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The generalizations of the Carathèodory Inequality for the holomorphic functions

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ABSTRACT

In this paper, the results of the Carathèodory inequality have been generalized. C. T. Rajagopal further strengthened the inequality (1.8) by considering the zeros of the function $f(z)$. We will obtain more general results for the inequalities (1.8) and (1.9) by considering both the zeros and the poles of the function $f(z)$.

Keywords: Holomorphic function, Poles and zeros

Holomorfik fonksiyonlar için Carathèodory eşitsizliğinin genellemesi

ÖZ

Bu makalede, Carathèodory eşitsizliğinin sonuçları genelleştirilmiştir. C. T. Rajagopal (1.8) eşitsizliğini, $f(z)$ fonksiyonun sıfırlarını da göz önüne alarak daha da güçlendirmiştir. Biz $f(z)$ fonksiyonun hem sıfırlarını hem de kutuplarını göz önünde bulundurarak, (1.8) ve (1.9) eşitsizlikleri için daha genel sonuçlar elde edeceğiz.

Anahtar Kelimeler: Holomorfik fonksiyon, Kutuplar ve sıfırlar

1. INTRODUCTION

Estimation of the holomorphic functions and their derivatives have a significant place in complex analysis and its applications. The real part of the holomorphic functions gets involved in the estimation of majorant. Among these inequalities are the Hadamard-Borel Carathèodory inequality for holomorphic functions in $D=\{z:|z|<1\}$ with $\Re f(z)$ bounded from above

$$|f(z)-f(0)|\leq\frac{2r}{1-r}\sup_{|\varsigma|<1}\Re\left(f(\varsigma)-f(0)\right),\ |z|=r \qquad (1.1)$$

and

$$|f(z)|\leq\frac{1+r}{1-r}|f(0)|+\frac{2r}{1-r}\sup_{|\varsigma|<1}\Re\left(f(\varsigma)\right),\ |z|=r \qquad (1.2)$$

frequently called the Borel-Carathèodory inequality [4].

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Similarly, if the function $f(z)$ is holomorphic on the unit disc D with $f(0)=0$ and $\Re f(z)\le A$ for $|z|<1$, then we have

$$|f(z)|\le\frac{2A|z|}{1-|z|} \tag{1.3}$$

holds for all $z\in\partial D$, and moreover

$$|f'(0)|\le 2A\,. \tag{1.4}$$

Equality is achieved in (1.3) (for some nonzero $z\in\partial D$) or in (1.4) if and only if $f(z)$ is the function of the form

$$f(z)=\frac{2Aze^{i\theta}}{1+ze^{i\theta}},$$

where θ is a real number ([4], p.3-4).

Sometimes, (1.1) and (1.2), as well as the related inequality for $\Re f(z)$

$$\Re f(z)\le\frac{1-r}{1+r}|f(0)|+\frac{2r}{1-r}\sup_{|\varsigma|<1}\Re\big(f(\varsigma)\big) \tag{1.5}$$

are called Hadamard-Borel-Carathèodory inequality.

Introducing the notation

$$A=\sup_{|z|<1}\Re f(z),\ M=\sup_{|z|<1}|f(z)|.$$

Lindelöf [6] obtained the following two-sided inequality similar to Hadamard-Borel-Carathèodory inequality.

$$\frac{1-r}{1+r}\Re f(0)-\frac{2r}{1-r}A\le\Re f(z)\le\frac{1-r}{1+r}\Re f(0)+\frac{2r}{1+r}A \tag{1.6}$$

Theorem 1. Let $f(z)$ be a bounded holomorphic and has no zeros in D with $f(0)=1$ and let $M=\sup_{|z|<1}|f(z)|$. Then for any z with $|z|=r<1$ two-sided inequality

$$M^{\frac{-2r}{1-r}}\le|f(z)|\le M^{\frac{2r}{1+r}} \tag{1.7}$$

holds ([4], p.14).

A similar estimate for $|f(z)|$ with $f(0)\neq 1$ can be obtained from (1.7) with $f(z)$ replaced by $\frac{f(z)}{f(0)}$. That is;

$$|f(z)|\le|f(0)|^{\frac{1-r}{1+r}}M^{\frac{2r}{1+r}}. \tag{1.8}$$

Similarly, from (1.7), we take

$$|f(z)|\ge|f(0)|^{\frac{1+r}{1-r}}M^{\frac{-2r}{1-r}}. \tag{1.9}$$

C. T. Rajagopal [5] further strengthened the inequality (1.8) by considering the zeros of the function $f(z)$.

Consider the following product:

$$B(z)=\prod_{\lambda=1}^{m}\frac{z-a_\lambda}{1-\overline{a_\lambda}z}.$$

$B(z)$ is called a finite Blaschke product, where $a_1,a_2,...,a_m\in\mathbb{C}$. $B(z)$ is holomorphic in the unit disc D, and

$$|B(z)|=1\ \ for\ z\in\partial D\,,$$

since $\left|\frac{z-a_\lambda}{1-\overline{a_\lambda}z}\right|=1$ when $|z|=1$. Therefore, the Maximum Modul Principle implies

$$|B(z)|<1\ \ for\ z\in D\,.$$

Similarly, the extremal function is often given by the Blaschke function

$$B_b(z)=\prod_{k=1}^{n}\frac{1-\overline{b_k}z}{z-b_k},$$

which is generally defined for any set $b_1,b_2,...,b_n$ of poles.

The following Theorem 2 is a simple example of the application of the maximum principle for holomorphic functions.

Theorem 2. Let $f(z)$ be a holomorphic function in the unit disc D except at the poles $b_1,b_2,...,b_n$. Suppose that none of the limiting values of $|f(z)|$ as z approaches the boundary of the unit disc D exceed 1. Then

$$|f(z)|\le\prod_{k=1}^{n}\left|\frac{1-\overline{b_k}z}{z-b_k}\right|,\ z\in D\,. \tag{1.10}$$

Equality at a point z (not a pole) is attained only if $f(z)$ is the function of the form

$$f(z)=c\prod_{k=1}^{n}\frac{1-\overline{b_k}z}{z-b_k},$$

where $|c|=1$ and $|b_k|<1$, $k=1,2,....,n$ ([1], p.286).

2. MAIN RESULTS

In this section, we will make this kind of improvement for the inequalities (1.8) and (1.9) by considering both the zeros and the poles of the function $f(z)$.

Theorem 3. Let $f(z)$ be a holomorphic function in the unit disc D except at the poles $b_1,b_2,...,b_n$ and $a_1,a_2,...,a_m$ are zeros of $f(z)$ in the unit disc D. Suppose that none of the limiting values of $|f(z)|$ as z approaches the boundary of the unit disc D exceed 1. Then for any $|z|=r$, we obtain

$$|f(z)|\le\prod_{k=1}^{n}\left|\frac{1-\overline{b_k}z}{z-b_k}\right|\prod_{\lambda=1}^{m}\left|\frac{z-a_\lambda}{1-\overline{a_\lambda}z}\right|\left(\frac{|f(0)|\prod_{k=1}^{n}|b_k|}{\prod_{\lambda=1}^{m}|a_\lambda|}\right)M^{\frac{2r}{1+r}},z\in D. \qquad (2.1)$$

Proof. Consider the auxiliary function

$$\phi(z)=\frac{f(z)}{\prod_{\lambda=1}^{m}\frac{z-a_\lambda}{1-\overline{a_\lambda}z}}\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}.$$

$\phi(z)$ is a holomorphic function in D and $|\phi(z)|\le 1$ for $z\in D$. That is; the function

$$\varphi(z)=f(z)\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}$$

is a holomorphic in D. As z approaches the boundary of D, the modulus of the limiting values of $\varphi(z)$ does not exceed 1. Applying the maximum principle implies that for each $z\in D$ we have $|\varphi(z)|\le 1$ (see, Theorem1). Now, consider the function

$$\phi(z)=\frac{\varphi(z)}{B(z)}, \qquad (2.2)$$

where $B(z)=\prod_{\lambda=1}^{m}\frac{z-a_\lambda}{1-\overline{a_\lambda}z}$. $B(z)$ is a holomorphic function in D, and $|B(z)|<1$ for $z\in D$. Therefore, the the maximum principle implies that for each $z\in D$ we obtain the inequality $|\varphi(z)|\le|B(z)|$ ([3], p.192-193). Thus, we have $|\phi(z)|\le 1$ for $z\in D$ and if we apply inequality (1.8) to the function $\phi(z)$, we obtain inequality (2.1).

Theorem 4. Under the hypotheses of the Theorem 3 and let $z=0$ be a simple zero in addition to the zeros in Theorem 3. Then we obtain

$$|f(z)|\le|z|\prod_{k=1}^{n}\left|\frac{1-\overline{b_k}z}{z-b_k}\right|\prod_{\lambda=1}^{m}\left|\frac{z-a_\lambda}{1-\overline{a_\lambda}z}\right|\left(\frac{|f'(0)|\prod_{k=1}^{n}|b_k|}{\prod_{\lambda=1}^{m}|a_\lambda|}\right)M^{\frac{2r}{1+r}},z\in D. \qquad (2.3)$$

Proof. Consider the function

$$\psi(z)=\frac{f(z)}{z\prod_{\lambda=1}^{m}\frac{z-a_\lambda}{1-\overline{a_\lambda}z}}\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}. \qquad (2.4)$$

$\psi(z)$ is a holomorphic function in D and $|\psi(z)|\le 1$ for $z\in D$ from proof of the Theorem 3. If we apply inequality (1.8) to the function $\psi(z)$, we obtain inequality (2.4).

Theorem 5. Under the hypotheses of the Theorem 3, we have

$$|f(z)|\ge\prod_{k=1}^{n}\left|\frac{1-\overline{b_k}z}{z-b_k}\right|\prod_{\lambda=1}^{m}\left|\frac{z-a_\lambda}{1-\overline{a_\lambda}z}\right|\left(\frac{|f(0)|\prod_{k=1}^{n}|b_k|}{\prod_{\lambda=1}^{m}|a_\lambda|}\right)M^{\frac{-2r}{1-r}},z\in D. \qquad (2.5)$$

Proof. Applying the inequality (1.9) to the function $\phi(z)$ which is defined in (2.2), we obtain inequality (2.5).

Theorem 6. Under the hypotheses of the Theorem 3 and let $z=0$ be a simple zero in addition to the zeros in Theorem 3. Then we obtain

$$|f(z)|\ge|z|\prod_{k=1}^{n}\left|\frac{1-\overline{b_k}z}{z-b_k}\right|\prod_{\lambda=1}^{m}\left|\frac{z-a_\lambda}{1-\overline{a_\lambda}z}\right|\left(\frac{|f'(0)|\prod_{k=1}^{n}|b_k|}{\prod_{\lambda=1}^{m}|a_\lambda|}\right)M^{\frac{-2r}{1-r}},z\in D. \qquad (2.6)$$

Proof. Applying the inequality (1.9) to the function $\psi(z)$ which is defined in (2.4), we obtain inequality (2.6).

Now, we will make this kind of improvement for the inequalities (1.3) and (1.4) by considering both the zeros and the poles of the function $f(z)$.

Theorem 7. Let $f(z)$ be a holomorphic function in the unit disc D except at the poles $b_1,b_2,...,b_n$, $f(0)=0$ and $a_1,a_2,...,a_m$ are zeros of $f(z)$ in the unit disc D that are different zero. Suppose that none of the limiting values of $\Re f(z)$ az z approaches the boundary of the unit disc D exceed A. Then, we obtain

$$|f(z)|\le\frac{2A|z|\prod_{k=1}^{n}\left|\frac{1-\overline{b_k}z}{z-b_k}\right|\prod_{\lambda=1}^{m}\left|\frac{z-a_\lambda}{1-\overline{a_\lambda}z}\right|}{1-|z|\prod_{k=1}^{n}\left|\frac{1-\overline{b_k}z}{z-b_k}\right|\prod_{\lambda=1}^{m}\left|\frac{z-a_\lambda}{1-\overline{a_\lambda}z}\right|} \qquad (2.7)$$

and

$$|f'(0)|\le\frac{2A\prod_{\lambda=1}^{m}|a_\lambda|}{\prod_{k=1}^{n}|b_k|}. \qquad (2.8)$$

Equality at a point z (which is not a pole) is achieved in (2.7) or in (2.8) if and only if $f(z)$ is the function of the form

$$f(z)=2A\frac{ze^{i\theta}\prod_{\lambda=1}^{m}\frac{z-a_\lambda}{1-\overline{a_\lambda}z}\prod_{k=1}^{n}\frac{1-\overline{b_k}z}{z-b_k}}{1+ze^{i\theta}\prod_{\lambda=1}^{m}\frac{z-a_\lambda}{1-\overline{a_\lambda}z}\prod_{k=1}^{n}\frac{1-\overline{b_k}z}{z-b_k}}$$

where $|a_\lambda|<1$, $|b_k|<1$ and θ is a real number.

Proof. Let

$$\Upsilon(z)=\frac{f(z)}{f(z)-2A}\frac{\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}}{\prod_{\lambda=1}^{m}\frac{z-a_\lambda}{1-\overline{a_\lambda}z}}.$$

$\Upsilon(z)$ is a holomorphic function in D and $|\Upsilon(z)|\le 1$ for $z\in D$. That is; the function

$$w(z)=\frac{f(z)}{f(z)-2A}\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}$$

is a holomorphic in D. Assume that any of limiting values of $\Re f(z)$ do not exceed A when z approaches the boundary of the unit disc D. Applying the maximum principle implies that for each $z \in D$ we have $|w(z)| \le 1$. Now, consider the function

$$\Upsilon(z) = \frac{w(z)}{\prod_{\lambda=1}^{m} \frac{z-a_\lambda}{1-\overline{a_\lambda} z}}.$$

The maximum principle implies that for each $z \in D$, we obtain the inequality

$$|w(z)| \le \prod_{\lambda=1}^{m} \left| \frac{z-a_\lambda}{1-\overline{a_\lambda} z} \right|.$$

Therefore, we have $|\Upsilon(z)| \le 1$ for $z \in D$ and $\Upsilon(0) = 0$. From the Schwarz lemma ([2], p.329), we take $|\Upsilon(z)| \le |z|$ and $|\Upsilon'(0)| \le 1$. So, we get

$$\left| \frac{f(z)}{f(z)-2A} \frac{\prod_{k=1}^{n} \frac{z-b_k}{1-\overline{b_k} z}}{\prod_{\lambda=1}^{m} \frac{z-a_\lambda}{1-\overline{a_\lambda} z}} \right| \le |z|$$

and

$$|f'(0)| \le \frac{2A \prod_{\lambda=1}^{m} |a_\lambda|}{\prod_{k=1}^{n} |b_k|}.$$

Thus, we obtain the inequality (2.7) and (2.8).

Now, we shall show that the inequality (2.7) and (2.8) are sharp. Introducing the notation

$$k = \prod_{\lambda=1}^{m} \left| \frac{z_0 - a_\lambda}{1-\overline{a_\lambda} z_0} \right|, \quad \delta = \prod_{k=1}^{n} \left| \frac{1-\overline{b_k} z_0}{z_0 - b_k} \right|.$$

If $|f(z_0)| = 2A \frac{k\delta |z_0|}{1-k\delta |z_0|}$, then

$$|f(z_0)| = k\delta |z_0| |f(z_0)| + 2Ak\delta |z_0|. \qquad (2.9)$$

We known that,

$$\left| \frac{f(z_0)}{f(z_0)-2A} \frac{\prod_{k=1}^{n} \frac{z_0-b_k}{1-\overline{b_k} z_0}}{\prod_{\lambda=1}^{m} \frac{z_0-a_\lambda}{1-\overline{a_\lambda} z_0}} \right| \le |z_0|$$

and

$$|f(z_0)| \le k\delta |z_0| |f(z_0) - 2A| \le +2Ak\delta |z_0| + k\delta |z_0| |f(z_0)|.$$

From (2.9), we take

$$|f(z_0)| = k\delta |z_0| |f(z_0) - 2A|.$$

Therefore, we obtain

$$\left| \frac{f(z_0)}{f(z_0)-2A} \frac{\prod_{k=1}^{n} \frac{z_0-b_k}{1-\overline{b_k} z_0}}{\prod_{\lambda=1}^{m} \frac{z_0-a_\lambda}{1-\overline{a_\lambda} z_0}} \right| = 1$$

and since z_0 is arbitrary

$$f(z) = 2A \frac{ze^{i\theta} \prod_{\lambda=1}^{m} \frac{z-a_\lambda}{1-\overline{a_\lambda} z} \prod_{k=1}^{n} \frac{1-\overline{b_k} z}{z-b_k}}{1 + ze^{i\theta} \prod_{\lambda=1}^{m} \frac{z-a_\lambda}{1-\overline{a_\lambda} z} \prod_{k=1}^{n} \frac{1-\overline{b_k} z}{z-b_k}}.$$

The sharpness of (2.8) can be shown analogously.

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