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Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 2, Pages 437–449 (2024)
DOI:10.31801/cfsuasmas.1359782
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: September 13, 2023; Accepted: February 7, 2024

$B-$RIESZ POTENTIAL IN $B-$LOCAL MORREY-LORENTZ SPACES

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Abstract. In this paper, the Riesz potential ($B-$Riesz potential) which are generated by the Laplace-Bessel differential operator will be studied. We obtain the necessary and sufficient conditions for the boundedness of the $B-$Riesz potential I_γ^α in the $B-$local Morrey-Lorentz spaces $M^{\rm loc}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ with the use of the rearrangement inequalities and boundedness of the Hardy operators H_ν^β and $\mathcal{H}_\nu^\beta$ with power weights.

1. Introduction

Lorentz spaces, which are very useful in the theory of interpolation, have first been introduced by Lorentz [18]. These spaces are Banach spaces and generalizations of Lebesgue spaces. The Lorentz space $L_{p,q}(\mathbb{R}^n)$, $0 < p, q \leq \infty$, is known as the set of all measurable functions f such that

$$\|f\|_{L_{p,q}(\mathbb{R}^n)} = \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L_q(0,\infty)} < \infty.$$

Here, by f^* we denote the nonincreasing rearrangement of f and

$$f^*(t) = \inf \{\lambda > 0 : |\{y \in \mathbb{R}^n : |f(y)| > \lambda\}| \leq t\}, \quad t \in (0,\infty).$$

The necessary and sufficient condition for the functional $\|\cdot\|_{L_{p,q}}$ be a norm is $1 \leq q \leq p$ or $p = q = \infty$. If $p = q = \infty$, then $L_{\infty,\infty}(\mathbb{R}^n) \equiv L_\infty(\mathbb{R}^n)$. One can easily observe that $L_{p,p}(\mathbb{R}^n) \equiv L_p(\mathbb{R}^n)$ and $L_{p,\infty}(\mathbb{R}^n) \equiv WL_p(\mathbb{R}^n)$. It is obvious that $L_{p,q} \subset L_p \subset L_{p,r} \subset WL_p$ for $0 < q \leq p \leq q \leq r \leq \infty$. For further details, we refer the interested reader to [5, 18, 19].

2020 *Mathematics Subject Classification.* 47G40, 46E30, 42B20, 42B35.
Keywords. Morrey spaces, Lorentz spaces, local Morrey-Lorentz spaces, Riesz potential.
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On Lorentz spaces, the boundedness of the Riesz potential and the boundedness of its version related to the Laplace-Bessel differential operator

$$\Delta_B := \sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i} + \sum_{i=k+1}^{n} \frac{\partial^2}{\partial x_i^2}, \quad 1 \le k \le n,$$

have been studied by many researchers [3, 4, 10–15, 21]. The Riesz potential connected with the Laplace-Bessel differential operator ($B-$Riesz potential) is generated by generalized shift operator

$$T^y f(x) := C_{\gamma,k} \int_0^\pi \dots \int_0^\pi f\left[(x_1, y_1)_{\alpha_1}, \dots, (x_k, y_k)_{\alpha_k}, x'' - y''\right] d\gamma(\alpha).$$

Here $C_{\gamma,k} = \pi^{-\frac{k}{2}}\Gamma(\frac{\gamma_i+1}{2})[\Gamma(\frac{\gamma_i}{2})]^{-1}$, $(x_i, y_i)_{\alpha_i} = (x_i^2 - 2x_i y_i \cos\alpha_i + y_i^2)^{\frac{1}{2}}$, $1 \le i \le k$, $1 \le k \le n$ and $d\gamma(\alpha) = \prod_{i=1}^{k} \sin^{\gamma_i - 1}\alpha_i \, d\alpha_i$ [16, 17].

The $B-$convolution operator is defined as:

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y) T^y g(x) (y')^\gamma dy.$$

Here, $\mathbb{R}^n_{k,+} = \{x \in \mathbb{R}^n : x_1 > 0, \dots, x_k > 0, 1 \le k \le n\}$, $\gamma = (\gamma_1, \dots, \gamma_k)$, $\gamma_1 > 0, \dots, \gamma_k > 0$, $|\gamma| = \gamma_1 + \dots + \gamma_k$. Let us set $x = (x', x'')$, $x' = (x_1, \dots, x_k) \in \mathbb{R}^k$, and $x'' = (x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-k}$.

The purpose of this paper is to obtain the boundedness of the $B-$Riesz potential operator I^α_γ on $B-$local Morrey-Lorentz spaces with the use of the rearrangement inequalities and the Hardy inequality. Local Morrey-Lorentz spaces $M^{\text{loc}}_{p,q,\lambda}(\mathbb{R}^n)$ which have first been introduced by Aykol et al. [2] and are generalizations of Lorentz spaces. One has $M^{\text{loc}}_{p,q,0}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$. They have also proved that the Riesz potential operator is bounded in these spaces. In this study, we consider the $B-$Riesz potential by

$$I^\alpha_\gamma f(x) = \int_{\mathbb{R}^n_{k,+}} T^y |x|^{\alpha - Q} f(y) (y')^\gamma dy, \quad 0 < \alpha < Q.$$

The maximal operator has a crucial role in the study of the regularity of some partial differential equations and in the study of the boundedness of some singular integrals and on the differentiability properties of functions. For a function $f \in L^{\text{loc}}_{1,\gamma}(\mathbb{R}^n_{k,+})$, the $B-$maximal operator and $B-$fractional maximal operator are defined by, (see [7]) respectively,

$$M_\gamma f(x) = \sup_{r>0} |B_+(0,r)|^{-1}_\gamma \int_{B_+(0,r)} T^y |f(x)| (y')^\gamma dy,$$
$$M^\alpha_\gamma f(x) = \sup_{r>0} |B_+(0,r)|^{\frac{\alpha}{Q}-1}_\gamma \int_{B_+(0,r)} T^y |f(x)| (y')^\gamma dy, \quad 0 \le \alpha < Q,$$

where $B_+(x,r) = \{y \in \mathbb{R}^n_{k,+} : |x-y| < r\}$. Let $B_+(0,r) \subset \mathbb{R}^n_{k,+}$ be a measurable set, then

$$|B_+(0,r)|_\gamma = \int_{B_+(0,r)} (x')^\gamma dx = \omega(n,k,\gamma) r^Q,$$

where $\omega(n,k,\gamma) = \dfrac{\pi^{\frac{n-k}{2}}}{2^k} \displaystyle\prod_{i=1}^{k} \dfrac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}$, $Q = n + |\gamma|$. It is easy to observe that $M^0_\gamma f = M_\gamma f$ for $\alpha = 0$ (see [7]). It is well known that the inequality $M^\alpha_\gamma \le C\, I^\alpha_\gamma$ holds.

On local Morrey-Lorentz space, the necessary and sufficient conditions for the boundedness of the Riesz potential operator are given in [13]. On the other hand, the $B-$ Riesz potential has been investigated in various function spaces by many mathematicians (see, for example [3,10–12]). The above results inspire us to investigate the boundedness of the $B-$Riesz potential defined on $B-$local Morrey-Lorentz spaces.

Throughout the paper, C denotes a positive constant independent of appropriate parameters and not necessary the same at each occurrence.

2. Preliminaries

Given any measurable set E with $|E|_\gamma = \displaystyle\int_E (x')^\gamma dx$ and a measurable function $f : \mathbb{R}^n_{k,+} \to \mathbb{R}$, the $\gamma-$rearrangement of f in decreasing order is defined as

$$f^*_\gamma(t) = \inf\{s > 0 : f_{*,\gamma}(s) \le t\}, \quad \forall t \in (0,\infty),$$

where $f_{*,\gamma}(s)$ denotes the $\gamma-$distribution function of f given by

$$f_{*,\gamma}(s) = \left|\left\{x \in \mathbb{R}^n_{k,+} : |f(x)| > s\right\}\right|_\gamma .$$

The average function of f^{**}_γ is defined as

$$f^{**}_\gamma(t) = \frac{1}{t}\int_0^t f^*_\gamma(s) ds, \quad t > 0,$$

and the following inequality holds (see [20]):

$$(f+g)^{**}_\gamma(t) \le f^{**}_\gamma(t) + g^{**}_\gamma(t).$$

Now, we give some characteristics of the $\gamma-$rearrangement of functions:

- if $0 < p < \infty$, then
$$\int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma dx = \int_0^\infty (f^*_\gamma(t))^p dt,$$
- for any $t > 0$,
$$\sup_{|E|_\gamma = t} \int_E |f(x)|(x')^\gamma dx = \int_0^t f^*_\gamma(s) ds, \tag{1}$$

- the following inequality holds:

$$\int_{\mathbb{R}^n_{k,+}} |f(x)g(x)|(x')^\gamma dx \le \int_0^\infty f^*_\gamma(t)g^*_\gamma(t)dt,$$

- the following inequality holds (see [5, 20, 22]):

$$(f+g)^*_\gamma(t) \le f^*_\gamma(t/2) + g^*_\gamma(t/2). \tag{2}$$

Definition 1. *[18] If $0 < p, q \le \infty$, then we define the Lorentz space $L_{p,q,\gamma}(\mathbb{R}^n_{k,+})$ is the set of all measurable functions $f \in \mathbb{R}^n_{k,+}$ such that*

$$\|f\|_{L_{p,q,\gamma}} = \left\| t^{\frac{1}{p}-\frac{1}{q}} f^*_\gamma(t) \right\|_{L_q(0,\infty)} < \infty.$$

If $0 < p \le \infty$, $q = \infty$, then $L_{p,\infty,\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,\gamma}(\mathbb{R}^n_{k,+})$, where $WL_{p,\gamma}(\mathbb{R}^n_{k,+})$ is weak Lebesgue space of all measurable functions f such that

$$\|f\|_{WL_{p,\gamma}(\mathbb{R}^n_{k,+})} = \sup_{t>0} t^{1/p} f^*_\gamma(t) < \infty, \quad 1 \le p < \infty.$$

If $p = q = \infty$ or $1 \le q \le p$, then the functional $\|f\|_{p,q,\gamma}$ is a norm [5, 11, 22]. However if $p = q = \infty$, then $L_{\infty,\infty,\gamma}(\mathbb{R}^n_{k,+}) = L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$.

In case $0 < p, q \le \infty$, a functional $\|\cdot\|^*_{L_{p,q,\gamma}}$ is given by

$$\|f\|^*_{L_{p,q,\gamma}} = \|f\|^*_{L_{p,q,\gamma}(0,\infty)} = \|t^{\frac{1}{p}-\frac{1}{q}} f^{**}_\gamma(t)\|_{L_q(0,\infty)},$$

which is a norm on $L_{p,q,\gamma}(\mathbb{R}^n_{k,+})$ for $1 \le q \le \infty$, $1 < p < \infty$ or $p = q = \infty$.

If $1 < p \le \infty$, $1 \le q \le \infty$, then

$$\|f\|_{p,q,\gamma} \le \|f\|^*_{p,q,\gamma} \le \frac{p}{p-1}\|f\|_{p,q,\gamma},$$

that is, $\|f\|_{p,q,\gamma}$ and $\|f\|^*_{p,q,\gamma}$ are equivalent.

Definition 2. *[8] Let $1 \le p < \infty$, and $0 \le \lambda \le Q$. The $B-$Morrey space $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ is the set of all measurable functions with $f \in L^{\text{loc}}_{p,\gamma}(\mathbb{R}^n_{k,+})$ such that*

$$\|f\|_{L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})} = \sup_{x\in\mathbb{R}^n_{k,+},\rho>0} \rho^{-\frac{\lambda}{p}} \|f\|_{L_{p,\gamma}(B_+(x,\rho))} < \infty.$$

If $\lambda = 0$, then $L_{p,0,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\gamma}(\mathbb{R}^n_{k,+})$; if $\lambda > Q$ or $\lambda < 0$, then $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n_{k,+}$. Also, the weak $B-$Morrey space $WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ is the set of all functions $f \in WL^{\text{loc}}_{p,\gamma}(\mathbb{R}^n_{k,+})$ with following norm

$$\|f\|_{WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})} = \sup_{x\in\mathbb{R}^n_{k,+},\rho>0} \rho^{-\frac{\lambda}{p}} \|f\|_{WL_{p,\gamma}(B_+(x,\rho))} < \infty.$$

Definition 3. *[6] Let $0 \le \lambda \le 1$ and $0 \le p < \infty$. The local Morrey space $LM_{p,\lambda} \equiv LM_{p,\lambda}(0,\infty)$ is the set of all functions $f \in L^{\text{loc}}_p(0,\infty)$ such that*

$$\|f\|_{LM_{p,\lambda}(0,\infty)} = \sup_{\rho>0} \rho^{-\frac{\lambda}{p}} \|f\|_{L_p(0,\rho)} < \infty.$$

Moreover, $WLM_{p,\lambda} \equiv WLM_{p,\lambda}(0,\infty)$ denotes the weak local Morrey space of all functions $f \in WL_p^{\rm loc}(0,\infty)$ such that

$$\|f\|_{WLM_{p,\lambda}(0,\infty)} = \sup_{\rho>0} \rho^{-\frac{\lambda}{p}} \|f\|_{WL_p(0,\rho)} < \infty.$$

Definition 4. *[9] Given a function $f \in L_{1,\gamma}^{\rm loc}(\mathbb{R}_{k,+}^n)$, and a ball $B_+(x,r)$. By $f_{B_+}(x)$ we denote the average of $T^y f$ on the ball B_+,*

$$f_{B_+}(x) = |B_+|_\gamma^{-1} \int_{B_+} T^y f(x)(y')^\gamma dy.$$

The $BMO-$Bessel space $BMO_\gamma(\mathbb{R}_{k,+}^n)$ is the set of all functions on $L_{1,\gamma}^{\rm loc}(\mathbb{R}_{k,+}^n)$ with

$$\|f\|_{*,\gamma} = \sup_{B_+} |B_+|_\gamma^{-1} \int_{B_+} |T^y f(x) - f_{B_+}|(y')^\gamma dy < \infty.$$

Definition 5. *Let $0 < p, q \le \infty$ and $0 \le \lambda \le 1$. The $B-$local Morrey-Lorentz space $M_{p,q,\lambda,\gamma}^{\rm loc}(\mathbb{R}_{k,+}^n)$ is set of all measurable functions with the quasinorm*

$$\|f\|_{M_{p,q,\lambda,\gamma}^{\rm loc}} = \sup_{\rho>0} \rho^{-\frac{\lambda}{q}} \|t^{\frac{1}{p}-\frac{1}{q}} f_\gamma^*(t)\|_{L_q(0,\rho)} < \infty.$$

If $\lambda > 1$ or $\lambda < 0$, then $M_{p,q,\lambda,\gamma}^{\rm loc}(\mathbb{R}_{k,+}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}_{k,+}^n$. Also,

$$M_{p,q,0,\gamma}^{\rm loc}(\mathbb{R}_{k,+}^n) = L_{p,q,\gamma}(\mathbb{R}_{k,+}^n) \quad \text{and} \quad M_{p,p,\lambda,\gamma}^{\rm loc}(\mathbb{R}_{k,+}^n) \equiv M_{p,0,\gamma}^{\rm loc}(\mathbb{R}_{k,+}^n).$$

The weak $B-$local Morrey-Lorentz space $WM_{p,q,\lambda,\gamma}^{\rm loc}(\mathbb{R}_{k,+}^n)$ is the set of all measurable functions with the quasinorm

$$\|f\|_{WM_{p,q,\lambda,\gamma}^{\rm loc}} = \sup_{\rho>0} \rho^{-\frac{\lambda}{q}} \|t^{\frac{1}{p}-\frac{1}{q}} f_\gamma^*(t)\|_{WL_q(0,\rho)} < \infty.$$

We need the boundedness of the Hardy operators which will be used in the proof of our main theorem.

Definition 6. *[21] Let φ be a measurable function on $(0,\infty)$ and $\beta \in \mathbb{R}$. The weighted Hardy operators H_ν^β and $\mathcal{H}_\nu^\beta$ with power weights are defined as*

$$H_\nu^\beta \varphi(t) = t^{\beta+\nu-1} \int_0^t \frac{\varphi(y)}{y^\nu} dy, \qquad \mathcal{H}_\nu^\beta \varphi(t) = t^{\beta+\nu} \int_t^\infty \frac{\varphi(y)}{y^{\nu+1}} dy.$$

In the following theorem, we state that the Hardy operators are bounded in local Morrey and weak local Morrey spaces.

Theorem 1. *[1, 21] Let $0 < \lambda < 1$, $0 < \beta < 1-\lambda$, $1 \le r < \dfrac{1-\lambda}{\beta}$ and $\dfrac{1}{r} - \dfrac{1}{s} = \dfrac{\beta}{1-\lambda}$.*

i. *If* $\nu < \dfrac{1}{r'} + \dfrac{\lambda}{r}$, *then*

$$\left\|H_{\nu}^{\beta}\varphi\right\|_{LM_{s,\lambda}(0,\infty)} \le C\left\|\varphi\right\|_{LM_{r,\lambda}(0,\infty)}.$$

ii. *If* $\nu = \dfrac{1}{r'} + \dfrac{\lambda}{r}$, *then*

$$\left\|H_{\nu}^{\beta}\varphi\right\|_{WLM_{s,\lambda}(0,\infty)} \le C\left\|\varphi\right\|_{LM_{r,\lambda}(0,\infty)}.$$

iii. *If* $\nu > \dfrac{\lambda-1}{r}$, *then*

$$\left\|\mathcal{H}_{\nu}^{\beta}\varphi\right\|_{LM_{s,\lambda}(0,\infty)} \le C\left\|\varphi\right\|_{LM_{r,\lambda}(0,\infty)}.$$

iv. *If* $\nu = \dfrac{\lambda-1}{r}$, *then*

$$\left\|\mathcal{H}_{\nu}^{\beta}\varphi\right\|_{WLM_{s,\lambda}(0,\infty)} \le C\left\|\varphi\right\|_{LM_{r,\lambda}(0,\infty)}.$$

3. $B-$Riesz Potential in $B-$Local Morrey-Lorentz Space

This section devoted to obtain the boundedness of the $B-$Riesz potential in $B-$local Morrey-Lorentz and weak $B-$local Morrey-Lorentz space.

For the $B-$Riesz potential, the following inequality

$$(I_{\gamma}^{\alpha}f)_{\gamma}^{*}(t) \le (I_{\gamma}^{\alpha}f)_{\gamma}^{**}(t) \le C_2\left(t^{\frac{\alpha}{Q}-1}\int_0^t f_{\gamma}^{*}(y)dy + \int_t^{\infty} y^{\frac{\alpha}{Q}-1}f_{\gamma}^{*}(y)dy\right) \tag{3}$$

holds, where $C_2 = C_{\gamma,k}(Q/\alpha)^2\omega(n,k,\gamma)^{(Q-\alpha)/Q}$ (see [10]).

Theorem 2. *Let* $0 \le \lambda < 1$, $0 < \alpha < Q$, $1 \le q \le \infty$, $1 \le r \le s \le \infty$, $\frac{r}{r+\lambda} \le p \le \left(\frac{\lambda}{r} + \frac{\alpha}{Q}\right)^{-1}$ *and* $f \in M_{p,r,\lambda,\gamma}^{\mathrm{loc}}$.

(i) *If* $\frac{r}{r+\lambda} < p < \left(\frac{\lambda}{r} + \frac{\alpha}{Q}\right)^{-1}$, *then* $\frac{1}{p} - \frac{1}{q} = \lambda\left(\frac{1}{r} - \frac{1}{s}\right) + \frac{\alpha}{Q}$ *is necessary and sufficient condition for the boundedness of* I_{γ}^{α} *from* $M_{p,r,\lambda,\gamma}^{\mathrm{loc}}$ *to* $M_{q,s,\lambda,\gamma}^{\mathrm{loc}}$.

(ii) *If* $p = \frac{r}{r+\lambda}$, *then* $1 - \frac{1}{q} = \frac{\alpha}{Q} - \frac{\lambda}{s}$ *is necessary and sufficient condition for the boundedness of* I_{γ}^{α} *from* $M_{p,r,\lambda,\gamma}^{\mathrm{loc}}$ *to* $WM_{q,s,\lambda,\gamma}^{\mathrm{loc}}$.

Proof. (i) *Sufficiency.* Let $\frac{r}{r+\lambda} < p < \left(\frac{\lambda}{r} + \frac{\alpha}{Q}\right)^{-1}$. From (3), we have

$$\begin{aligned}
\|I_{\gamma}^{\alpha}f\|_{M_{q,s,\lambda,\gamma}^{\mathrm{loc}}(\mathbb{R}_{k,+}^{n})} &= \sup_{\rho>0}\rho^{-\frac{\lambda}{s}}\left\|t^{\frac{1}{q}-\frac{1}{s}}(I_{\gamma}^{\alpha}f)_{\gamma}^{*}(t)\right\|_{L_s(0,\rho)} \\
&\le \sup_{\rho>0}\rho^{-\frac{\lambda}{s}}\left\|t^{\frac{1}{q}-\frac{1}{s}}(I_{\gamma}^{\alpha}f)_{\gamma}^{**}(t)\right\|_{L_s(0,\rho)} \\
&\le C_2\sup_{\rho>0}\rho^{-\frac{\lambda}{s}}\left\|t^{\frac{1}{q}-\frac{1}{s}}\left(t^{\frac{\alpha}{Q}-1}\int_0^t f_{\gamma}^{*}(y)dy + \int_t^{\infty} y^{\frac{\alpha}{Q}-1}f_{\gamma}^{*}(y)dy\right)\right\|_{L_s(0,\rho)}
\end{aligned}$$

$$\leq C_2 \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| t^{\frac{1}{q}-\frac{1}{s}+\frac{\alpha}{Q}-1} \int_0^t f_\gamma^*(y)dy \right\|_{L_s(0,\rho)}$$
$$+ C_2 \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| t^{\frac{1}{q}-\frac{1}{s}} \int_t^\infty y^{\frac{\alpha}{Q}-1} f_\gamma^*(y)dy \right\|_{L_s(0,\rho)}$$
$$= I_1 + I_2.$$

We take $\nu = \dfrac{1}{p} - \dfrac{1}{r}$ and $\varphi(y) = y^{\frac{1}{p}-\frac{1}{r}} f_\gamma^*(y)$. Then, we have

$$\beta = \frac{1}{q} - \frac{1}{s} + \frac{1}{r} - \frac{1}{p} + \frac{\alpha}{Q}.$$

From Theorem 1, we can write $\beta = (1-\lambda)\left(\dfrac{1}{r} - \dfrac{1}{s}\right)$. Then we get $\dfrac{1}{p} - \dfrac{1}{q} = \lambda\left(\dfrac{1}{r} - \dfrac{1}{s}\right) + \dfrac{\alpha}{Q}$. Therefore, again by Theorem 1, we get

$$\begin{aligned}
I_1 &= C_2 \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| t^{\frac{1}{q}-\frac{1}{s}+\frac{\alpha}{Q}-1} \int_0^t f_\gamma^*(y)dy \right\|_{L_s(0,\rho)} \\
&= C_2 \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| t^{\beta+\nu-1} \int_0^t \frac{\varphi(y)}{y^\nu} dy \right\|_{L_s(0,\rho)} \\
&= C \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| H_\nu^\beta \varphi \right\|_{L_s(0,\rho)} \\
&= C \left\| H_\nu^\beta \varphi \right\|_{LM_{s,\lambda}(0,\infty)} \\
&\leq C \left\| \varphi \right\|_{LM_{r,\lambda}(0,\infty)} = C \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \left\| \varphi \right\|_{L_r(0,\rho)} \\
&= C \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \| t^{\frac{1}{p}-\frac{1}{r}} f_\gamma^*(y) \|_{L_r(0,\rho)} = C \left\| f \right\|_{M^{\mathrm{loc}}_{p,r,\lambda,\gamma}(\mathbb{R}^n_{k,+})}.
\end{aligned}$$

We now estimate I_2. We take $\nu = \dfrac{1}{p} - \dfrac{1}{r} - \dfrac{\alpha}{Q}$ and $\varphi(y) = y^{\frac{1}{p}-\frac{1}{r}} f_\gamma^*(y)$. Then, we get

$$\beta = \frac{1}{q} - \frac{1}{s} + \frac{1}{r} - \frac{1}{p} + \frac{\alpha}{Q}.$$

Therefore, by using Theorem 1, we obtain

$$\begin{aligned}
I_2 &= C_2 \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| t^{\frac{1}{q}-\frac{1}{s}} \int_t^\infty y^{\frac{\alpha}{Q}-1} f_\gamma^*(y)dy \right\|_{L_s(0,\rho)} \\
&= C_2 \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| t^{\beta+\nu} \int_t^\infty \frac{\varphi(y)}{y^{\nu+1}} dy \right\|_{L_s(0,\rho)} \\
&= C_2 \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| \mathcal{H}_\nu^\beta \varphi \right\|_{L_s(0,\rho)}
\end{aligned}$$

$$= C \left\|\mathcal{H}_\nu^\beta \varphi\right\|_{LM_{s,\lambda}(0,\infty)}$$
$$\leq C \left\|\varphi\right\|_{LM_{r,\lambda}(0,\infty)} = C \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \|\varphi\|_{L_{r,\lambda}(0,\rho)}$$
$$= C \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \|t^{\frac{1}{p}-\frac{1}{r}} f_\gamma^*(\tau)\|_{L_r(0,\rho)} = C \left\|f\right\|_{M^{\rm loc}_{p,r,\lambda,\gamma}(\mathbb{R}^n_{k,+})}.$$

Hence, we obtain that the $B-$Riesz potential I_γ^α bounded from $M^{\rm loc}_{p,r,\lambda,\gamma}$ to $M^{\rm loc}_{q,s,\lambda,\gamma}$.

Necessity. Suppose that the $B-$Riesz potential I_γ^α is bounded from $M^{\rm loc}_{p,r,\lambda,\gamma}$ to $M^{\rm loc}_{q,s,\lambda,\gamma}$ and $\dfrac{r}{r+\lambda} \leq p \leq \left(\dfrac{\lambda}{r}+\dfrac{\alpha}{Q}\right)^{-1}$. For $\tau>0$, we define $f_\tau(x):=f(\tau x)$. Then $(f_\tau)_\gamma^*(t) = f_\gamma^*(t\tau^Q)$ and

$$\begin{aligned}
\|f_\tau\|_{M^{\rm loc}_{p,r,\lambda,\gamma}} &= \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \|t^{\frac{1}{p}-\frac{1}{r}} (f_\tau)_\gamma^*(t)\|_{L_r(0,\rho)} \\
&= \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \|t^{\frac{1}{p}-\frac{1}{r}} f_\gamma^*(t\tau^Q)\|_{L_r(0,\rho)} \\
&= \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \tau^{-\frac{Q}{p}} \|t^{\frac{1}{p}-\frac{1}{r}} f_\gamma^*(t)\|_{L_r(0,\rho\tau^Q)} \\
&= \tau^{-\frac{Q}{p}+\frac{Q\lambda}{r}} \sup_{\rho>0} (\rho\tau^Q)^{-\frac{\lambda}{r}} \|t^{\frac{1}{p}-\frac{1}{r}} f_\gamma^*(t)\|_{L_r(0,\rho\tau^Q)} \\
&= \tau^{-Q(\frac{1}{p}-\frac{\lambda}{r})} \|f\|_{M^{\rm loc}_{p,r,\lambda,\gamma}}.
\end{aligned}$$

Also, $(I_\gamma^\alpha f_\tau)(x) = \tau^{-\alpha}(I_\gamma^\alpha f)(\tau^Q x)$ and $(I_\gamma^\alpha f_\tau)_\gamma^*(t) = \tau^{-\alpha}(I_\gamma^\alpha f)_\gamma^*(t\tau^Q)$. Then, we get

$$\begin{aligned}
\|I_\gamma^\alpha f_\tau\|_{M^{\rm loc}_{q,s,\lambda,\gamma}} &= \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \|t^{\frac{1}{q}-\frac{1}{s}} (I_\gamma^\alpha f_\tau)_\gamma^*(t)\|_{L_s(0,\rho)} \\
&= \tau^{-\alpha} \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \|t^{\frac{1}{q}-\frac{1}{s}} (I_\gamma^\alpha f)_\gamma^*(t\tau^Q)\|_{L_s(0,\rho)} \\
&= \tau^{-\alpha} \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left(\int_0^\infty (t\tau^Q)^{\frac{s}{q}-1} ((I_\gamma^\alpha f)_\gamma^*(t\tau^Q))^s d((t\tau^Q))\right)^{\frac{1}{s}} \tau^{-\frac{Q}{q}} \\
&= \tau^{-\alpha-\frac{Q}{q}-\frac{Q\lambda}{s}} \sup_{\rho>0} (\rho\tau^Q)^{-\frac{\lambda}{s}} \|t^{\frac{1}{q}-\frac{1}{s}} I_\gamma^\alpha f_\gamma^*(t)\|_{L_s(0,\rho)} \\
&= \tau^{-\alpha-Q(\frac{1}{q}-\frac{\lambda}{s})} \|I_\gamma^\alpha f\|_{M^{\rm loc}_{q,s,\lambda,\gamma}}.
\end{aligned}$$

Since the $B-$Riesz potential I_γ^α is bounded from $M^{\rm loc}_{p,r,\lambda,\gamma}$ to $M^{\rm loc}_{q,s,\lambda,\gamma}$, we can write $\|I_\gamma^\alpha f\|_{M^{\rm loc}_{q,s,\lambda,\gamma}} \leq C\|f\|_{M^{\rm loc}_{p,r,\lambda,\gamma}}$, where $C>0$ is a constant. Then

$$\begin{aligned}
\|I_\gamma^\alpha f\|_{M^{\rm loc}_{q,s,\lambda,\gamma}} &= \tau^{\alpha+Q(\frac{1}{q}-\frac{\lambda}{s})} \|I_\gamma^\alpha f_\tau\|_{M^{\rm loc}_{q,s,\lambda,\gamma}} \\
&\leq C\tau^{\alpha+Q(\frac{1}{q}-\frac{\lambda}{s})} \|f_\tau\|_{M^{\rm loc}_{p,r,\lambda,\gamma}}
\end{aligned}$$

$$= \tau^{\alpha+Q(\frac{1}{q}-\frac{\lambda}{s})}\|f\|_{M^{\rm loc}_{p,r,\lambda,\gamma}}$$
$$= \tau^{\alpha+Q(\frac{1}{q}-\frac{1}{p})+Q\lambda(\frac{1}{r}-\frac{1}{s})}\|f\|_{M^{\rm loc}_{p,r,\lambda,\gamma}}.$$

- If $\frac{1}{p} < \frac{1}{q} + \lambda\left(\frac{1}{r} - \frac{1}{s}\right) + \frac{\alpha}{Q}$, then we have $\|I^\alpha_\gamma f\|_{M^{\rm loc}_{q,s,\lambda,\gamma}} = 0$ as $\tau \to 0$ for all $f \in M^{\rm loc}_{p,r,\lambda,\gamma}$.
- If $\frac{1}{p} > \frac{1}{q} + \lambda\left(\frac{1}{r} - \frac{1}{s}\right) + \frac{\alpha}{Q}$, then we have $\|I^\alpha_\gamma f\|_{M^{\rm loc}_{q,s,\lambda,\gamma}} = 0$ as $\tau \to \infty$ for all $f \in M^{\rm loc}_{p,r,\lambda,\gamma}$.
- If $\frac{1}{p} - \frac{1}{q} \neq \lambda\left(\frac{1}{r} - \frac{1}{s}\right) + \frac{\alpha}{Q}$, then we have $I^\alpha_\gamma f(x) = 0$ for all $f \in M^{\rm loc}_{p,r,\lambda,\gamma}$ and a.e. $x \in \mathbb{R}^n_{k,+}$, which is impossible.

Hence, we obtain $\frac{1}{p} - \frac{1}{q} = \lambda\left(\frac{1}{r} - \frac{1}{s}\right) + \frac{\alpha}{Q}$.

(ii) *Sufficiency.* Let $\frac{r}{r+\lambda} < p < \left(\frac{\lambda}{r} + \frac{\alpha}{Q}\right)^{-1}$. From (3), we have

$$\begin{aligned}
\|I^\alpha_\gamma f\|_{WM^{\rm loc}_{q,s,\lambda,\gamma}(\mathbb{R}^n_{k,+})} &= \sup_{\rho>0}\rho^{-\frac{\lambda}{s}}\left\|t^{\frac{1}{q}-\frac{1}{s}}(I^\alpha_\gamma f)^*_\gamma(t)\right\|_{WL_s(0,\rho)} \\
&\le \sup_{\rho>0}\rho^{-\frac{\lambda}{s}}\left\|t^{\frac{1}{q}-\frac{1}{s}}(I^\alpha_\gamma f)^{**}_\gamma(t)\right\|_{WL_s(0,\rho)} \\
&\le C_2\sup_{\rho>0}\rho^{-\frac{\lambda}{s}}\left\|t^{\frac{1}{q}-\frac{1}{s}}\left(t^{\frac{\alpha}{Q}-1}\int_0^t f^*_\gamma(y)dy + \int_t^\infty y^{\frac{\alpha}{Q}-1}f^*_\gamma(y)dy\right)\right\|_{WL_s(0,\rho)} \\
&\le C_2\sup_{\rho>0}\rho^{-\frac{\lambda}{s}}\left\|t^{\frac{1}{q}-\frac{1}{s}+\frac{\alpha}{Q}-1}\int_0^t f^*_\gamma(y)dy\right\|_{WL_s(0,\rho)} \\
&\quad + C_2\sup_{\rho>0}\rho^{-\frac{\lambda}{s}}\left\|t^{\frac{1}{q}-\frac{1}{s}}\int_t^\infty y^{\frac{\alpha}{Q}-1}f^*_\gamma(y)dy\right\|_{WL_s(0,\rho)} \\
&= J_1 + J_2.
\end{aligned}$$

We take $\nu = 1 + \dfrac{\lambda-1}{r}$ and $\varphi(y) = y^{1+\frac{\lambda-1}{r}}f^*_\gamma(y)$ in the Hardy operator. Then, we get

$$\beta = \frac{1}{q} - \frac{1}{s} + \frac{1}{r} + \frac{\alpha}{Q} - 1 - \frac{\lambda}{r}.$$

From Theorem 1, we can write $\beta = (1-\lambda)\left(\dfrac{1}{r} - \dfrac{1}{s}\right)$. Then we have $1 - \dfrac{1}{q} = \dfrac{\alpha}{Q} - \dfrac{\lambda}{s}$. Therefore, again by Theorem 1, we get

$$\begin{aligned}
J_1 &= C_2\sup_{\rho>0}\rho^{-\frac{\lambda}{s}}\left\|t^{\frac{1}{q}-\frac{1}{s}+\frac{\alpha}{Q}-1}\int_0^t f^*_\gamma(y)dy\right\|_{WL_s(0,\rho)} \\
&= C_2\sup_{\rho>0}\rho^{-\frac{\lambda}{s}}\left\|t^{\beta+\nu-1}\int_0^t \frac{\varphi(y)}{y^\nu}dy\right\|_{WL_s(0,\rho)}
\end{aligned}$$

$$= C \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| H_\nu^\beta \varphi \right\|_{WL_s(0,\rho)}$$
$$= C \left\| H_\nu^\beta \varphi \right\|_{WLM_{s,\lambda}(0,\infty)}$$
$$\leq C \left\| \varphi \right\|_{LM_{r,\lambda}(0,\infty)}$$
$$= C \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \left\| \varphi \right\|_{L_r(0,\rho)}$$
$$= C \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \| y^{1+\frac{\lambda-1}{r}} f_\gamma^*(y) \|_{L_r(0,\rho)}$$
$$= C \left\| f \right\|_{M^{\rm loc}_{p,r,\lambda,\gamma}(\mathbb{R}^n_{k,+})}.$$

We now estimate J_2. We take $\nu = 1 + \dfrac{\lambda-1}{r} - \dfrac{\alpha}{Q}$ and $\varphi(y) = y^{1+\frac{\lambda-1}{r}} f_\gamma^*(y)$ in the Hardy operator. Then, we get

$$\beta = \frac{1}{q} - \frac{1}{s} + \frac{1}{r} + \frac{\alpha}{Q} - 1 - \frac{\lambda}{r}.$$

Therefore, by using Theorem 1, we obtain

$$J_2 = C_2 \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| t^{\frac{1}{q}-\frac{1}{s}} \int_t^\infty y^{\frac{\alpha}{Q}-1} f_\gamma^*(y) dy \right\|_{WL_s(0,\rho)}$$
$$= C_2 \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| t^{\beta+\nu} \int_t^\infty \frac{\varphi(y)}{y^{\nu+1}} dy \right\|_{WL_s(0,\rho)}$$
$$= C_2 \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| \mathcal{H}_\nu^\beta \varphi \right\|_{WL_s(0,\rho)}$$
$$= C \left\| \mathcal{H}_\nu^\beta \varphi \right\|_{WLM_{s,\lambda}(0,\infty)}$$
$$\leq C \left\| \varphi \right\|_{LM_{r,\lambda}(0,\infty)}$$
$$= C \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \left\| \varphi \right\|_{L_{r,\lambda}(0,\rho)}$$
$$= C \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \| y^{1+\frac{\lambda-1}{r}} f_\gamma^*(y) \|_{L_r(0,\rho)}$$
$$= C \left\| f \right\|_{M^{\rm loc}_{p,r,\lambda,\gamma}(\mathbb{R}^n_{k,+})}.$$

Necessity. Suppose that the $B-$ Riesz potential is I_γ^α bounded from $M^{\rm loc}_{p,r,\lambda,\gamma}$ to $WM^{\rm loc}_{q,s,\lambda,\gamma}$ and $p = \dfrac{r}{r+\lambda}$. Again, for $\tau > 0$, we define $f_\tau(x) := f(\tau x)$. Then $\|f_\tau\|_{M^{\rm loc}_{r/(r+\lambda),r,\lambda,\gamma}} = \tau^{-Q} \|f\|_{M^{\rm loc}_{r/(r+\lambda),r,\lambda,\gamma}}$ and

$$\|I_\gamma^\alpha f_\tau\|_{WM^{\rm loc}_{q,s,\lambda,\gamma}} = \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \|y^{\frac{1}{q}-\frac{1}{s}} (I_\gamma^\alpha f_\tau)_\gamma^*(y)\|_{WL_s(0,\rho)}$$
$$= \tau^{-\alpha} \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \|y^{\frac{1}{q}-\frac{1}{s}} (I_\gamma^\alpha f)_\gamma^*(y\tau^Q)\|_{WL_s(0,\rho)}$$

$$= \tau^{-\alpha-\frac{Q}{q}-\frac{Q\lambda}{s}} \sup_{\rho>0} (\rho\tau^{Q})^{-\frac{\lambda}{s}} \| y^{\frac{1}{q}-\frac{1}{s}} I_\gamma^\alpha f_\gamma^*(y)\|_{WL_s(0,\rho)}$$
$$= \tau^{-\alpha-Q(\frac{1}{q}-\frac{\lambda}{s})} \|I_\gamma^\alpha f\|_{WM^{\rm loc}_{q,s,\lambda,\gamma}}.$$

Since the $B-$Riesz potential I_γ^α is bounded from $M^{\rm loc}_{p,r,\lambda,\gamma}$ to $WM^{\rm loc}_{q,s,\lambda,\gamma}$, we have $\|I_\gamma^\alpha f\|_{WM^{\rm loc}_{q,s,\lambda,\gamma}} \le C\|f\|_{M^{\rm loc}_{p,r,\lambda,\gamma}}$, where $C>0$ is a constant. Then we get

$$\begin{aligned} \|I_\gamma^\alpha f\|_{WM^{\rm loc}_{q,s,\lambda,\gamma}} &= \tau^{\alpha+Q(\frac{1}{q}-\frac{\lambda}{s})}\|I_\gamma^\alpha f_\tau\|_{WM^{\rm loc}_{q,s,\lambda,\gamma}} \\ &\le C\tau^{\alpha+Q(\frac{1}{q}-\frac{\lambda}{s})}\|f_\tau\|_{M^{\rm loc}_{p,r,\lambda,\gamma}} \\ &= \tau^{\alpha+Q(\frac{1}{q}-\frac{\lambda}{s})}\|f\|_{M^{\rm loc}_{p,r,\lambda,\gamma}} \\ &= \tau^{\alpha+Q(\frac{1}{q}-1-\frac{\lambda}{r})+Q\lambda(\frac{1}{r}-\frac{1}{s})}\|f\|_{M^{\rm loc}_{p,r,\lambda,\gamma}}. \end{aligned}$$

- If $1 < \frac{1}{q}+\frac{\alpha}{Q}-\frac{\lambda}{s}$, then we have $\|I_\gamma^\alpha f\|_{WM^{\rm loc}_{q,s,\lambda,\gamma}} = 0$ as $\tau \to 0$ for all $f \in M^{\rm loc}_{r/(r+\lambda),r,\lambda,\gamma}$.
- If $1 > \frac{1}{q}+\frac{\alpha}{Q}-\frac{\lambda}{s}$, then we have $\|I_\gamma^\alpha f\|_{WM^{\rm loc}_{q,s,\lambda,\gamma}} = 0$ as $\tau \to \infty$ for all $f \in M^{\rm loc}_{r/(r+\lambda),r,\lambda,\gamma}$.
- If $1 \neq \frac{1}{q}+\frac{\alpha}{Q}-\frac{\lambda}{s}$, then we have $I_\gamma^\alpha f(x) = 0$ for all $f \in M^{\rm loc}_{p,r,\lambda,\gamma}$ and a.e. $x \in \mathbb{R}^n_{k,+}$, which is impossible.

Hence, we obtain $1-\frac{1}{q} = \frac{\alpha}{Q}-\frac{\lambda}{s}$. This completes the proof.

□

The following corollary is easily obtained from the inequality $M_\gamma^\alpha \le C\, I_\gamma^\alpha$ and Theorem 2, .

Corollary 1. *Let $0 \le \lambda < 1$, $0 < \alpha < Q$, $1 \le q \le \infty$, $1 \le r \le s \le \infty$, $\frac{r}{r+\lambda} \le p \le \left(\frac{\lambda}{r}+\frac{\alpha}{Q}\right)^{-1}$.*

(i) *If $\frac{r}{r+\lambda} < p < \left(\frac{\lambda}{r}+\frac{\alpha}{Q}\right)^{-1}$, then $\frac{1}{p}-\frac{1}{q} = \lambda\left(\frac{1}{r}-\frac{1}{s}\right)+\frac{\alpha}{Q}$ is necessary and sufficient condition for the boundedness of the $B-$fractional maximal operator M_γ^α from $M^{\rm loc}_{p,r,\lambda,\gamma}$ to $M^{\rm loc}_{q,s,\lambda,\gamma}$.*

(ii) *If $p = \frac{r}{r+\lambda}$, then $1-\frac{1}{q} = \frac{\alpha}{Q}-\frac{\lambda}{s}$ is necessary and sufficient condition for the boundedness of the $B-$fractional maximal operator M_γ^α from $M^{\rm loc}_{p,r,\lambda,\gamma}$ to $WM^{\rm loc}_{q,s,\lambda,\gamma}$.*

Author Contribution Statements The authors jointly worked on the results and they read and approved the final manuscript.

Declaration of Competing Interests The authors declare no potential conflict of interests.

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