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Research Article

A new ridge type estimator and its performance for the linear regression model: Simulation and application

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Abstract

Ridge regression is employed to address the issue of multicollinearity among independent variables. The shrinkage parameter (k) plays a key role in balancing the bias and variance tradeoff. This paper reviewed several promising existing ride regression estimators designed for estimating the ridge or shrinkage parameter k within the Gaussian linear regression model. In addition, we have proposed a new estimator (CK), which is a function of number of independent variables, sample size and standard error of regression model. The performance of our proposed estimator with OLS and existing shrinkage estimators, is compared using extensive Monte Carlo simulations in terms of minimum mean squared error (MSE). Simulation results demonstrated that the proposed CK estimator outperformed other in the majority of the considered simulation scenarios. A real-life data is analyzed to illustrate the findings of the paper.

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1. Introduction

The multiple linear regression model plays an essential role in statistical inference and is used extensively in various discipline: business, environmental, industrial, medical, and social sciences. One of the important assumptions in linear regression model is that the explanatory variables are independent. However, in practice, there may have moderate to strong linear relationships among the explanatory variables, which causes the multicollinearity problem. It is difficult to estimate the unique effects of individual variables in the regression equation when the explanatory variables are multicollinear. Moreover, the estimated regression coefficient may produce wrong sign and will have unduly large sampling variance, which affects both inference and prediction [6].

In literature, there are various methods exist to solve the multicollinearity problem. Among them, "ridge regression" proposed by [7] is the most popular one which has much usefulness in real life. They suggested ridge regression (RR) as an alternative approach

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to ordinary least squares (OLS) by including some bias in the regression model. To apply RR, a researcher needs to determine the value of the ridge or shrinkage parameter k. The choice of the optimal value of the shrinkage parameter k has an important role in reducing the value of mean squared error (MSE). Since 1970, researchers are focusing on the estimation of shrinkage parameter using different methods and various models and then comparing their results with the OLS estimator. There is much research available in literature on the estimation of the ridge parameter or shrinkage parameter for the linear regression model, see for example [1–4, 8, 9, 11, 13, 14, 20, 23], and very recently [12] among others. The literature on shrinkage parameter is colossal, but less attention is on the presence of multicollinearity when errors are no longer normally distributed. Under this condition, the OLS and ridge regression standard error becomes large. Therefore, considering the diversity in the behavior of error terms, we propose a new way of estimating shrinkage parameter. The proposed CK estimator is a novel composition of sample size, standard error of regression model, and the number of independent variables. Our proposed estimator has a more efficient performance than OLS and existing shrinkage estimators when errors are distributed as normal and non-normal.

The organization of this paper is as follows: Statistical models, different ridge parameters and proposed estimators are given in Section 2. A simulation study has been conducted in Section 3. To illustrate the merit of the proposed estimator, a real-life data are analyzed in Section 4. This paper ends up with some concluding remarks in Section 5.

2. Models and estimators

2.1. Linear model

To describe the ridge regression models, we consider the following multiple linear regression model:

$$y = X\beta + \varepsilon, \tag{2.1}$$

where y is an $(nx1)$ response vector of observations on the dependent variable, X be the (nxp) design matrix of rank p, β is an $(px1)$ vector of unknown parameters to be estimated, and ε is an $(nx1)$ vector of unobservable errors with 0 mean and variance σ^2. The regression parameters vector, is mostly estimated using the method of Least Squares (LS) when there is no violation of any of the classical linear regression Model (CLRM) assumptions [7]. The ordinary least squares (OLS) of is defined as follows:

$$\hat{\beta}_{LS} = (X'X)^{-1}X'y \tag{2.2}$$

with covariance matrix, $Cov(\hat{\beta}) = \sigma^2(X'X)^{-1}$. It can be seen that both $\hat{\beta}$ and $Cov(\hat{\beta})$ are heavily dependent on characteristics of the matrix $X'X$. The OLS estimator is an unbiased and has minimum variance among the class of all such unbiased linear estimator. In a multiple linear regression model, it is generally assumed that predictors must be uncorrelated with each other. However, in many practical situations (e.g. engineering in particular [7], often find that the regressors are nearly dependent. In that case X'X matrix becomes ill conditioned (i.e. $|X'X| \approx 0$). If $X'X$ is ill conditioned, then $\hat{\beta}$ is sensitive to a number of errors i.e. regression coefficients may have wrong signs with large, the sampling variance. In this situation, the meaningful statistical inference about the regression coefficients becomes very difficult for practitioners. To overcome this problem, Hoerl and Kennard [7] proposed ridge regression estimator (RRE) which can be obtained by augmenting Equation (2.1) with and consequently applying the method of least squares to estimate β. Thus, the ridge estimator of β is obtained as

$$\hat{\beta}_k = (X'X + kI)^{-1}X'y; \quad k \geq 0, \tag{2.3}$$

where k is the shrinkage or biasing parameter. This is known as the ridge regression estimator (RRE). The RRE provides biased but smaller variance than the OLS estimator [7]. From (2.5), we observe that as $k \to 0$, $\hat{\beta} \to \hat{\beta}_{LS}$ and as $k \to \infty$, $\hat{\beta} \to 0$. The canonical form of regression model (2.1) is defined as follows:

$$y = Z\alpha + u, \tag{2.4}$$

where $Z = XP$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p) = P'\beta$, P is an orthogonal matrix shuch that $P'P = I_p$ and $P'X'XP = \Lambda$, where $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_p)$ is the orthogonal matrix consisting of the eigen values of $X'X$. Then the OLS and the generalized ridge regression (RR) estimators in canonical form are respectively given as follows:

$$\hat{\alpha}_{LS} = (Z'Z)^{-1}Z'y, \tag{2.5}$$
$$\hat{\alpha}_{RR} = (Z'Z + K)^{-1}Z'y, \tag{2.6}$$

where $K = diag(k_1, k_2, \ldots, k_p)$, $k_i > 0$, $i = 1, 2, \ldots, p$. The MSE of the OLS and generalized RR estimator can be written respectively as follows:

$$MSE(\hat{\alpha}_{LS}) = \sigma^2 \sum_{i=1}^{p} \frac{1}{\lambda_i}, \tag{2.7}$$
$$MSE(\hat{\alpha}_{RR}) = \sigma^2 \sum_{i=1}^{p} \left[\frac{\lambda_i}{(\lambda_i + k_i)^2} + \frac{k_i \alpha_i^2}{(\lambda_i + k_i)^2} \right]. \tag{2.8}$$

The first term on the right side of (2.10) is the variance and second term is the square of the bias introduced by the RR estimator. This second term will be zero when $k = 0$, however, it is a monotonic increasing function of k. On the other hand, the variance is a monotonic decreasing function of k. Thus, when k increases, the variance decrease and bias increase. Hoerl and Kennard [7] shows that there always exists a value of $k > 0$, for which MSE of RR estimator is smaller than that of the mean square error of OLS estimator. Some existing RR estimators are given in the next section.

2.2. Different shrinkage estimators

The parameter k is known as the "biased" or "ridge" or "shrinkage" parameter and it must be estimated using real data. Most of recent efforts in the area of multicollinearity and ridge regression estimators have concentrated on estimating value of k. We will review many statistical methodology used to analyze the estimation of k in this section.

2.2.1. Existing shrinkage estimators. For the linear regression model, Hoerl and Kennard [7] obtained the optimal values of k_i as the ratio of estimated error variance $\hat{\sigma}^2$ and *ith* estimate of α using OLS as follows:

$$\hat{k}_i = \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}, \tag{2.9}$$

where $\hat{\sigma}^2 = \sum_{j=1}^{n} \hat{e}_j^2/(n-p)$. To determine a single value of k, Hoerl and Kennard [7] suggest the following estimator.

$$\hat{k}_{HK} = \frac{\hat{\sigma}^2}{\hat{\alpha}_{max}^2}, \tag{2.10}$$

where $\hat{\alpha}_{max} = \max(\hat{\alpha}_1, \ldots, \hat{\alpha}_p)$. The RR estimator in (2.10) will give smaller MSEs than the OLS estimator. Kibria [11] proposed estimators of k by using the arithmetic mean

(KAM) and geometric mean (KGM) of the values of k_i values in (2.9) as follows:

$$\hat{k}_{KAM} = \frac{\hat{\sigma}^2}{p}\sum_{i=1}^{p}\frac{1}{\hat{\alpha}_i^2},\ i = 1, 2, \ldots, p, \tag{2.11}$$

$$\hat{k}_{KGM} = \frac{\hat{\sigma}^2}{(\prod_{i=1}^{p}\hat{\alpha}_i^2)^{1/p}}. \tag{2.12}$$

Muniz and Kibria [17] introduced the following estimator.

$$\hat{k}_{KGM} = \left(\prod_{i=1}^{p}\sqrt{(\hat{\sigma}^2/\hat{\alpha}_i^2)}\right)^{1/p} \tag{2.13}$$

Khalaf et al. [10] suggested a new ridge estimator as

$$\hat{k}_{KM} = \frac{\hat{\sigma}^2\lambda_{max}}{\hat{\alpha}_{max}^2\sum_{i=1}^{p}|\hat{\alpha}_i|}, \tag{2.14}$$

where λ_{max} denotes the maximum eigen value of the $X'X$ matrix. Yasin et al. [22] proposed a modified RR estimator as follows:

$$\hat{k}_{AY} = \frac{p^2\hat{\sigma}^2}{\lambda_{max}^2\sum_{i=1}^{p}\hat{\alpha}_i^2}. \tag{2.15}$$

Dorugade [5] introduced a straightforward estimator based upon the standard error of regression model ($\hat{\sigma}$) only which is described as

$$\hat{k}_D = \hat{\sigma}, \tag{2.16}$$

where $\hat{\sigma} = (\sum_{j=1}^{n}\hat{u}_j^2/(n-p-1))^{1/2}$ and $\hat{u}_i$ denotes the OLS residuals. Following [5], Lukman and Olatunji [15] developed the following estimator.

$$\hat{k}_{AL} = p\hat{\sigma} \tag{2.17}$$

Being motivated by different researchers, Dar et al. [4] proposed a new ridge estimator based on condition index, error variance and number of independent variables as follows:

$$\hat{k}_{KCI} = \hat{\sigma}^2 p^m\left(\frac{\lambda_{max}}{\lambda_{min}}\right)^{1/2}, \tag{2.18}$$

where λ_{min} is the minimum eigen value and m is a arbitrary constant i.e. $0 \le m \le 1$. The optimal value of m is obtained by mean squared criterion (MSE). In this next section, we propose a new estimators for shrinkage parameter k. The Stein-type estimator [19] is given as follows:

$$\hat{\beta}_s = c\hat{\beta}_{LS}, \tag{2.19}$$

where the estimator of c is given as

$$\hat{c} = \frac{\hat{\beta}_{LS}'\hat{\beta}_{LS}}{\hat{\beta}_{LS}'\hat{\beta}_{LS} + \hat{\sigma}^2 tr(X'X)^{-1}}. \tag{2.20}$$

2.2.2. Proposed shrinkage estimator. Following [5] and [15], we suggest the following new ridge estimator which depends on both n and p. The proposed shrinkage estimators are given as follows:

$$\hat{k}_{CK_1} = \hat{\sigma}p^{\left(1+\frac{p}{n}\right)} \tag{2.21}$$

and

$$\hat{k}_{CK} = \hat{\sigma} \times \max\left(p^{\left(1+\frac{p}{n}\right)}, p^{\left(1+\frac{1}{p}\right)}\right). \tag{2.22}$$

The proposed estimators depend on the standard error of the regression model, ($\hat{\sigma}$), the sample size, n and the number of independent variables, p. The primary idea for the development estimators (CK_1 and CK) is based on the need of high shrinkage in the presence of multicollinearity using (i) $(1 + p/n)$ exponent term when size of sample is small and the number of dimensions (independent variables) is low and (ii) $(1 + 1/p)$ exponent term for low dimensional data respectively. In addition, for large value of p (i.e. $p \to \infty$), when the value of $p^(1 + 1/p)$ is maximum, the $\hat{k}_{CK}$ will approach to $\hat{k}_{AL}$ estimator. Also when $n \to \infty$ and p is small, the proposed $\hat{k}_{CK_1}$ estimator behave similarly as $\hat{k}_{AL}$ estimator.

2.3. Performance criterion

To assess the performance of the proposed shrinkage estimators, we will use the MSE criterion. Several research, such as [4, 11, 21], have used this criterion to evaluate the performance of estimators. The MSE is expressed as follows:

$$MSE(\hat{\alpha}) = E(\hat{\alpha} - \alpha)'(\hat{\alpha} - \alpha) \tag{2.23}$$

$$= \frac{1}{p}\sum_{i=1}^{p}(\hat{\alpha}_i - \alpha_i)^2. \tag{2.24}$$

It is difficult to compare (2.10 - 2.22) theoretically, so we will compare the estimators performance using Monte Carlo simulation in the next section.

3. Monte Carlo simulation

The process of Monte Carlo simulation is given in this section to evaluate the performance of proposed shrinkage estimators with OLS and existing shrinkage estimators.

3.1. Design of simulation

Following [16], the number of independent variables are generated using:

$$x_{ji} = \sqrt{(1-\rho^2)}z_{ji} + \rho z_{j,p+1},\ j = 1, 2, \ldots, n,\ i = 1, 2, \ldots, p, \tag{3.1}$$

where ρ^2 denotes the specified correlation between the independent variables and z_{ji} is generated from normal distribution with zero mean and unit standard deviation. The dependent variable is generated using the following equation:

$$y_j = \beta_0 + \beta_1 X_{j1} + \ldots + \beta_p X_{jp} + u_j,\ j = 1, 2, \ldots, n, \tag{3.2}$$

where n denotes the number of observations and u_j is the error term and. Assuming zero intercept i.e. $\beta_0 = 0$ in the model in (3.2) and selecting β values by considering $\beta'\beta = 1$. In this study, the correlation values considered are $\rho = 0.70, 0.80, 0.90$ and 0.99. Other factor we have considered to vary are the number of independent variables (p), error variance, sample size (n) and error distribution. The values used are: $n = 15, 30, 50, 100$, $p = 2, 5, 10$, $\sigma = 1, 2, 5$. u_i follows $N(0, \sigma)$ and χ^2_1. To compute the MSE, we used the following algorithm.

Algorithm 1: MSE Computation

Step 1 *Generate the matrix of number of independent variables matrix using Equation (3.1) and standardize it.*
Step 2 *Determine the eigenvalues* $(\lambda_1, \ldots, \lambda_p)$ *and corresponding eigenvectors* $(e_1, \ldots, e_p)$ *of the matrix* $X'X$ *given that* $\sum_{i=1}^{p}\lambda_i = p$.
Step 3 *Compute the regression coefficients in the canonical form* α *using* $\alpha = e_{max}P$ *where* $P = [e_1, \ldots, e_p]$ *and eigen vector of largest eigen value* λ_{max} *is denoted by* e_{max}.
Step 4 *Generate the random error term using the cases stated above i.e.* $N(0, 1)$, $N(0, 2)$, $N(0, 5)$ *and* χ^2_3.

Step 5 *Determine the values of dependent variable using Equation (2.6).*
Step 6 *Using Equations (2.5) and (2.6), calculate the OLS and RR estimates of α respectively.*
Step 7 *Repeat the steps (2) - (6) for each of the N Monte Carlo runs in the simulation study.*
Step 8 *Calculate the MSE of all the considered estimators using*:

$$MSE(\hat{\alpha}) = \frac{1}{N}\sum_{v=1}^{N}\sum_{i=1}^{p}(\hat{\alpha}_{vi} - \alpha_i)^2, \; i = 1, 2, \ldots, p. \tag{3.3}$$

The simulation study is based on $N = 10000$ runs. The estimated MSE values for different values of ρ, n and p is shown in Tables 1-2 and supplementary tables for different parametric conditions are presented in the Tables A.1-A.5. The simulation study is conducted using R language for evaluation of estimators. In Section 3.2, the simulation results and discussion is given.

Table 1. Estimated MSE values of the estimators with different values of σ and ρ when $u_i \sim N(0,1)$, $p = 2$.

σ	1				5				10				
ρ	0.7	0.8	0.9	0.99	0.7	0.8	0.9	0.99	0.7	0.8	0.9	0.99	Mean
							$n = 15$						
OLS	0.189	0.296	0.611	6.273	0.189	0.293	0.593	6.368	0.190	0.294	0.603	6.113	1.834
HK	0.424	0.539	0.796	1.585	0.392	0.421	0.482	1.642	0.408	0.414	0.592	0.959	0.721
KAM	0.357	0.435	0.538	0.602	0.332	0.368	0.431	0.506	0.361	0.371	0.398	0.512	0.434
KMS	0.453	0.527	0.544	2.238	0.429	0.459	0.591	1.514	0.457	0.482	0.524	1.337	0.796
KM	0.149	0.207	0.322	0.409	0.153	0.171	0.262	0.421	0.153	0.183	0.352	0.431	0.268
AY	0.168	0.204	0.528	1.637	0.152	0.176	0.356	1.707	0.152	0.238	0.365	1.749	0.619
D	0.144	0.189	0.324	0.421	0.148	0.181	0.243	0.252	0.148	0.193	0.263	0.362	0.239
AL	0.133	0.140	0.160	0.197	0.131	0.141	0.160	0.176	0.131	0.144	0.159	0.198	0.155
GM	0.215	0.218	0.285	0.960	0.220	0.242	0.286	0.982	0.220	0.224	0.284	0.903	0.420
KCI	0.171	0.155	0.141	0.137	0.154	0.146	0.141	0.076	0.158	0.148	0.141	0.078	0.137
Stein	0.996	0.925	0.846	0.799	0.958	0.955	0.947	0.887	0.958	0.955	0.946	0.861	0.919
CK	0.132	0.126	0.112	0.072	0.127	0.119	0.112	0.072	0.129	0.123	0.115	0.074	0.109
							$n = 30$						
OLS	0.173	0.221	0.368	3.245	0.178	0.223	0.365	3.343	0.178	0.226	0.365	3.384	1.022
HK	0.289	0.294	0.392	0.504	0.289	0.294	0.331	0.919	0.290	0.295	0.308	0.850	0.421
KAM	0.250	0.266	0.332	0.374	0.249	0.264	0.323	0.384	0.255	0.266	0.325	0.377	0.305
KMS	0.329	0.332	0.408	1.264	0.330	0.332	0.345	1.396	0.337	0.340	0.404	1.224	0.587
KM	0.135	0.180	0.262	0.370	0.134	0.158	0.253	0.377	0.133	0.174	0.259	0.365	0.233
AY	0.148	0.195	0.387	1.221	0.147	0.162	0.273	1.089	0.146	0.208	0.351	1.649	0.498
D	0.143	0.169	0.252	0.314	0.141	0.176	0.224	0.238	0.141	0.183	0.249	0.259	0.207
AL	0.114	0.134	0.143	0.152	0.113	0.132	0.139	0.152	0.112	0.131	0.142	0.187	0.138
GM	0.153	0.193	0.259	0.752	0.153	0.190	0.258	0.795	0.154	0.188	0.254	0.835	0.349
KCI	0.163	0.151	0.074	0.051	0.110	0.109	0.073	0.051	0.109	0.108	0.075	0.052	0.094
Stein	0.910	0.908	0.808	0.786	0.911	0.908	0.903	0.869	0.911	0.909	0.902	0.859	0.882
CK	0.097	0.092	0.051	0.045	0.097	0.086	0.071	0.036	0.110	0.105	0.102	0.036	0.077

3.2. Simulation results and discussion

Table 1 shows the simulation results for $p = 2$ and $n = \{15, 30\}$ with different values of σ and ρ when $\varepsilon_i \sim \mathcal{N}(0, 1)$. Table 2 shows the simulation results for $p = 2$ and $n = \{15, 30\}$ with different values of p and ρ when $\varepsilon_i \sim \chi^2_{(1)}$. The remaining results for different values of n with $p = \{5, 10\}$ when $\varepsilon_i \sim \mathcal{N}(0, 1)$ and $p = \{5, 10, 15\}$ when $\varepsilon_i \sim \chi^2_{(1)}$ are provided in the supplementary materials Tables (A.1 - A.5).

Table 2. Estimated MSE values of the estimators with different values of p and ρ when $\varepsilon_i \sim \chi^2_1$.

p	2				5				10			
ρ	0.7	0.8	0.9	0.99	0.7	0.8	0.9	0.99	0.7	0.8	0.9	0.99
	$n=15$											
OLS	0.419	0.576	1.069	10.720	3.523	5.134	10.117	94.692	8.983	13.701	29.560	340.044
HK	0.247	0.268	0.320	1.559	1.020	1.306	2.384	21.545	1.786	2.457	6.224	68.093
KAM	0.421	0.399	0.377	0.298	0.453	0.429	0.402	0.397	0.626	0.535	0.421	0.383
KMS	0.286	0.319	0.407	3.005	1.209	1.638	3.357	46.363	2.588	3.502	9.117	148.435
KM	0.274	0.312	0.378	0.451	0.597	0.686	0.873	1.325	0.677	0.871	1.152	2.086
AY	0.260	0.312	0.476	3.662	1.355	2.123	4.510	44.011	1.852	3.763	10.233	133.844
D	0.261	0.291	0.319	0.398	0.815	0.832	0.870	0.977	1.323	1.544	1.755	1.868
AL	0.200	0.202	0.219	0.241	0.210	0.196	0.162	0.048	0.195	0.190	0.094	0.040
GM	0.313	0.316	0.340	0.592	0.302	0.342	0.498	2.440	0.260	0.323	0.416	2.522
KCI	0.183	0.169	0.146	0.140	0.198	0.140	0.417	0.238	0.209	0.145	0.133	0.124
Stein	0.951	0.948	0.942	0.899	0.988	0.985	0.975	14.249	0.979	0.979	0.867	60.609
CK	0.178	0.171	0.150	0.086	0.158	0.143	0.112	0.058	0.348	0.285	0.232	0.195
	$n=30$											
OLS	0.288	0.397	0.687	5.072	1.435	2.219	4.734	50.634	4.632	7.491	15.554	160.090
HK	0.238	0.246	0.279	1.231	0.533	0.685	1.184	10.601	1.329	2.257	4.480	45.001
KAM	0.343	0.326	0.313	0.289	0.435	0.373	0.335	0.292	0.467	0.393	0.322	0.319
KMS	0.243	0.282	0.317	1.833	0.516	0.673	1.299	18.965	1.334	2.303	5.027	75.994
KM	0.194	0.236	0.312	0.416	0.396	0.472	0.643	1.305	0.664	0.726	1.112	2.052
AY	0.183	0.234	0.356	2.012	0.656	1.042	2.241	24.184	1.844	3.334	7.502	83.110
D	0.197	0.236	0.287	0.294	0.647	0.735	0.863	0.932	1.315	1.525	1.698	1.727
AL	0.150	0.165	0.171	0.203	0.208	0.160	0.123	0.045	0.189	0.167	0.088	0.028
GM	0.213	0.230	0.271	0.547	0.208	0.212	0.294	1.550	0.176	0.209	0.344	2.435
KCI	0.120	0.113	0.092	0.089	0.179	0.136	0.129	0.064	0.204	0.129	0.066	0.052
Stein	0.910	0.908	0.898	0.861	0.963	0.958	0.898	2.782	0.959	0.946	0.806	9.671
CK	0.134	0.128	0.128	0.061	0.139	0.113	0.109	0.034	0.098	0.079	0.056	0.025

From the simulation results, we have the following findings:

(1) When the sample size increases, the estimated Mean Squared Error (MSE) values of all estimators are likely to fall, which is also reported in the literature (see [4, 11, 20].
(2) The estimated MSE of all estimators increases with the increase in the degree of collinearity when $p = 2$, $\sigma = \{1, 2, 5\}$, and $n = \{15, 30\}$, except for the KCI, Stein, and our proposed CK estimator.
(3) The proposed CK estimator is efficient when the degree of multicollinearity (ρ) is between 0.70 and 0.99.
(4) The Stein estimator has shown the worst performance when $n = 15$, $p = 2$, $\sigma = \{1, 2, 5\}$ when the collinearity is between 0.7 and 0.99. However, for a severe degree of collinearity ($\rho = 0.99$), the OLS is the worst performer. Similar findings are obtained when $n = 30$.
(5) When $p = 2$, the proposed CK estimator has the best performance, followed by AL and KCI estimators, respectively. Similarly, in the case of other values of p, the CK, AL, and KCI are the top performers.
(6) When the errors are non-normal, the estimated MSE values of KAM, KCI, Stein, and CK estimators are likely to fall with an increase in the correlation levels among independent variables. However, the estimated MSE of OLS, HK, KMS, KM, AY, D, AL, and GM increases with an increase in ρ when $p = 2$.
(7) In most simulation conditions, the increase in the values of σ increases the estimated MSE values of the estimators, which is also a finding of [18] and [20].

In general, all estimators, except the Stein estimator, outperform the OLS estimator in the presence of multicollinearity. Our developed CK estimator exhibits the best performance among all the cases considered.

4. Application

In this section, we aim to illustrate the application of the proposed estimator and its comparison with the considered competing estimators through a real-life dataset. For this purpose, an empirical example of precipitation data (www.pmd.gov.pk) is presented. The dependent variable is precipitation (PRE), and there are three regressors: minimum temperature ($MnTMP$), maximum temperature ($MxTMP$), and wind speed (WSP). This dataset consists of 243 observations.

Consider the following linear relation:

$$PRE = \beta_0 + \beta_1 MnTMP + \beta_2 MxTMP + \beta_3 WSP + \varepsilon_j, \quad j = 1, 2, \ldots, n.$$

Figure 1 depicts the correlation among the variables of the precipitation data. It is evident from Figure 1 that a moderate to strong correlation exist among the independent variables.

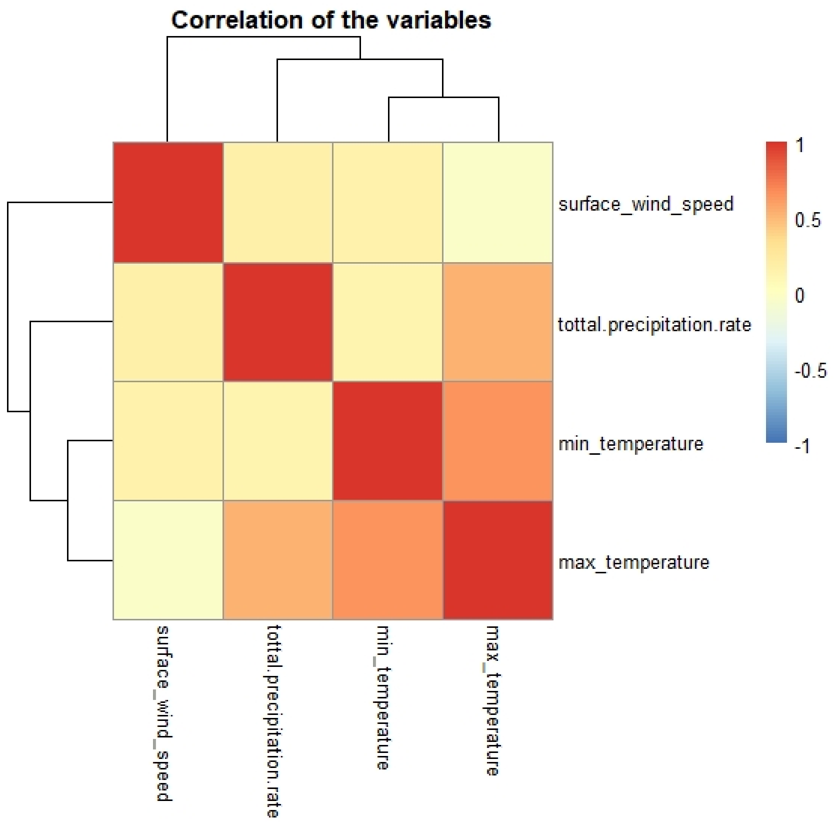

Figure 1. Correlation plot for precipitation data.

The estimated MSE values along with the estimated regression coefficients for each of the proposed and the competing estimators are presented in Table 3. It is evident from Table 3 that the proposed CK estimator has the smallest MSE. This suggests the superiority of our estimator over the considered ridge estimators.

Table 3. Regression estimates and MSE for precipitation data.

	MSE	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$
OLS	0.1953	-0.44	0.82	0.25
HK	0.1917	-0.05	0.31	0.10
AM	0.2603	-0.18	0.51	0.16
KMS	0.1913	-0.03	0.27	0.08
KM	0.1857	-0.42	0.80	0.24
AY	0.1865	-0.42	0.80	0.24
D	0.1898	-0.44	0.82	0.25
AL	0.1850	-0.43	0.81	0.24
GM	0.1929	-0.38	0.75	0.23
KCI	0.1946	-0.44	0.82	0.25
Stein	0.3902	0.00	0.00	0.00
CK	0.1848	-0.42	0.81	0.24

5. Some concluding remarks

In this study, we have developed a new shrinkage estimator to address the issue of multicollinearity in the linear regression model. The new CK estimator is a function of sample size, the number of independent variables, and the standard error of the regression model. Intensive simulations are conducted to compare and evaluate the performance of the proposed shrinkage estimator against existing estimators in scenarios involving with normal and non-normal errors associate with multicollinearity. The proposed estimator with an appropriate choice of error distribution and number of independent variables outperforms all comparative estimators. The conclusion of this paper is constrained by the limitations of the simulation study. To make definitive statements about the performance of the estimators, a more extensive simulation study encompassing a broader range of parametric conditions would be necessary.

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Appendix

A.1. Estimated MSE values of the estimators with different values of σ and ρ when $u_i \sim N(0,1)$, $p = 5$.

p	1				2				5			
ρ	0.7	0.8	0.9	0.99	0.7	0.8	0.9	0.99	0.7	0.8	0.9	0.99
						$n = 15$						
OLS	1.493	2.162	4.163	38.078	1.481	2.149	4.149	37.930	1.474	2.169	4.161	37.308
HK	0.768	0.989	1.642	12.666	0.759	0.968	1.634	12.659	0.760	0.991	1.612	12.151
KAM	0.415	0.379	0.394	1.064	0.414	0.384	0.392	1.080	0.420	0.385	0.388	1.074
KMS	0.760	0.978	1.715	18.283	0.750	0.955	1.706	18.308	0.752	0.979	1.682	17.596
KM	0.380	0.442	0.592	1.160	0.373	0.436	0.584	1.173	0.374	0.442	0.586	1.140
AY	0.804	1.224	2.478	23.330	0.792	1.208	2.463	23.214	0.793	1.226	2.466	22.633
D	0.653	0.734	0.852	0.542	0.647	0.725	0.839	0.549	0.645	0.737	0.845	0.531
AL	0.200	0.188	0.162	0.051	0.198	0.185	0.159	0.051	0.197	0.188	0.160	0.051
GM	0.274	0.313	0.484	2.766	0.267	0.306	0.479	2.807	0.271	0.311	0.480	2.726
KCI	0.327	0.262	0.163	0.064	0.323	0.256	0.160	0.064	0.323	0.262	0.161	0.066
Stein	0.933	0.926	0.946	9.513	0.933	0.927	0.945	9.442	0.933	0.926	0.947	9.238
CK	0.135	0.120	0.095	0.037	0.134	0.118	0.094	0.037	0.134	0.120	0.095	0.037
						$n = 30$						
OLS	0.631	0.952	1.868	18.091	0.629	0.939	1.866	18.438	0.632	0.951	1.866	17.944
HK	0.430	0.404	0.883	5.860	0.430	0.397	0.884	6.118	0.432	0.405	0.889	5.876
KAM	0.357	0.317	0.289	0.692	0.364	0.317	0.283	0.686	0.359	0.310	0.286	0.665
KMS	0.383	0.388	0.789	7.612	0.382	0.381	0.790	7.899	0.384	0.389	0.796	7.637
KM	0.247	0.299	0.409	0.979	0.245	0.296	0.405	0.970	0.247	0.300	0.407	0.956
AY	0.421	0.629	1.205	11.035	0.419	0.623	1.203	11.287	0.421	0.629	1.202	10.921
D	0.422	0.527	0.690	0.753	0.422	0.523	0.685	0.751	0.423	0.528	0.690	0.745
AL	0.179	0.183	0.179	0.068	0.179	0.183	0.176	0.068	0.180	0.184	0.178	0.067
GM	0.161	0.177	0.254	1.437	0.159	0.175	0.251	1.413	0.160	0.177	0.252	1.383
KCI	0.271	0.243	0.166	0.026	0.271	0.242	0.164	0.026	0.272	0.244	0.166	0.026
Stein	0.977	0.973	0.963	1.670	0.977	0.973	0.963	1.720	0.977	0.973	0.963	1.658
CK	0.141	0.138	0.127	0.042	0.141	0.138	0.124	0.042	0.141	0.139	0.126	0.042
						$n = 50$						
OLS	0.226	0.340	0.659	6.868	0.232	0.338	0.672	6.895	0.229	0.344	0.678	6.818
HK	0.196	0.271	0.435	2.506	0.201	0.270	0.446	2.511	0.198	0.274	0.448	2.446
KAM	0.402	0.337	0.267	0.408	0.404	0.333	0.269	0.403	0.403	0.336	0.265	0.409
KMS	0.182	0.239	0.357	2.735	0.187	0.237	0.368	2.739	0.184	0.241	0.370	2.669
KM	0.133	0.170	0.244	0.755	0.136	0.168	0.250	0.740	0.135	0.171	0.250	0.748
AY	0.173	0.257	0.477	4.380	0.178	0.256	0.490	4.380	0.175	0.260	0.493	4.326
D	0.196	0.271	0.424	0.901	0.201	0.270	0.432	0.887	0.199	0.274	0.434	0.888
AL	0.128	0.150	0.172	0.112	0.131	0.149	0.176	0.109	0.130	0.151	0.175	0.110
GM	0.121	0.111	0.130	0.714	0.121	0.111	0.134	0.696	0.121	0.111	0.134	0.709
KCI	0.176	0.206	0.201	0.025	0.180	0.205	0.205	0.025	0.178	0.208	0.205	0.025
Stein	0.990	0.989	0.986	0.969	0.990	0.989	0.986	0.970	0.990	0.989	0.986	0.968
CK	0.211	0.124	0.113	0.068	0.113	0.122	0.135	0.066	0.112	0.124	0.135	0.067
						$n = 100$						
OLS	0.147	0.230	0.483	5.059	0.147	0.227	0.477	4.977	0.148	0.224	0.475	5.086
HK	0.132	0.164	0.346	1.894	0.132	0.158	0.342	1.840	0.133	0.161	0.339	1.912
KAM	0.348	0.280	0.216	0.313	0.351	0.281	0.212	0.307	0.354	0.284	0.216	0.307
KMS	0.121	0.189	0.277	1.937	0.121	0.185	0.273	1.877	0.122	0.189	0.271	1.955
KM	0.092	0.123	0.194	0.654	0.092	0.122	0.192	0.647	0.092	0.120	0.191	0.652
AY	0.123	0.189	0.375	3.259	0.123	0.187	0.369	3.205	0.124	0.184	0.367	3.293
D	0.132	0.194	0.340	0.869	0.133	0.192	0.336	0.863	0.133	0.189	0.335	0.877
AL	0.095	0.120	0.155	0.127	0.096	0.119	0.155	0.127	0.096	0.118	0.154	0.129
GM	0.075	0.071	0.091	0.540	0.075	0.070	0.091	0.532	0.077	0.070	0.091	0.532
KCI	0.117	0.145	0.154	0.018	0.118	0.143	0.153	0.018	0.118	0.141	0.152	0.018
Stein	0.995	0.995	0.994	0.980	0.995	0.995	0.994	0.980	0.995	0.995	0.994	0.979
CK	0.084	0.102	0.123	0.078	0.085	0.101	0.122	0.078	0.085	0.100	0.122	0.079

A.2. Estimated MSE values of the estimators with different values of σ and ρ when $u_i \sim N(0,1)$, $p = 10$.

p	1				2				5			
ρ	0.7	0.8	0.9	0.99	0.7	0.8	0.9	0.99	0.7	0.8	0.9	0.99
						$n = 15$						
OLS	3.034	4.43	8.673	84.663	3.022	4.437	8.626	84.424	3.061	4.4	8.579	84.946
HK	1.638	2.194	3.917	35.202	1.625	2.201	3.905	35.26	1.661	2.179	3.883	35.905
KAM	0.463	0.408	0.378	1.111	0.465	0.403	0.367	1.158	0.456	0.413	0.374	1.111
KMS	1.557	2.062	3.868	48.399	1.543	2.068	3.855	48.522	1.582	2.044	3.832	48.984
KM	0.634	0.761	0.999	2.246	0.633	0.754	0.985	2.242	0.65	0.746	0.992	2.226
AY	1.73	2.772	5.83	59.512	1.72	2.777	5.783	59.266	1.76	2.746	5.759	59.754
D	1.374	1.582	1.821	1.375	1.372	1.577	1.813	1.368	1.396	1.565	1.812	1.358
AL	0.204	0.186	0.147	0.041	0.204	0.184	0.145	0.041	0.208	0.183	0.147	0.041
GM	0.315	0.399	0.655	4.641	0.312	0.393	0.635	4.601	0.326	0.391	0.647	4.617
KCI	0.736	0.622	0.378	0.052	0.733	0.618	0.373	0.051	0.754	0.613	0.377	0.052
Stein	0.945	0.943	1.009	19.372	0.945	0.943	1.012	18.875	0.944	0.943	1.006	19.508
CK	0.163	0.135	0.104	0.077	0.165	0.134	0.105	0.077	0.164	0.134	0.104	0.077
						$n = 30$						
OLS	1.178	1.711	3.265	30.993	1.184	1.717	3.257	30.826	1.191	1.728	3.229	30.976
HK	0.798	1.023	1.29	11.773	0.797	1.028	1.314	11.704	0.799	1.037	1.289	11.798
KAM	0.469	0.407	0.321	0.39	0.473	0.403	0.316	0.391	0.475	0.41	0.309	0.381
KMS	0.714	0.868	1.079	13.603	0.713	0.871	1.101	13.555	0.715	0.881	1.077	13.674
KM	0.35	0.408	0.545	1.505	0.348	0.406	0.549	1.518	0.348	0.407	0.544	1.5
AY	0.66	1.036	2.089	20.571	0.66	1.037	2.098	20.47	0.663	1.047	2.069	20.628
D	0.772	0.93	1.2	1.87	0.772	0.932	1.207	1.88	0.775	0.937	1.196	1.864
AL	0.2	0.198	0.189	0.069	0.199	0.198	0.19	0.069	0.199	0.199	0.189	0.068
GM	0.149	0.153	0.213	1.368	0.148	0.152	0.215	1.382	0.149	0.154	0.212	1.362
KCI	0.513	0.451	0.325	0.027	0.51	0.453	0.329	0.027	0.511	0.454	0.326	0.027
Stein	0.984	0.981	0.972	1.787	0.984	0.981	0.972	1.804	0.984	0.981	0.972	1.819
CK	0.109	0.098	0.08	0.024	0.109	0.098	0.08	0.024	0.109	0.098	0.08	0.024
						$n = 50$						
OLS	0.541	0.801	1.611	16.938	0.540	0.798	1.615	16.744	0.539	0.799	1.618	16.927
HK	0.452	0.609	0.685	7.253	0.451	0.608	0.700	6.946	0.449	0.608	0.704	7.128
KAM	0.458	0.375	0.281	0.329	0.459	0.378	0.279	0.324	0.461	0.378	0.276	0.319
KMS	0.398	0.495	0.484	7.587	0.396	0.494	0.498	7.265	0.395	0.493	0.502	7.472
KM	0.248	0.305	0.447	1.411	0.245	0.305	0.452	1.402	0.245	0.303	0.451	1.401
AY	0.387	0.591	1.196	12.239	0.385	0.589	1.203	12.05	0.383	0.589	1.200	12.202
D	0.453	0.603	0.94	2.024	0.451	0.602	0.946	2.011	0.450	0.602	0.946	2.01
AL	0.188	0.199	0.205	0.076	0.186	0.199	0.206	0.076	0.186	0.198	0.206	0.075
GM	0.106	0.103	0.146	1.080	0.105	0.103	0.149	1.070	0.107	0.102	0.148	1.071
KCI	0.379	0.401	0.361	0.026	0.377	0.402	0.365	0.026	0.376	0.401	0.365	0.026
Stein	0.994	0.993	0.989	0.998	0.994	0.993	0.989	0.996	0.994	0.992	0.989	1.000
CK	0.135	0.135	0.124	0.035	0.134	0.134	0.125	0.035	0.134	0.134	0.125	0.035
						$n = 100$						
OLS	0.304	0.454	0.930	9.461	0.297	0.455	0.933	9.427	0.301	0.457	0.930	9.549
HK	0.268	0.373	0.649	4.003	0.263	0.373	0.650	4.038	0.266	0.376	0.650	4.096
KAM	0.418	0.342	0.244	0.206	0.416	0.339	0.253	0.208	0.416	0.346	0.246	0.211
KMS	0.232	0.296	0.472	3.695	0.227	0.296	0.472	3.731	0.230	0.298	0.473	3.792
KM	0.154	0.197	0.307	1.103	0.152	0.197	0.305	1.109	0.154	0.197	0.306	1.12
AY	0.242	0.362	0.722	6.765	0.237	0.362	0.723	6.736	0.24	0.364	0.72	6.851
D	0.268	0.372	0.628	1.92	0.263	0.373	0.628	1.921	0.266	0.375	0.627	1.936
AL	0.142	0.163	0.196	0.111	0.140	0.163	0.196	0.110	0.142	0.164	0.196	0.111
GM	0.066	0.062	0.089	0.620	0.065	0.062	0.087	0.628	0.066	0.061	0.088	0.638
KCI	0.226	0.257	0.274	0.029	0.222	0.257	0.274	0.029	0.225	0.258	0.274	0.029
Stein	0.997	0.997	0.996	0.983	0.997	0.997	0.996	0.983	0.997	0.997	0.996	0.983
CK	0.126	0.141	0.162	0.075	0.124	0.141	0.162	0.075	0.126	0.142	0.162	0.076

A.3. Estimated MSE values of the estimators with different values of σ and ρ when $\varepsilon_i \sim N(0,1)$, $p = 15$.

σ	1				2				5			
ρ	0.7	0.8	0.9	0.99	0.7	0.8	0.9	0.99	0.7	0.8	0.9	0.99
						$n = 50$						
OLS	1.182	1.857	3.847	40.237	1.189	1.871	3.849	40.564	1.199	1.843	3.832	40.390
HK	0.835	1.144	1.943	16.573	0.836	1.160	1.958	16.404	0.847	1.137	1.962	16.410
KAM	0.466	0.376	0.276	0.262	0.464	0.374	0.279	0.255	0.463	0.379	0.268	0.267
KMS	0.688	0.886	1.483	17.880	0.689	0.900	1.499	17.678	0.701	0.880	1.502	17.679
KM	0.360	0.447	0.669	2.063	0.361	0.455	0.667	2.053	0.367	0.447	0.671	2.052
AY	0.731	1.206	2.593	27.536	0.732	1.215	2.593	27.813	0.743	1.194	2.579	27.670
D	0.807	1.027	1.492	3.085	0.808	1.038	1.488	3.075	0.817	1.024	1.487	3.068
AL	0.202	0.206	0.197	0.058	0.203	0.209	0.196	0.057	0.205	0.206	0.196	0.057
GM	0.097	0.112	0.189	1.474	0.098	0.115	0.189	1.465	0.100	0.112	0.190	1.470
KCI	0.497	0.461	0.341	0.017	0.497	0.467	0.341	0.016	0.504	0.461	0.340	0.016
Stein	0.994	0.992	0.987	1.354	0.994	0.992	0.987	1.372	0.994	0.992	0.987	1.354
CK	0.096	0.088	0.069	0.016	0.097	0.089	0.069	0.015	0.098	0.088	0.069	0.015
						$n = 100$						
OLS	0.573	0.866	1.725	17.086	0.568	0.857	1.739	17.094	0.568	0.868	1.735	17.028
HK	0.482	0.663	1.115	7.518	0.478	0.659	1.124	7.595	0.478	0.664	1.125	7.618
KAM	0.410	0.336	0.246	0.176	0.421	0.339	0.246	0.180	0.409	0.336	0.245	0.176
KMS	0.388	0.492	0.779	6.998	0.385	0.489	0.786	7.082	0.384	0.492	0.786	7.095
KM	0.244	0.311	0.467	1.629	0.242	0.308	0.471	1.628	0.242	0.308	0.471	1.613
AY	0.444	0.672	1.320	12.631	0.440	0.666	1.333	12.655	0.440	0.673	1.329	12.563
D	0.480	0.654	1.027	3.016	0.477	0.647	1.035	3.011	0.477	0.653	1.035	2.989
AL	0.181	0.198	0.211	0.086	0.180	0.197	0.212	0.086	0.180	0.197	0.212	0.085
GM	0.058	0.067	0.107	0.829	0.058	0.066	0.108	0.833	0.058	0.066	0.109	0.818
KCI	0.355	0.379	0.357	0.028	0.353	0.376	0.360	0.028	0.353	0.376	0.360	0.028
Stein	0.998	0.997	0.996	0.985	0.998	0.997	0.996	0.985	0.998	0.997	0.996	0.985
CK	0.136	0.142	0.136	0.042	0.135	0.141	0.137	0.042	0.135	0.141	0.137	0.042

A.4. Estimated MSE values of the estimators with different values of n and ρ when $\varepsilon_i \sim \chi_1^2$, $p = 10$.

n	15				30			
ρ	0.7	0.8	0.9	0.99	0.7	0.8	0.9	0.99
OLS	8.983	13.701	29.56	340.044	4.632	7.491	15.554	160.09
HK	1.786	2.457	6.224	68.093	1.329	2.257	4.48	45.001
KAM	0.626	0.535	0.421	0.383	0.467	0.393	0.322	0.469
KMS	2.588	3.502	9.117	148.435	1.334	2.303	5.027	75.994
KM	0.677	0.771	1.012	2.086	0.664	0.826	1.152	2.52
AY	1.852	3.763	10.233	133.844	1.844	3.334	7.502	83.11
D	1.223	1.244	1.255	0.568	1.315	1.525	1.798	1.127
AL	0.165	0.132	0.094	0.04	0.189	0.167	0.123	0.028
GM	0.26	0.233	0.306	2.122	0.176	0.209	0.344	2.435
KCI	0.194	0.145	0.133	0.34	0.204	0.129	0.066	0.11
Stein	0.979	0.979	1.087	60.609	0.989	0.986	0.986	9.671
CK	0.348	0.285	0.232	0.195	0.098	0.079	0.056	0.025

A.5. Estimated MSE values of the estimators with different values of p and ρ when $\varepsilon_i \sim \chi^2_1$.

p	5				10				15			
ρ	0.7	0.8	0.9	0.99	0.7	0.8	0.9	0.99	0.7	0.8	0.9	0.99
	$n = 50$											
OLS	0.746	1.163	2.418	24.849	1.439	2.263	4.79	50.529	2.322	3.418	6.613	66.12
HK	0.387	0.507	0.82	6.019	0.776	1.058	1.793	16.396	1.163	1.521	2.509	22.394
KAM	0.417	0.348	0.289	0.461	0.498	0.418	0.312	0.348	0.568	0.477	0.352	0.254
KMS	0.342	0.444	0.77	9.172	0.655	0.865	1.54	21.969	1.011	1.25	2.099	28.566
KM	0.286	0.359	0.51	1.202	0.451	0.565	0.824	2.149	0.594	0.715	0.987	2.722
AY	0.396	0.621	1.261	12.325	0.728	1.238	2.73	29.303	1.039	1.775	3.821	40.905
D	0.474	0.593	0.78	0.756	0.898	1.13	1.553	1.813	1.374	1.646	2.131	2.878
AL	0.197	0.203	0.194	0.061	0.224	0.22	0.188	0.044	0.213	0.202	0.17	0.041
GM	0.159	0.158	0.207	1.039	0.138	0.134	0.189	1.294	0.131	0.124	0.17	1.229
KCI	0.196	0.156	0.088	0.053	0.359	0.276	0.138	0.03	0.509	0.379	0.196	0.022
Stein	0.995	0.994	0.991	1.085	0.997	0.996	0.993	1.123	0.997	0.996	0.994	1.089
CK	0.155	0.153	0.135	0.039	0.146	0.134	0.102	0.023	0.101	0.084	0.061	0.017
	$n = 100$											
OLS	0.253	0.379	0.777	8.288	0.727	1.130	2.354	24.865	1.101	1.759	3.742	40.014
HK	0.199	0.265	0.417	2.266	0.486	0.646	1.054	7.953	0.735	1.013	1.724	14.490
KAM	0.425	0.357	0.263	0.323	0.466	0.375	0.279	0.250	0.486	0.390	0.282	0.203
KMS	0.175	0.218	0.324	2.703	0.380	0.472	0.764	9.272	0.553	0.713	1.211	16.455
KM	0.156	0.200	0.309	0.949	0.298	0.378	0.581	1.843	0.429	0.562	0.871	2.682
AY	0.179	0.263	0.503	4.540	0.451	0.711	1.472	15.049	0.708	1.166	2.499	26.502
D	0.221	0.306	0.504	0.999	0.558	0.746	1.138	2.231	0.838	1.144	1.787	3.426
AL	0.146	0.171	0.201	0.108	0.209	0.223	0.226	0.070	0.234	0.238	0.213	0.048
GM	0.126	0.107	0.112	0.539	0.091	0.082	0.117	0.811	0.077	0.080	0.129	1.030
KCI	0.166	0.172	0.137	0.017	0.293	0.263	0.169	0.012	0.421	0.367	0.211	0.010
Stein	0.998	0.998	0.998	0.989	0.999	0.999	0.998	0.990	0.999	0.999	0.998	0.993
CK	0.126	0.140	0.152	0.064	0.178	0.183	0.174	0.047	0.162	0.154	0.123	0.023