all $s \in \Pi_0^p$. From hence it follows that $\operatorname{Re}\left\{ \dfrac{G^{(p)}(s)}{\mathbf{H}G(s)} \right\} > 0$, i. e. $G \in PSD_{0,1}$ and $O_{1,\mathbf{H}}(E) \subset PSD_{0,1}$.

From above it follows that $O^*_{1,\mathbf{H}}(E) \subset PSD_{0,1}$. On the other hand, if $G \in D$ and $G \in PSD_{0,1}$ then by Corollary 2 $\sum_{\lambda_{(n)}>h} \Lambda_{(n)}|g_{(n)}| \leq \mathbf{H}$, i. e. $G \in O^*_{1,\mathbf{H}}(E)$. Thus, $PSD_{0,1} \bigcap D \subset O^*_{1,\mathbf{H}}(E)$ and $PSD_{0,1} \bigcap D^* = O^*_{1,\mathbf{H}}(E)$.
Since $G \in PCD_{0,1}$ if and only if $G^{(p)}/\mathbf{H} \in PSD_{0,1}$, and by Lemma 3.1 $G \in O_{2,\mathbf{H}\delta}(E)$ if and only if $G^{(p)}/\mathbf{H} \in O_{1,\delta}(E/\mathbf{H}) = O_{1,\delta}(E)$, one can easily obtain the corresponding results for pseudoconvex functions.
For example, if $G \in O_{2,\mathbf{H}^2}(E)$ then $G^{(p)}/\mathbf{H} \in O_{1,\mathbf{H}}(E)$ and, thus, $G^{(p)}/\mathbf{H} \in PSD_{0,1}$ and $G^{(p)} \in PSD_{0,1}$. Therefore, $O_{2,\mathbf{H}^2}(E) \subset PCD_{0,1}$.
The proof of Theorem 3.2 is completed. □

Now we investigate the neighborhoods of a pseudostarlike function of the order α. The following theorem is true.

Theorem 3.3. *Let $0 \leq \alpha_1 < \alpha < \mathbf{H}$, $\Lambda = \min\{\Lambda_{(n)} : \lambda_{(n)} > h\}$, $\delta_1 = (\alpha - \alpha_1)\dfrac{\Lambda - \mathbf{H}}{\Lambda - \alpha}$, $\delta_2 = \Lambda\left(\dfrac{\mathbf{H} - \alpha}{\Lambda - \alpha} + \dfrac{\mathbf{H} - \alpha_1}{\Lambda - \alpha_1}\right)$ and $F \in D^* \bigcap PSD_{\alpha,1}$. Then $O^*_{1,\delta_1}(F) \subset PSD_{\alpha_1,1}$ and $D^* \bigcap PSD_{\alpha_1,1} \subset O^*_{1,\delta_2}(F)$, $O^*_{2,\mathbf{H}\delta_1}(F) \subset PSD_{\alpha_1,1}$ and $D^* \bigcap PSD_{\alpha_1,1} \subset O^*_{1,\mathbf{H}\delta_2}(F)$.*

Proof. Let $G \in O^*_{1,\delta_1}(F)$. Since $F \in PSD_{\alpha,1}$, by Corollary 2.3 condition (2.6) holds. Therefore, for $\alpha_1 < \alpha$ we get

$$\begin{aligned}
\sum_{\lambda_{(n)}>h} \{\Lambda_{(n)} - \alpha_1\}|g_{(n)}| &\leq \sum_{\lambda_{(n)}>h} \{\Lambda_{(n)} - \alpha_1\}|g_{(n)} - f_{(n)}| + \sum_{\lambda_{(n)}>h} \{\Lambda_{(n)} - \alpha_1\}|f_{(n)}| \\
&= \sum_{\lambda_{(n)}>h} \{\Lambda_{(n)} - \alpha_1\}|g_{(n)} - f_{(n)}| + \sum_{\lambda_{(n)}>h} \{\Lambda_{(n)} - \alpha\}|f_{(n)}| + (\alpha - \alpha_1)\sum_{\lambda_{(n)}>h} |f_{(n)}| \\
&\leq \delta_1 + \mathbf{H} - \alpha + (\alpha - \alpha_1)\sum_{\lambda_{(n)}>h} |f_{(n)}|.
\end{aligned}$$

But in view of (2.6)

$$\sum_{\lambda_{(n)}>h} |f_{(n)}| \leq \sum_{\lambda_{(n)}>h} \frac{\Lambda_{(n)} - \alpha}{\Lambda - \alpha}|f_{(n)}| \leq \frac{\mathbf{H} - \alpha}{\Lambda - \alpha}.$$

Therefore,

$$\sum_{\lambda_{(n)}>h} \{\Lambda_{(n)} - \alpha_1\}|g_{(n)}| \leq \delta_1 + \mathbf{H} - \alpha + (\alpha - \alpha_1)\frac{\mathbf{H} - \alpha}{\Lambda - \alpha} \leq \mathbf{H} - \alpha_1$$

i. e. by Corollary 2.3 the function $G \in PSD_{\alpha_1,1}$ and, thus, $O^*_{1,\delta_1}(F) \subset PSD_{\alpha_1,1}$.
Now suppose that $G \in D^*_0 \bigcap PSD_{\alpha_1,1}$. Then in view of (2.6) we have

$$\begin{aligned}
\sum_{\lambda_{(n)}>h} \Lambda_{(n)}|g_{(n)} - f_{(n)}| &= \sum_{\lambda_{(n)}>h} \frac{\Lambda_{(n)}}{\Lambda_{(n)} - \alpha_1}(\Lambda_{(n)} - \alpha_1)|g_{(n)} - f_{(n)}| \\
&\leq \frac{\Lambda}{\Lambda - \alpha_1}\sum_{\lambda_{(n)}>h} (\Lambda_{(n)} - \alpha_1)|g_{(n)} - f_{(n)}| \\
&\leq \frac{\Lambda}{\Lambda - \alpha_1}\left(\sum_{\lambda_{(n)}>h} (\Lambda_{(n)} - \alpha_1)|g_{(n)}| + \sum_{\lambda_{(n)}>h} \frac{\Lambda_{(n)} - \alpha_1}{\Lambda_{(n)} - \alpha}(\Lambda_{(n)} - \alpha)|f_{(n)}|\right) \\
&\leq \frac{\Lambda}{\Lambda - \alpha_1}\left(\mathbf{H} - \alpha_1 + \frac{\Lambda - \alpha_1}{\Lambda - \alpha}(\mathbf{H} - \alpha)\right) = \delta_2,
\end{aligned}$$

i. e. $G \in O^*_{1,\delta_2}(F)$ and, thus, $D^* \bigcap PSD_{\alpha_1,1} \subset O^*_{1,\delta_2}(F)$.
Finally, since $F \in PCD_{0,1}$ if and only if $\dfrac{F^{(p)}}{\mathbf{H}} \in PSD_{0,1}$, and by Lemma 1 $G \in O_{2,\mathbf{H}\delta}(F)$ if and only if $\dfrac{G^{(p)}}{\mathbf{H}} \in O_{1,\delta}\left(\dfrac{F^{(p)}}{\mathbf{H}}\right)$, one can easily obtain the corresponding results for pseudoconvex functions. For example, if $G \in O_{2,\mathbf{H}\delta_1}(F)$ then $\dfrac{G^{(p)}}{\mathbf{H}} \in O_{1,\delta_1}\left(\dfrac{F^{(p)}}{\mathbf{H}}\right)$ and, thus, $\dfrac{G^{(p)}}{\mathbf{H}} \in PSD_{\alpha_1,1}$ and $G \in PCD_{\alpha_1,1}$. Therefore, $O_{2,\mathbf{H}\delta_1}(F) \subset PCD_{\alpha_1,1}$. The proof of Theorem 3.3 is completed. □

Finally, we consider the generalized case when the function F is a pseudostarlike in joint variables of the order α and the type β. The following theorem is true.

Theorem 3.4. *Let $0 \leq \alpha < \mathbf{H}$, $0 < \beta < \beta_1 \leq 1$,*

$$Q = \frac{(1+\beta_1)\Lambda - 2\beta_1\alpha - \mathbf{H}(1-\beta_1)}{(1+\beta)\Lambda - 2\beta\alpha - \mathbf{H}(1-\beta)}, \quad \delta_1 = \frac{2(\mathbf{H} - \alpha)(\beta_1 - Q\beta)}{1+\beta_1}$$

$$\delta_2 = \frac{2\beta_1(\mathbf{H} - \alpha)\Lambda}{(1+\beta_1)\Lambda - \mathbf{H}(1-\beta_1) - 2\alpha\beta_1} + \frac{2\beta(\mathbf{H} - \alpha)\Lambda}{(1+\beta)\Lambda - \mathbf{H}(1-\beta) - 2\alpha\beta},$$

and

$$F \in D^* \bigcap PSD_{\alpha,\beta}.$$

Then, $O^*_{1,\delta_1}(F) \subset PSD_{\alpha,\beta_1}$, and $D^* \bigcap PSD_{\alpha,\beta_1} \subset O^*_{1,\delta_2}(F)$, $O^*_{2,\mathbf{H}\delta_1}(F) \subset PSD_{\alpha,\beta_1}$ and $D^* \bigcap PSD_{\alpha,\beta_1} \subset O^*_{2,\mathbf{H}\delta_2}(F)$.

Proof. At first we remark that in view of the conditions $0 \le \alpha < \mathbf{H}$ and $0 < \beta < \beta_1 \le 1$

$$\max_{\lambda_{(n)}>h} \frac{(1+\beta_1)\Lambda_{(n)} - 2\beta_1\alpha - \mathbf{H}(1-\beta_1)}{(1+\beta)\Lambda_{(n)} - 2\beta\alpha - \mathbf{H}(1-\beta)} = Q$$

and $\beta_1 - Q\beta > 0$. For $0 < \beta < \beta_1 \le 1$, we have

$$\begin{aligned} \sum_{\lambda_{(n)}>h} \{(1+\beta_1)\Lambda_{(n)} - 2\beta\alpha - \mathbf{H}(1-\beta_1)\}|g_{(n)}| &\le \sum_{\lambda_{(n)}>h} \{(1+\beta_1)\Lambda_{(n)} - 2\beta_1\alpha - \mathbf{H}(1-\beta_1)\}|g_{(n)} - f_{(n)}| \\ &+ \sum_{\lambda_{(n)}>h} \{(1+\beta_1)\Lambda_{(n)} - 2\beta_1\alpha - \mathbf{H}(1-\beta_1)\}|f_{(n)}| \end{aligned} \tag{3.1}$$

If $G \in O^*_{1,\delta_1}(F)$, then

$$\sum_{\lambda_{(n)}>h} \{(1+\beta_1)\Lambda_{(n)} - 2\beta_1\alpha - \mathbf{H}(1-\beta_1)\}|g_{(n)} - f_{(n)}| \le (1+\beta_1) \sum_{\lambda_{(n)}>h} \Lambda_{(n)}|g_{(n)} - f_{(n)}| \le (1+\beta_1)\delta_1,$$

and, since $F \in D^* \bigcap PSD_{\alpha,\beta}$, by Theorem 2.1

$$\begin{aligned} \sum_{\lambda_{(n)}>h} \{(1+\beta_1)\Lambda_{(n)} - 2\beta_1\alpha - \mathbf{H}(1-\beta_1)\}|f_{(n)}| &= \sum_{\lambda_{(n)}>h} \frac{(1+\beta_1)\Lambda_{(n)} - 2\beta_1\alpha - \mathbf{H}(1-\beta_1)}{(1+\beta)\Lambda_{(n)} - 2\beta\alpha - \mathbf{H}(1-\beta)} \{(1+\beta)\Lambda_{(n)} - 2\beta\alpha - \mathbf{H}(1-\beta)\}|f_{(n)}| \\ &\le Q \sum_{\lambda_{(n)}>h} \{(1+\beta)\Lambda_{(n)} - 2\beta\alpha - \mathbf{H}(1-\beta)\}|f_{(n)}| \le 2Q\beta(\mathbf{H}-\alpha). \end{aligned}$$

Therefore, (3.1) implies

$$\sum_{\lambda_{(n)}>h} \{(1+\beta_1)\Lambda_{(n)} - 2\beta\alpha - \mathbf{H}(1-\beta_1)\}|g_{(n)}| \le (1+\beta_1)\delta_1 + 2Q\beta(\mathbf{H}-\alpha) = 2\beta_1(\mathbf{H}-\alpha),$$

i. e. by Theorem 2.2 $G \in PSD_{\alpha,\beta_1}$. Theorem 3.4 is proved. □

4. Hadamard Compositions of Multiple Dirichlet Series

For Dirichlet series $F_j(s) = e^{(s,h)} + \sum\limits_{\lambda_{(n)}>h} f_{(n),j} \exp\{(\lambda_{(n)}, s)\}$ $(j = 1,2)$ the Hadamard composition has the form

$$(F_1 * F_2(s) = e^{(s,h)} + \sum_{\lambda_{(n)}>h} f_{(n),1} f_{(n),2} \exp\{(\lambda_{(n)}, s)\}. \tag{4.1}$$

Theorem 2.1 and Corollary 2.3 imply the following statements.

Corollary 4.1. *If the functions $F_j \in D^*$ are pseudostarlike of the orders $\alpha_j \in [0,\mathbf{H})$ then Hadamard composition $F_1 * F_2$ is pseudostarlike of the order $\alpha = \max\{\alpha_1, \alpha_2\}$.*

If the functions $F_j \in D^*$ are pseudoconvex of the orders $\alpha_j \in [0,\mathbf{H})$ then Hadamard composition $F_1 * F_2$ is pseudoconvex of the order $\alpha = \max\{\alpha_1, \alpha_2\}$.
Indeed, since $F_j \in D^*$ that is $f_{(n),j} \le 0$ for all n and j, from (2.6) it follows that $|f_{(n),j}| \le (\mathbf{H}-\alpha_j)/(\Lambda_{(n)} - \alpha_j) < 1$ for $\lambda_{(n)} > h$ and therefore,

$$\sum_{\lambda_{(n)}>h} \frac{\Lambda_{(n)} - \alpha_1}{\mathbf{H} - \alpha_1} |f_{(n),1} f_{(n),2}| \le \sum_{k=1}^{\infty} \frac{\Lambda_{(n)} - \alpha_1}{\mathbf{H} - \alpha_1} |f_{(n),k}| \le 1$$

for each $k = 1$ and $k = 2$, i. e. the function $F_1 * F_2$ is pseudostarlike of the order α_1 and of the order α_2, and thus, $F_1 * F_2$ is pseudostarlike of the order $\alpha = \max\{\alpha_1, \alpha_2\}$.
The proof of the pseudoconvexity of $F_1 * F_2$ is similar.

Corollary 4.2. *If the functions $F_j \in D^*$ are pseudostarlike of the order $\alpha \in [0,\mathbf{H})$ and the type $\beta_j > 0$ then Hadamard composition $F_1 * F_2$ is pseudostarlike of the order α and the type $\beta = \min\{\beta_1, \beta_2\}$.*

If the functions $F_j \in D^*$ are pseudoconvex of the order $\alpha \in [0,\mathbf{H})$ and the type $\beta_j > 0$ then Hadamard composition $F_1 * F_2$ is pseudoconvex of the order α and the type $\beta = \min\{\beta_1, \beta_2\}$.
Indeed, from (2.3) it follows that

$$|f_{(n),j}| \le \frac{2\beta_j(\mathbf{H}-\alpha)}{(1+\beta_j)\Lambda_{(n)} - (1-\beta_j)\mathbf{H} - 2\beta_j\alpha} < 1$$

for $\lambda_{(n)} > h$ and therefore, as above we have

$$\sum_{\lambda_{(n)}>h} \frac{(1+\beta_j)\Lambda_{(n)} - (1-\beta_j)\mathbf{H} - 2\beta_j\alpha}{2\beta_j(\mathbf{H}-\alpha)} |f_{(n),1}||f_{(n),2}| \le 1, \quad j = 1,2.$$

Hence it follows that $F_1 * F_2$ is pseudostarlike of the order α and the type β_j for each j and thus, $F_1 * F_2$ is pseudostarlike of the order α and the type $\beta = \min\{\beta_1, \beta_2\}$.
The proof of the pseudoconvexity of $F_1 * F_2$ is similar.

5. Differential Equation

Here we consider a differential equation

$$\frac{\partial^p w}{\partial s_1,\dots,\partial s_p}+(\gamma_0 e^{2(s,h)}+\gamma_1 e^{(s,h)}+\gamma_2)w=0 \tag{5.1}$$

and we will find out at what conditions on the parameters γ_0, γ_1, γ_2 this equation has solution (2.1) pseudostarlike in joint variables of the order $\alpha\in[0,\mathbf{H})$ and the type $\beta>0$.
We will look for a solution to the equation in the form

$$F(s)=e^{(s,h)}+\sum_{n=1}^{\infty} f_{(\mathbf{n})}\exp\{((n+1)h,s)\}=e^{(s,h)}+\sum_{n=1}^{\infty} f_{(\mathbf{n})}\exp\{(n+1)(h,s)\}, \tag{5.2}$$

where $(\mathbf{n})=(n,\dots,n)$ (p times) and $\lambda_{(n)}=(n+1)h=((n+1)h_1,\dots,(n+1)h_p)$. Since

$$F^{(p)}(s)=\mathbf{H}e^{(s,h)}+\sum_{n=1}^{\infty}(n+1)^p\mathbf{H}f_{(\mathbf{n})}\exp\{(n+1)(h,s)\},$$

we have

$$\mathbf{H}e^{(s,h)}+\sum_{n=1}^{\infty}(n+1)^p\mathbf{H}f_{(\mathbf{n})}\exp\{(n+1)(h,s)\}+\gamma_0 e^{3(s,h)}+\gamma_0\sum_{n=1}^{\infty} f_{(\mathbf{n})}\exp\{(n+3)(h,s)\}+\gamma_1 e^{2(s,h)}$$
$$+\gamma_1\sum_{n=1}^{\infty} f_{(\mathbf{n})}\exp\{(n+2)(h,s)\}+\gamma_2 e^{(s,h)}+\gamma_2\sum_{n=1}^{\infty} f_{(\mathbf{n})}\exp\{(n+1)(h,s)\}\equiv 0,$$

i. e.

$$(\mathbf{H}+\gamma_2)e^{(s,h)}+\gamma_1 e^{2(s,h)}+\gamma_0 e^{3(s,h)}+(2^p\mathbf{H}+\gamma_2)f_{(\mathbf{1})}e^{2(s,h)}+(3^p\mathbf{H}+\gamma_2)f_{(\mathbf{2})}e^{3(s,h)}+\gamma_1 f_{(\mathbf{1})}e^{3(s,h)}+\sum_{n=3}^{\infty}\{((n+1)^p\mathbf{H}+\gamma_2)f_{(\mathbf{n})}$$
$$\geq\mathbf{3}+\gamma_1 f_{(\mathbf{n-1})}+\gamma_0 f_{(\mathbf{n-2})}\}\exp\{(n+1)(h,s)\}\equiv 0.$$

Hence it follows that

$$\mathbf{H}+\gamma_2=0,\quad (2^p\mathbf{H}+\gamma_2)f_{(\mathbf{1})}+\gamma_1=0,\quad (3^p\mathbf{H}+\gamma_2)f_{(\mathbf{2})}+\gamma_1 f_{(\mathbf{1})}+\gamma_0=0$$

and

$$((n+1)^p\mathbf{H}+\gamma_2)f_{(\mathbf{n})}+\gamma_1 f_{(\mathbf{n-1})}+\gamma_0 f_{(\mathbf{n-2})}\}\exp\{(n+1)(h,s),\quad n\geq 3.$$

Therefore, the following lemma is correct.

Lemma 5.1. *Function (5.2) satisfies differential equation (5.1) if and only if*

$$\gamma_2=-\mathbf{H},\quad f_{(\mathbf{1})}=-\frac{\gamma_1}{(2^p-1)\mathbf{H}},\quad f_{(\mathbf{2})}=-\frac{\gamma_1 f_{((\mathbf{1})}+\gamma_0}{(3^p-1)-1)\mathbf{H}}$$

and

$$f_{(\mathbf{n})}=-\frac{\gamma_1 f_{((\mathbf{n-1}))}+\gamma_0 f_{((\mathbf{n-2}))}}{((n+1)^p-1)\mathbf{H}}\quad (n\geq 3). \tag{5.3}$$

Using Lemma 5.1 now we prove the following theorem.

Theorem 5.2. *Let $\alpha\in[0,\mathbf{H})$, $\beta>0$, $\gamma_2=-\mathbf{H}$ and $|\gamma_0|+|\gamma_1|\leq\dfrac{2\beta}{1+\beta}(\mathbf{H}-\alpha)$. Then differential equation (5.1) has entire solution (5.2) pseudostarlike in joint variables of the order α and the type β.*

Proof. Recall that the function (2.1) is called pseudostarlike in joint variables of the order $\alpha\in[0,\mathbf{H})$ and type $\beta>0$ if

$$|F^{(p)}(s)/F(s)-\mathbf{H}|<\beta|F^{(p)}(s)/F(s)-(2\alpha-\mathbf{H})|.$$

Also, we remark that function F is a solution of differential equation (5.1) if and only if

$$F^{(p)}(s)+(\gamma_0 e^{2(s,h)}+\gamma_1 e^{(s,h)}+\gamma_2)F(s)\equiv 0.$$

Hence it follows that F is pseudostarlike in joint variables of the order $\alpha\in[0,\mathbf{H})$ and the type $\beta>0$ if and only if

$$|-(\gamma_0 e^{2(s,h)}+\gamma_1 e^{(s,h)}+\gamma_2)-\mathbf{H}|<\beta|-(\gamma_0 e^{2(s,h)}+\gamma_1 e^{(s,h)}+\gamma_2)-(2\alpha-\mathbf{H})|,\quad s\in\Pi_0^p,$$

i. e.

$$|\gamma_0 e^{2(s,h)}+\gamma_1 e^{(s,h)}|<\beta|\gamma_0 e^{2(s,h)}+\gamma_1 e^{(s,h)}-2(\mathbf{H}-\alpha)|,\quad s\in\Pi_0^p,$$

and thus,

$$\frac{1}{\beta} < \left| 1 - \frac{2(\mathbf{H}-\alpha)}{\gamma_0 e^{2(s,h)} + \gamma_1 e^{(s,h)}} \right|, \quad s \in \Pi_0^p.$$

Since $|w-1| \geq |w|-1$, this inequality holds if

$$\frac{2(\mathbf{H}-\alpha)}{|\gamma_0 e^{2(s,h)} + \gamma_1 e^{(s,h)}|} > 1 + \frac{1}{\beta},$$

i. e. if

$$|\gamma_0 e^{2(s,h)} + \gamma_1 e^{(s,h)}| < \frac{2\beta(\mathbf{H}-\alpha)}{1+\beta}, \quad s \in \Pi_0^p.$$

The last condition holds because

$$|\gamma_0 e^{2(s,h)} + \gamma_1 e^{(s,h)}| \leq |\gamma_0| e^{2(\sigma,h)} + |\gamma_1| e^{(\sigma,h)} < |\gamma_0| + |\gamma_1| \leq \frac{2\beta}{1+\beta}(\mathbf{H}-\alpha)$$

and thus, function (5.2) is pseudostarlike in joint variables of the order α and the type β.
Finally, since for every $\sigma \in \mathbb{R}^p$ there exits $n_0 = n_0(\sigma) \geq 1$ such that

$$\frac{(|\gamma_0| + |\gamma_1|)\exp\{2(h,\sigma)\}}{n^p \mathbf{H}} \leq \frac{1}{2},$$

in view of (5.3) we have

$$\begin{aligned}
\sum_{n=n_0}^{\infty} |f_{(\mathbf{n})}| \exp\{(n+1)(\sigma,h)\} &\leq \sum_{n=n_0}^{\infty} \frac{|\gamma_1| |f_{(\mathbf{n-1})}|}{((n+1)^p - 1)\mathbf{H}} e^{n(\sigma,h)} e^{(\sigma,h)} + \sum_{n=n_0}^{\infty} \frac{|\gamma_0| |f_{(\mathbf{n-2})}|}{((n+1)^p - 1)\mathbf{H}} e^{(n-1)(\sigma,h)} e^{2(\sigma,h)} \\
&= \sum_{n=n_0-1}^{\infty} \frac{|\gamma_1| |f_{(\mathbf{n})}|}{((n+2)^p - 1)\mathbf{H}} e^{(n+1)(\sigma,h)} e^{(\sigma,h)} + \sum_{n=n_0-2}^{\infty} \frac{|\gamma_0| |f_{(\mathbf{n})}|}{((n+3)^p - 1)\mathbf{H}} e^{(n+1)(\sigma,h)} e^{2(\sigma,h)} \\
&\leq \sum_{n=n_0}^{\infty} \frac{(|\gamma_1| + |\gamma_0|) e^{2(\sigma,h)}}{n^p \mathbf{H}} |f_{(\mathbf{n})}| e^{(n+1)(\sigma,h)} + \frac{|\gamma_1| |f_{(\mathbf{n_0-1})}|}{((n_0+1)^p - 1)\mathbf{H}} e^{(n_0+1)(\sigma,h)} \\
&\quad + \frac{|\gamma_0| |f_{(\mathbf{n_0-2})}|}{((n_0+1)^p - 1)\mathbf{H}} e^{(n_0+1)(\sigma,h)} + \frac{|\gamma_0| |f_{(\mathbf{n_0-1})}|}{((n+2)^p - 1)\mathbf{H}} e^{(n_0+2)(\sigma,h)} \\
&\leq \frac{1}{2} \sum_{n=n_0}^{\infty} |f_{(\mathbf{n})}| \exp\{(n+1)(\sigma,h)\} + \text{const},
\end{aligned}$$

i. e. Dirichlet series (5.2) is entire (absolutely convergent in $\mathbb{C}^p$). The proof of Theorem 5.2 is completed. □

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References

[1] G. M. Golusin, *Geometrical theory of functions of complex variables*, M. Nauka, 1966. (in Russian); Engl. Transl.: AMS: Translations of Mathematical Monograph, **26** (1969).
[2] A. W. Goodman, *Univalent functions and nonanalytic curves*, Proc. Amer. Math. Soc., **8**(3) (1957), 597–601.
[3] M. M.Sheremeta, *Geometric Properties of Analytic Solutions of Differential Equations*, Lviv: Publisher I. E. Chyzhykov, 2019.
[4] I. S. Jack, *Functions starlike and convex of order α*, J. London Math. Soc., **3** (1971), 469–474.
[5] V. P. Gupta, *Convex class of starlike functions*, Yokohama Math. J., **32** (1984), 55–59.
[6] A. W. Goodman, *Univalent functions and nonanalytic curves*, Proc. Amer. Math. Soc., **8** (1957), 598–601.
[7] S. Ruscheweyh, *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc., **81**(4) (1981), 521–527.
[8] R. Fournier, *A note on neighborhoods of univalent functions*, Proc. Amer. Math. Soc., **87**(1) (1983), 117–121.
[9] H. Silverman, *Neighborhoods of a class of analytic functions*, Far East J. Math. Sci., **3**(2) (1995), 165–169.

[10] O. Altıntaş, *Neighborhoods of certain analytic functions with negative coefficients*, Int. J. Math. and Math. Sci. **13**(4) (1996), 210–219.
[11] O. Altıntaş, O. Özkan, H. M.Srivastava, *Neighborhoods of a class of analytic functions with negative coefficients*, Applied Math. Lettr., **13** (2000), 63–67.
[12] B. A. Frasin, M. Daras, *Integral means and neighborhoods for analytic functions with negative coefficients*, Soochow Journal Math., **30**(2) (2004), 217–223.
[13] G. Murugusundaramoorthy, H. M. Srivastava, *Neighborhoods of certain classes of analytic functions of complex order*, J. Inequal. Pure and Appl. Math., **5**(2) (2004), Article 24.
[14] M. N. Pascu, N. R. Pascu, *Neighborhoods of univalent functions*, Bull. Amer. Math. Soc., **83** (2011), 510–219.
[15] J. Hadamard, *Theoreme sur le series entieres*, Acta Math., **22** (1899), 55–63.
[16] J. Hadamard, *La serie de Taylor et son prolongement analitique*, Scientia Phys. Math., **12** (1901), 43–62.
[17] L. Bieberbach, *Analytische Fortzetzung*, Berlin, 1955.
[18] Yu. F. Korobeinik, N. N. Mavrodi, *Singular points of the Hadamard composition*, Ukr. Math. J., **42**(12) (1990), 1711–1713. (in Russian); Engl. Transl.: Ukr. Math. J., **42**(12) (1990), 1545–1547.
[19] L. Zalzman, *Hadamard product of shlicht functions*, Proc. Amer. Math. Soc., **19**(3) (1968), 544–548.
[20] M. L. Mogra, *Hadamard product of certain meromorphic univalent functions*, J. Math. Anal. Appl., **157** (1991), 10–16.
[21] J. H. Choi, Y. C. Kim, S. Owa, *Generalizations of Hadamard products of functions with negative coefficients*, J. Math. Anal. Appl., **199** (1996), 495–501.
[22] M. K. Aouf, H. Silverman, *Generalizations of Hadamard products of meromorphic univalent functions with positive coefficients*, Demonstr. Math. **51**(2) (2008), 381–388.
[23] J. Liu, P. Srivastava, *Hadamard products of certain classes of p-valent starlike functions*, RACSM. **113** (2019), 2001–205.
[24] O. M. Holovata, O. M. Mulyava, M. M. Sheremeta, *Pseudostarlike, pseudoconvex and close-to-pseudoconvex Dirichlet series satisfying differential equations with exponential coefficients*, Mat. Methods and Fiz.-Mech. Polya., **61**(1) (2018), 57–70.
[25] M. M. Sheremeta, *Pseudostarlike and pseudoconvex Dirichlet series of the order α and the type β*, Mat. Stud. **54**(1) (2020), 23–31.
[26] S. M. Shah, *Univalence of a function f and its successive derivatives when f satisfies a differential equation, II*, J. Math. Anal. and Appl., **142** (1989), 422–430.
[27] Z. M. Sheremeta, *On entire solutions of a differential equation*, Mat. Stud., **14**(1) (2000), 54–58.
[28] Z. M. Sheremeta, M. M. Sheremeta, *Convexity of entire solutions of a differential equation*, Mat. Methods and Fiz.-Mech. Polya., **47**(2) (2004), 181–185.
[29] M. M. Sheremeta, *Pseudostarlike and pseudoconvex solutions of a differential equation with exponential coefficients*, Mat. Stud., **56**(1) (2021), 39–47.
[30] M. M. Sheremeta, *On certain subclass of Dirichlet series absolutely convergent in half-plane*, Mat. Stud., **57**(1)(2022), 32–44.
[31] M. M. Sheremeta, O. B. Skaskiv, *Pseudostarlike and pseudoconvex in a direction multiple Dirichlet series*, Mat. Stud., **58**(2) (2023), 182–200.