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STABILITY ANALYSIS OF NEUTRAL VOLTERRA INTEGRO-DIFFERENTIAL EQUATION

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Abstract. The study establishes the stability bounds of the second-order neutral Volterra integro-differential equation concerning both the right-side and initial conditions. The examples are given to show the applicability of the method and confirm the predicted theoretical analysis.

1. Introduction

Numerous scientific models and many disciplines lead to integro-differential equations (IDEs). This makes it attractive to use different methods to solve them (see, e.g. [1–5]).

IDEs are categorized by the interval of their integral terms. Volterra integro-differential equation (VIDE) is those where integration limits are variables, whereas Fredholm IDE is integration limits that only involve constants. VIDEs were first introduced by Vito Volterra in 1926, and since then many studies have been carried out on the VIDEs.

In recent years, many researchers have investigated the qualitative behaviors of solutions to these equations. For example, in [6], the authors proposed a method for obtaining sufficient conditions for the stability of solutions of systems of linear VIDEs. They give adequate criteria for the stability of the solutions of VIDE when the initial conditions are perturbed. In [7], presented some explicit criteria for the uniform asymptotic and the exponential stability of the nonlinear VIDE using spectral properties of Metzler matrices and the comparison principle. Amirali, in [8], establishes the stability inequalities for the linear nonhomogeneous Volterra

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delay integro-differential equations (VDIDEs). The author shows that the solution continuously depends on the right-side and initial data. Alahmadi et al. [9], utilize the Lyapunov functionals combined with the Laplace transform to obtain boundedness and stability results about nonlinear VIDE solutions. Yu et al. [10], are concerned with the numerical stability of Runge-Kutta methods for nonlinear neutral VDIDE. The stability analysis of exact solutions of the linear neutral VDIDE system is considered in [11]. The method authors use, shows that it preserves the delay-independent stability of exact solutions. In [12], the authors present some estimates for the exact solution of the neutral VDIDE, which show the stability of the problem for the right-side and initial condition. In [13], using the positivity of linear VIDE, the authors give an explicit criterion for the uniform asymptotic stability of positive equations. Amirali et al. [14], consider the stability inequalities which can be established for any order of derivative for high-order linear VDIDEs. Yapman et al. [15],study stability analysis of exact solution and convergence analysis of a fitted numerical method for a singularly perturbed nonlinear VIDE with delay. In [16], the authors give the stability inequalities for the following neutral VIDE with respect to the initial conditions and the right-hand side. Panda et al. [17], present stability analysis of first-order singularly perturbed VIDE.

The goal of this paper is to present the stability inequalities for the neutral second-order VIDE:

$$u''(t) + a(t)\,u(t) - \int_0^t \left[K_1(t,s)\,u''(s) + K_2(t,s)\,u(s)\right] ds = f(t),\ \ t \in \Omega = (0,T] \tag{1}$$

$$u(0) = A, \quad u'(0) = B, \tag{2}$$

where $a(t), f(t)\left(t \in \bar{\Omega} \equiv [0,T]\right)$ and $K_i(t,s), i = 1,2, \left((t,s) \in \bar{\Omega}^2\right)$ are the sufficiently smooth functions satisfying certain regularity conditions to be specified.

2. Stability Bounds for the Differential Problem

Here we establish stability bounds regarding the right-side and initial conditions for the problem (1)-(2).

For any function $g(t) \in C\left(\bar{\Omega}\right)$ we use $\|g\|_\infty \equiv \|g\|_{\infty,\bar{\Omega}} := \max_{\bar{\Omega}} |g(t)|$.

Theorem 1. *If $a(t), f(t) \in C\left(\bar{\Omega}\right)$, $K_1, K_2 \in C\left(\bar{\Omega}^2\right)$, then for the solution $u(t)$ of (1)-(2) holds the following inequality:*

$$\|u\|_\infty \le \alpha e^{\beta}, \tag{3}$$

where

$$\alpha = T^2\left(1 + \bar{K}_1 e^{\bar{K}_1 T} T\right)\|f\|_\infty + |A| + T\,|B|,$$

$$\beta = T^2\left(\|a\|_\infty + \mu T\right),$$
$$\mu = \bar{K}_2 + \|a\|_\infty \bar{K}_1 e^{\bar{K}_1 T} + T\bar{K}_2\bar{K}_1 e^{\bar{K}_1 T}$$

and

$$\bar{K}_1 = \max_{\bar{\Omega}^2}\left|K_1(t,s)\right|,$$
$$\bar{K}_2 = \max_{\bar{\Omega}^2}\left|K_2(t,s)\right|.$$

Proof. Denoting $\delta(t) = |u''(t)|$, we get

$$\delta(t) \le \rho(t) + \int_0^t \bar{K}_1 \delta(s)\, ds,$$

where

$$\rho(t) = |f(t)| + |a(t)|\,|u(t)| + \int_0^t |K_2(t,s)|\,|u(s)|\, ds.$$

Then by Gronwall's inequality we have

$$\delta(t) \le \rho(t) + \bar{K}_1 \int_0^t \rho(s)\, e^{\bar{K}_1(t-s)} ds.$$

Since

$$\rho(t) \le \|a\|_\infty |u(t)| + \bar{K}_2 \int_0^t |u(s)| ds + \|f\|_\infty$$

we get

$$\begin{aligned}
|u''(t)| \le & \|a\|_\infty |u(t)| + \|f\|_\infty + \bar{K}_2 \int_0^t |u(s)|\, ds \\
& + \bar{K}_1 e^{\bar{K}_1 T} \int_0^t [\|a\|_\infty |u(s)| + \|f\|_\infty] ds + \bar{K}_2 \bar{K}_1 e^{\bar{K}_1 T} \int_0^t \int_0^s |u(\zeta)|\, d\zeta ds \\
= & \|a\|_\infty |u(t)| + \|f\|_\infty + \left(\bar{K}_2 + \|a\|_\infty \bar{K}_1 e^{\bar{K}_1 T}\right) \int_0^t |u(s)|\, ds \\
& + \bar{K}_2 \bar{K}_1 e^{\bar{K}_1 T} \int_0^t (t-s)\, |u(s)|\, ds + \bar{K}_1 e^{\bar{K}_1 T} \int_0^t \|f\|_\infty ds
\end{aligned}$$

$$\leq \|a\|_{\infty} |u(t)| + \left(1 + \bar{K}_1 e^{\bar{K}_1 T} T\right) \|f\|_{\infty} + \mu \int_0^t |u(s)|\, ds. \tag{4}$$

Next, using the following relations which is true for any $g \in C^2$

$$g(t) = g(0) + t g'(0) + \int_0^t (t-s) g''(s)\, ds,$$

$$|g(t)| \leq |g(0)| + T|g'(0)| + T \int_0^t |g''(s)|\, ds$$

and

$$\int_0^t |u''(s)| \geq \frac{1}{T} |u(t)| - \frac{1}{T} |u(0)| - |u'(0)|,$$

the inequality (4) reduces to

$$|u(t)| \leq T^2 \left(1 + \bar{K}_1 e^{\bar{K}_1 T} T\right) \|f\|_{\infty} + |A| + T|B|$$
$$+ T\left(\|a\|_{\infty} + \mu T\right) \int_0^t |u(s)|\, ds.$$

Finally, applying the Gronwall's inequality we get

$$|u(t)| \leq \left[T^2 \left(1 + \bar{K}_1 e^{\bar{K}_1 T} T\right) \|f\|_{\infty} + |A| + T|B|\right] e^{Tt\left(\|a\|_{\infty} + \mu T\right)},$$

which proves Theorem (1).

□

3. Numerical Examples

This section includes examples that confirm the theoretical methodology.

Example 1. *Consider the following problem:*

$$u''(t) + u(t) - \int_0^t \left[\frac{t}{20} u''(s) + \left(\frac{t+s}{40}\right) u(s)\right] ds = \frac{t - \sin t}{40}, \quad 0 < t \leq 1,$$
$$u(0) = 0, \quad u'(0) = 1.$$

The solution is given by

$$u\,(t) = \sin t.$$

Since

$$T = 1, \quad \bar{K}_1 = 0.05, \quad \bar{K}_2 = 0.05, \quad \mu = 0.1052,$$

$$\|f\|_\infty = 0.004, \quad |A| = 0, \quad |B| = 1, \quad \|a\|_\infty = 1,$$

the bound will be

$$|u\,(t)| \leq 1.0042 \times e^{1.1052t}.$$

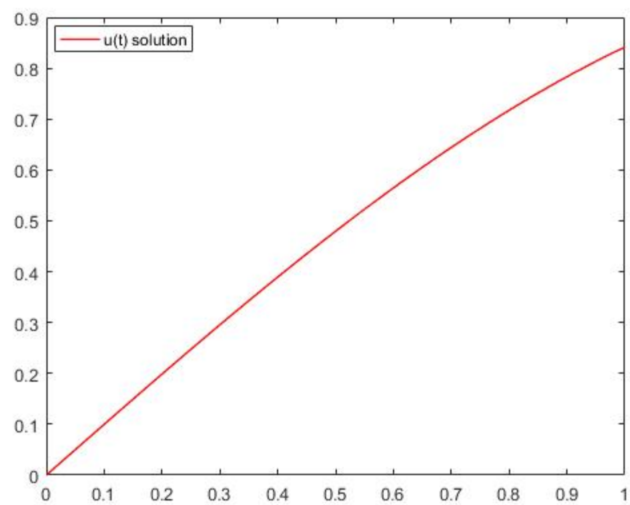

FIGURE 1. $u(t)$ solution

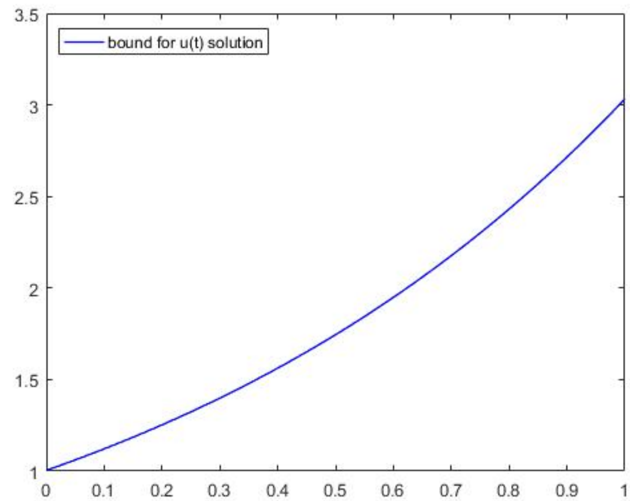

FIGURE 2. $\bar{u}(t) = 1.0042 \times exp(1.1052t)$

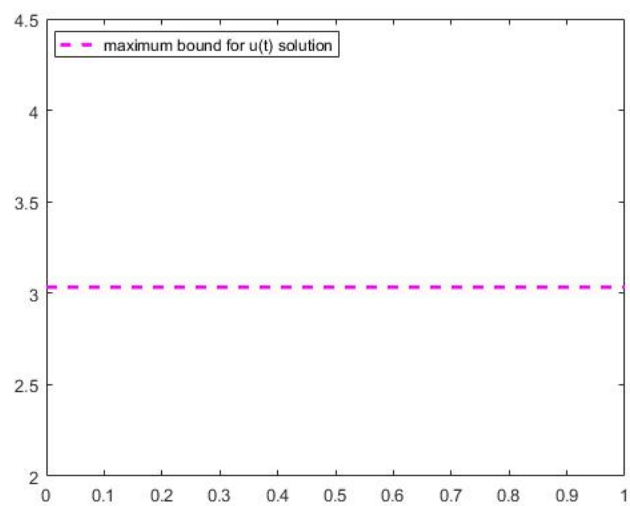

FIGURE 3. Maximum bound for the solution

Example 2. *Now give an another problem, which is defined as follows:*

$$u''(t)+\frac{t}{12}u(t)-\int\limits_0^t\left[\sqrt{\frac{ts}{2}}u''(s)+s^2u(s)\right]ds=\frac{t}{24}\left(t-1-3t^3+4t^2\right),\ 0<t\leq 0.75,$$
$$u(0)=\frac{-1}{2},\quad u'(0)=\frac{1}{2}.$$

The solution of the problem is

$$u(t)=\frac{t-1}{2}.$$

Since

$$T=0.75,\quad \bar{K}_1=0.5303,\quad \bar{K}_2=0.5625,\quad \mu=0.9448,$$

$$\|f\|_\infty=0.0229,\quad |A|=\frac{1}{2},\quad |B|=\frac{1}{2},\quad \|a\|_\infty=0.0625,$$

bound for the solution $u(t)$ will be

$$|u(t)|\leq 0.8955\times e^{0.5783t}.$$

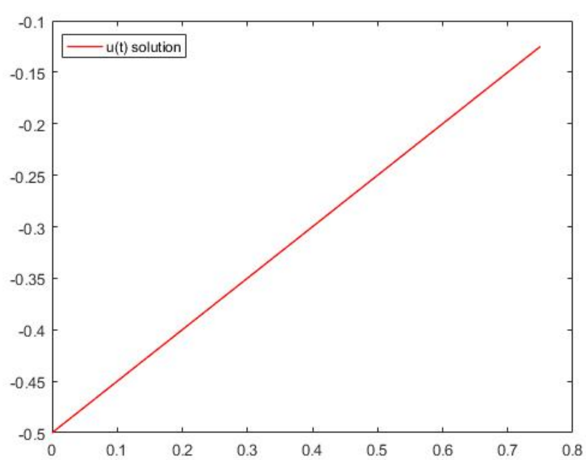

FIGURE 4. $u(t)$ solution

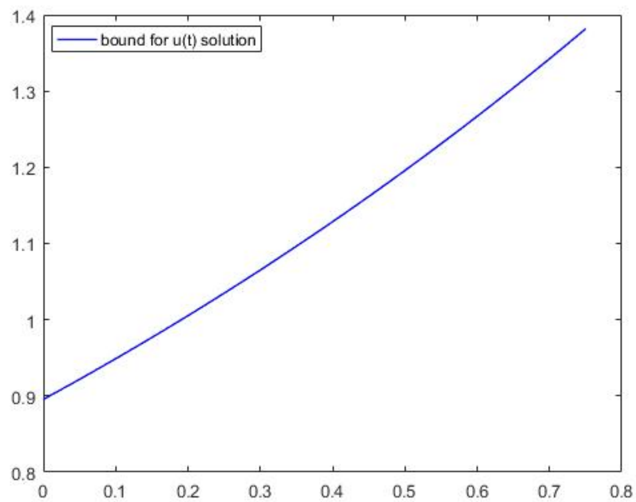

FIGURE 5. $\bar{u}(t) = 0.8955 \times exp(0.5783t)$

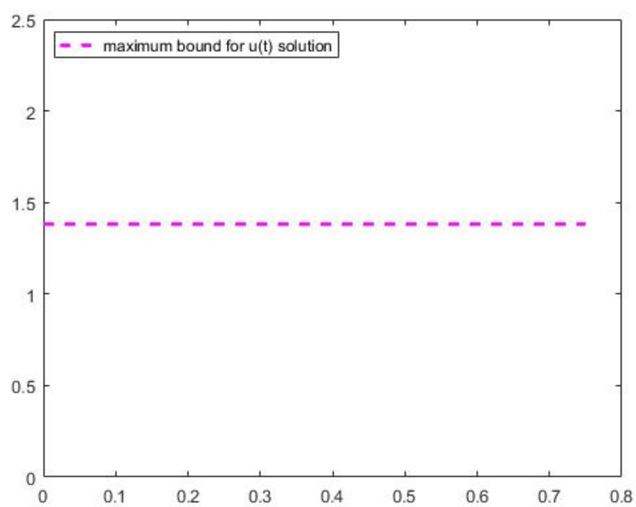

FIGURE 6. Maximum bound for the solution

Figure (1)- Figure (6) show that as t values increase, the bound of the solution expands.

4. Conclusion

This work presented the stability inequalities in respect to the right-side and initial conditions for the second-order neutral Volterra integro-differential equation. We showed that the bound of solution expressed by the inequality (3). Theoretical results are supported with examples.

Author Contribution Statements All authors contributed equally to the writing of this paper.

Declaration of Competing Interests The authors declare that they have no competing interest Author's contributions. All authors read and approved the final manuscript.

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