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SPECTRAL PROPERTIES OF A FUNCTIONAL BINOMIAL MATRIX

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Abstract. In this article, we consider the definition of the Fibonacci polynomial sequence with the second-order linear recurrence relation, where coefficients and initial conditions depend on the variable t. And then, we introduce the functional binomial matrix depending on the coefficients of the second-order linear recurrence relation. In the following, we study the spectral properties of the functional binomial matrix using the Fibonacci polynomial sequence and we obtain a diagonal decomposition for it using the Vandermunde matrix. Finally, by applying some linear algebra tools we obtain a number of combinatorial identities involving the Fibonacci polynomial sequence.

1. Introduction

The Fibonacci sequence and the Lucas sequence are among the most well-known second-order linear recurrence sequences that are of particular importance in number theory and combinatorics (see [17]):

$$F_n = F_{n-1} + F_{n-2}, \qquad F_0 = 0, \quad F_1 = 1, \tag{1}$$

$$L_n = L_{n-1} + L_{n-2}, \qquad L_0 = 2, \quad L_1 = 1. \tag{2}$$

Usually, second-order linear recurrence relations are generalized with two ideas, first by preserving the recurrence relation and second by preserving the initial conditions. The most prominent examples of Fibonacci-Like sequences are given as follows:

- The Jacobsthal sequence [11] is defined by the recurrence relation

$$J_n = J_{n-1} + J_{n-2} \quad (n \geq 2), \qquad J_0 = 1, \quad J_1 = 1. \tag{3}$$

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- The Jacobsthal-Lucas sequence [11] is defined by the recurrence relation
$$j_n = j_{n-1} + 2j_{n-2} \quad (n \geq 2), \qquad j_0 = 2, \quad j_1 = 1. \tag{4}$$
- Singh et al. [26] defined Fibonacci-Like sequence
$$S_n = S_{n-1} + S_{n-2} \quad (n \geq 2), \qquad S_0 = 2, \quad S_1 = 2. \tag{5}$$
- Horadam [11], Kalman [14], Stanimirović [27] and Gupta [10] generalized the Fibonacci sequence by considering a new initial condition and a new recurrence relation:
$$F_n = AF_{n-1} + BF_{n-2} \quad (n \geq 2), \qquad F_0 = a, \quad F_1 = b, \tag{6}$$
where A, B, a and b are positive integers.

A natural way to generalize the Fibonacci sequence is to use the Fibonacci polynomials. For over a century, both Fibonacci and Lucas polynomials have appeared in the literature in the study of several subjects such as algebra, geometry, combinatorics, approximation theory, statistics, and number theory [23]. Fibonacci polynomials were studied in 1883 by Catalan and Jacobsthal [8, 13]. Many works dealt with different properties of these polynomials and their applications. Fibonacci polynomials appear in different frameworks. Fibonacci polynomials are special cases of Chebyshev polynomials and have been studied on a more advanced level by many mathematicians. Large classes of Fibonacci-Like polynomials can be defined with the help of recurrence relations and the properties of the resulting Fibonacci numbers can be studied [17].

The most prominent examples of Fibonacci polynomials sequences are given as follows:

- The polynomials $F_n(t)$ studied by Catalan [8] are defined by the recurrence relation:
$$F_n(t) = tF_{n-1}(t) + F_{n-2}(t) \quad (n \geq 2), \qquad F_0(t) = 1, \quad F_1(t) = t. \tag{7}$$
- The Fibonacci polynomials studied by Jacobsthal [13] were defined by
$$J_n(t) = J_{n-1}(t) + tJ_{n-2}(t) \quad (n \geq 2), \qquad J_0(t) = 1, \quad J_1(t) = 1. \tag{8}$$
- The Pell polynomials [12] are defined by
$$P_n(t) = 2tP_{n-1}(t) + P_{n-2}(t) \quad (n \geq 2), \qquad P_0(t) = 0, \quad P_1(t) = 1. \tag{9}$$
- The Lucas polynomials [5] are defined by
$$L_n(t) = tL_{n-1}(t) + L_{n-2}(t) \quad (n \geq 2), \qquad L_0(t) = 2, \quad L_1(t) = t. \tag{10}$$

Many authors have studied Fibonacci polynomials with different ideas [4, 19, 21, 22, 26]. But recently Kaygisiz and Sahin [15] have presented new generalizations of Lucas numbers with matrix representation using generalized Lucas polynomials. Also, Lee and Asci [18] have defined a new generalization of Fibonacci polynomial called (A, B)-Fibonacci polynomial with the help of Pascal matrix. They obtain combinatorial identities and, using Riordan's method, obtain Pascal matrix factorizations

including (A, B)-Fibonacci polynomials. In this paper, we present generalization of the Fibonacci and Lucas polynomials by changing the initial terms and the recurrence relation.
In [7], Carlits (for $a = b = 1$) and in [1], Akkuse studied the $(n+1) \times (n+1)$ matrix $\mathcal{B}_n = \left[a^{i+j-n} b^{n-j} \binom{i}{n-j}\right]_{0 \le i,j \le n}$, and derived many interesting results on spectral and powers of these matrices. In this paper, introducing a generalized functional matrix $\mathcal{B}_n[x(t), y(t)]$ which call it the generalized functional binomial matrix of two variables $x(t)$ and $y(t)$ (both variables are dependent on t), we find the eigenvalues, eigenvectors and characteristic polynomial of it. We also obtain a decomposition for the matrix $\mathcal{B}_n[x(t), y(t)]$ and some identities for the polynomials Fibonacci sequence.

Definition 1. *The functional binomial matrix of two variables of order* $(n+1) \times (n+1)$ *is defined by*

$$\mathcal{B}_n[x(t), y(t)] = \left[x(t)^{i+j-n} y(t)^{n-j} \binom{i}{n-j}\right]_{0 \le i,j \le n}. \tag{11}$$

Example 1. *The functional binomial matrix of two variables of order* 4×4 *is as follows*

$$\mathcal{B}_3[x(t), y(t)] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & y(t) & x(t) \\ 0 & y(t)^2 & 2x(t)y(t) & x(t)^2 \\ y(t)^3 & 3y(t)^2x(t) & 3y(t)x(t)^2 & x(t)^3 \end{pmatrix}.$$

In the following lemma, we can easily obtain a decomposition for the functional binomial matrix of two variables, considering $\mathcal{B}_n[x(t), 1] = \mathcal{B}_n[x(t)]$.

Lemma 1.

$$\mathcal{B}_n[x(t), y(t)] = \mathcal{B}_n[x(t)]\mathrm{diag}\big(y(t)^n, \cdots, y(t), 1\big).$$

For finding $\mathcal{B}_n^{-1}[x(t), y(t)]$, it is enough to find $\mathcal{B}_n^{-1}[x(t)]$. Now, consider the matrix $\tilde{I} = [\delta_{i,n-j}]_{0 \le i,j \le n}$, where $\delta_{i,n-j}$ is the Kronecker delta. It is easy to see that $\mathcal{B}_n[x(t)] = \mathcal{P}_n[x(t)]\tilde{I}_{n+1}$, where $\mathcal{P}_n[x(t)] = \left[\binom{i}{j} x(t)^{i-j}\right]_{0 \le i,j \le n}$ is the Pascal matrix with one variable, has the following properties (see [2, 3, 6, 16]):

(1) $\mathcal{P}_n[x(t)]\mathcal{P}_n[y(t)] = \mathcal{P}_n[x(t) + y(t)]$,
(2) $\mathcal{P}_n[x(t)]\mathcal{P}_n[-x(t)] = \mathcal{P}_n[0] = I_{n+1}$ namely $\mathcal{P}_n^{-1}[x(t)] = \mathcal{P}_n[-x(t)]$.

Therefore

$$\mathcal{B}_n^{-1}[x(t)] = \tilde{I}\mathcal{P}_n[-x(t)] = \left[\binom{n-i}{j}(-x(t))^{n-i-j}\right]_{0 \le i,j \le n}.$$

According to above topics, we present the inverse of the functional binomial matrix of two variables as follows $\mathcal{B}_n^{-1}\left[x(t), y(t)\right] = \left[(-x(t))^{n-i-j}(y(t))^{i-n}\binom{n-i}{j}\right]_{0 \le i,j \le n}$.

Example 2.

$$\mathcal{B}_4^{-1}[x(t),y(t)] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & y(t) & -x(t) \\ 0 & y(t)^2 & -2x(t)y(t) & x(t)^2 \\ y(t)^3 & -3y(t)^2x(t) & 3y(t)x(t)^2 & -x(t)^3 \end{pmatrix}.$$

2. The generalized Fibonacci polynomial and the functional binomial matrix

According to relations (1)-(5), a natural and general definition (6) can be presented, where coefficients and initial conditions are positive integers. Now, with the same idea and according to the recurrence relations (8)-(10), the following general definition can be presented, where the coefficients of the recursive and initial relation are considered as polynomials with integer coefficients.

Definition 2. *Let $A(t), B(t), a(t)$ and $b(t)$ be polynomials with integer coefficients. The generalized Fibonacci polynomials $\{F_n\big(a(t), b(t); A(t), B(t)\big)\}_{n\geq 0}$ (we shall often drop the argument $\big(a(t), b(t); A(t), B(t)\big)$ and simply write $\{F_n(t)\}_{n\geq 0}$ are defined by the recurrence relation*

$$F_n(t) = A(t)F_{n-1}(t) + B(t)F_{n-2}(t) \quad (n \geq 2), \tag{12}$$

$$F_0(t) = a(t), \quad F_1(t) = b(t). \tag{13}$$

For easy notation, we shall sometimes write A, B, a, b for $A(t), B(t), a(t)$ and $b(t)$. We display some special cases of the sequence $\{F_n(t)\}_{n\geq 0}$, in Table 1.

Table 1. Some special cases of $\{F_n(t)\}_{n\geq 0}$

Polynomial Type	$F_n\big(a,b;A,B\big)$	$A(t)$	$B(t)$	$a(t)$	$b(t)$
generalized Fibonacci	$F_n(t)$	t	1	1	t
generalized Lucas	$L_n(t)$	t	1	2	t
generalized Pell	$P_n(t)$	$2t$	1	0	1
Jacobsthal	$J_n(t)$	1	t	1	1
1st kind Chebyshev	$T_n(t)$	$2t$	-1	1	t
2nd kind Chebyshev	$U_n(t)$	$2t$	-1	1	$2t$
3th kind Chebyshev	$V_n(t)$	$2t$	-1	1	$2t-1$
4th kind Chebyshev	$W_n(t)$	$2t$	-1	1	$2t+1$

Theorem 1. *The non-degenerated second-order recurrent sequence $F_n(t)$, defined in* (12)*, satisfies the following generalization of the Binet formula*

$$F_n(t) = \left(\frac{b - a\beta}{\alpha - \beta}\right)\alpha^n + \left(\frac{a\alpha - b}{\alpha - \beta}\right)\beta^n \quad (n \geq 0), \tag{14}$$

where α and β are the roots of the characteristic equation $\lambda^2 - A\lambda - B = 0$.

Corollary 1. *For $a = 0$ and $b \neq 0$, we have*

$$F_n(t) = \frac{b\big(\alpha^n - \beta^n\big)}{\alpha - \beta} \quad (n \geq 0), \tag{15}$$

and for $a \neq 0$ and $b = 0$, we have

$$F_n(t) = \frac{-a\alpha\beta\big(\alpha^{n-1} - \beta^{n-1}\big)}{\alpha - \beta} \quad (n \geq 0), \tag{16}$$

and also for $b = ka$ where k is a non-zero fixed number, we have

$$F_n(t) = \frac{a\big[\alpha^n - \beta^n - k\alpha\beta\big(\alpha^{n-1} - \beta^{n-1}\big)\big]}{\alpha - \beta} \quad (n \geq 0). \tag{17}$$

Corollary 2. *For $n \geq 1$ and $k \geq 0$, we have*

$$F_{k(n+1)}(t) = \mathcal{A}_k F_{kn}(t) - (-B)^k F_{k(n-1)}(t),$$

where $\mathcal{A}_k$ satisfy $\mathcal{A}_{k+1} = A\mathcal{A}_k + B\mathcal{A}_{k-1}$ with the boundary conditions $\mathcal{A}_0 = 2$ and $\mathcal{A}_1 = A$.

Proof. By the Binet formula (14) and since $\mathcal{A}_k = \alpha^k + \beta^k$ and $\alpha\beta = -B$, we have

$$\begin{aligned}
\mathcal{A}_k F_{kn}(t) - (-B)^k F_{k(n-1)}(t) = \\
&= \big(\alpha^k + \beta^k\big)\left[\left(\frac{b - a\beta}{\alpha - \beta}\right)\alpha^{kn} + \left(\frac{a\alpha - b}{\alpha - \beta}\right)\beta^{kn}\right] \\
&\quad - \big(\alpha\beta\big)^k\left[\left(\frac{b - a\beta}{\alpha - \beta}\right)\alpha^{k(n-1)} + \left(\frac{a\alpha - b}{\alpha - \beta}\right)\beta^{k(n-1)}\right] \\
&= \left(\frac{b - a\beta}{\alpha - \beta}\right)\alpha^{k(n+1)} + \left(\frac{a\alpha - b}{\alpha - \beta}\right)\beta^{k(n+1)} \\
&= F_{k(n+1)}(t).
\end{aligned}$$

□

The following theorem is the main result of this paper which gives the relation of the characteristic polynomial of the generalized binomial matrix of two variables $\mathcal{B}_n[A, B]$ with the generalized Fibonacci sequence $\big\{F_n(t)\big\}_{n\geq 0}$.

Theorem 2. *If $\Big(F_\ell(t)^{n-i}(t)F_{\ell+1}^i(t)\Big)_{0\leq i\leq n}$ be a column vector of $(n+1)$-dimension, then*

$$\mathcal{B}_n[A, B]\Big(F_\ell(t)^{n-i}F_{\ell+1}^i(t)\Big)_{0\leq i\leq n} = \Big(F_{\ell+1}^{n-i}(t)F_{\ell+2}^i(t)\Big)_{0\leq i\leq n}. \tag{18}$$

Proof. Let $\mathcal{B}_n[A,B]\Big(F_\ell^{n-i}(t)F_{\ell+1}^i(t)\Big)_{0\le i\le n} = [a_i]$, we have

$$\begin{aligned} a_i &= \sum_{k=0}^{n}\binom{i}{n-k}A^{i+k-n}B^{n-k}F_\ell^{n-k}(t)F_{\ell+1}^k(t) \\ &= F_{\ell+1}^{n-i}(t)\sum_{k=n-i}^{n}\binom{i}{n-k}\big(BF_\ell(t)\big)^{n-k}\big(AF_{\ell+1}(t)\big)^{i+k-n}, \end{aligned}$$

which substituting $r = k-n+i$, we obtain

$$\begin{aligned} a_i &= F_{\ell+1}^{n-i}(t)\sum_{r=0}^{i}\binom{i}{r}\Big(BF_\ell(t)\Big)^{i-r}\Big(AF_{\ell+1}(t)\Big)^{r} \\ &= F_{\ell+1}^{n-i}(t)\Big(BF_\ell(t)+AF_{\ell+1}(t)\Big)^{i} \\ &= F_{\ell+1}^{n-i}(t)F_{\ell+2}^{i}(t). \end{aligned}$$

□

Example 3.

$$\begin{aligned} \mathcal{B}_3[A,B]\Big(F_\ell^{3-i}(t)F_{\ell+1}^i(t)\Big)_{0\le i\le 3} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & B & A \\ 0 & B^2 & 2AB & A^2 \\ B^3 & 3B^2A & 3BA^2 & A^3 \end{pmatrix}\begin{pmatrix} F_\ell^3(t) \\ F_\ell^2(t)F_{\ell+1}(t) \\ F_\ell(t)F_{\ell+1}^2(t) \\ F_{\ell+1}^3(t) \end{pmatrix} \\ &= \begin{pmatrix} F_{\ell+1}^3(t) \\ F_{\ell+1}^2(t)F_{\ell+2}(t) \\ F_{\ell+1}(t)F_{\ell+2}^2(t) \\ F_{\ell+2}^3(t) \end{pmatrix}. \end{aligned}$$

Corollary 3.

$$F_{\ell+1}^{n-i}(t)F_{\ell+2}^i(t) = \sum_{i_1,\cdots,i_\ell}\binom{i}{n-i_1}\binom{i_1}{n-i_2}\cdots\binom{i_{\ell+1}}{n-i_\ell}A^{i+i_\ell-n\ell+2\sum_{r=1}^{\ell-1}i_r}B^{n\ell-\sum_{r=1}^{\ell}i_r}a^{n-i_\ell}b^{i_\ell}.$$

Proof. By induction on ℓ and using (18), we have

$$\mathcal{B}_n^\ell[A,B]\Big(a^{n-i}b^i\Big)_{0\le i\le n} = \Big(F_{\ell+1}^{n-i}(t)F_{\ell+2}^i(t)\Big)_{0\le i\le n}.$$

Now, if we consider the i-th rows, we get

$$F_{\ell+1}^{n-i}(t)F_{\ell+2}^i(t) = \Big(\mathcal{B}_n^\ell[A,B]\big(a^{n-s}b^s\big)_{s=0}^n\Big)_i = \sum_{i_1,\cdots,i_\ell}a_{i,i_1}\cdots a_{i_{\ell-1},i_\ell}a^{n-i_\ell}b^{i_\ell}$$

$$= \sum_{i_1,\cdots,i_\ell} \binom{i}{n-i_1}\binom{i_1}{n-i_2}\cdots\binom{i_{\ell-1}}{n-i_\ell} A^{i+i_\ell-n\ell+2\sum_{r=1}^{\ell-1} i_r} B^{n\ell-\sum_{r=1}^{\ell} i_r} a^{n-i_\ell} b^{i_\ell}.$$

□

The matrix $\left[F_j^{n-i}(t)F_{j+1}^i(t)\right]_{0\le i,j\le n}$ is invertible.

Proof. If we divide the j-th column by $F_{j+1}^n(t)$, we obtain the Vandermonde matrix $\left[\left(\frac{F_j(t)}{F_{j+1}(t)}\right)^{n-i}\right]$ which has nonzero determinant. □

Theorem 3. *For the sequence* $\left\{F_n(t)\right\}_{n\ge 0}$ *and* $k\ge 2$, *we have*

$$\begin{aligned}&\left(xF_k(t)+BF_{k-1}(t)\right)^r\left(xF_{k+1}(t)+BF_k(t)\right)^{n-r}\\ &\qquad= \sum_{r_0,r_1,\cdots,r_k}\binom{r}{r_0}\binom{n-r}{r_1}\cdots\binom{n-r_{k-1}}{r_k} A^{(k-1)n-r_0-r_1-2\sum_{\ell=2}^k r_\ell - r_{k+1}}\\ &\qquad\qquad \times B^{\sum_{\ell=0}^k r_\ell} a^{r_0} b^{r-r_0+r_1}\left(Ab+Ba\right)^{n-r_1-r} x^{n-r_k}.\end{aligned}\tag{19}$$

Proof. Using the binomial expansion, we have

$$\begin{aligned}&\left(xF_1(t)+BF_0(t)\right)^r\left(xF_2(t)+BF_1(t)\right)^{n-r}\\ &\qquad= \sum_{r_0,r_1}\binom{r}{r_0}\binom{n-r}{r_1} B^{r_0+r_1} a^{r_0} b^{r-r_0+r_1}\left(Ab+Ba\right)^{n-r_1-r} x^{n-r_0-r_1}.\end{aligned}\tag{20}$$

For all integers $k\ge 2$, we prove equality (19) by induction. For $k=2$, in (20), we replace x by $A+Bx^{-1}$ and multiply the result by x^n, and the conclusion is obtained. Assuming that (19) holds for the value k, we replace x by $A+Bx^{-1}$ and multiply the result by x^n. The left side of the formula is as follows

$$\begin{aligned}&\left(AF_k(t)x+BF_{k-1}(t)x+BF_k(t)\right)^r\left(AF_{k+1}(t)x+BF_k(t)x+BF_{k+1}(t)\right)^{n-r}\\ &\qquad= \left(F_{k+1}(t)x+BF_k(t)\right)^r\left(F_{k+2}(t)x+BF_{k+1}(t)\right)^{n-r},\end{aligned}$$

the right side of the formula is as follows

$$\begin{aligned}&\sum_{r_0,r_1,\cdots,r_{k+1}}\binom{r}{r_0}\binom{n-r}{r_1}\cdots\binom{n-r_{k-1}}{r_k}\binom{n-r_k}{r_{k+1}}\\ &\qquad \times A^{kn-r_0-r_1-2\sum_{\ell=2}^k r_\ell - r_{k+1}} B^{\sum_{\ell=0}^{k+1} r_\ell} a^{r_0} b^{r-r_0+r_1}\left(Ab+Ba\right)^{n-r_1-r} x^{n-r_{k+1}}.\end{aligned}$$

This evidently completes the proof of (19). □

Corollary 4. *For* $k\ge 2$, *we have*

$$\begin{aligned}&\left(xF_k(t)+BF_{k-1}(t)\right)^r\left(xF_{k+1}(t)+BF_k(t)\right)^{n-r}\\ &\qquad= \sum_{r_1,\cdots,r_k}\binom{n-r}{r_1}\cdots\binom{n-r_{k-1}}{r_k} A^{(k+1)n-2\sum_{\ell=1}^{k-1} r_\ell - r} B^{\sum_{\ell=1}^k r_\ell} x^{n-r_k},\end{aligned}\tag{21}$$

where $F_0(t) = 0$ *and* $F_1(t) = A$.

Lemma 2. *For all* $k \geq 1$, *we have*

$$\operatorname{tr}\left(\mathcal{B}_n^k[A,B]\right) = \frac{F_{k(n+1)}(t)}{F_k(t)}, \tag{22}$$

where $F_0(t) = 0$ *and* $F_1(t) = A$.

Proof. We multiply (21) by x^r and sum over r. This gives

$$\sum_{r=0}^{n} x^r \big(xF_k(t) + BF_{k-1}(t)\big)^r \big(xF_{k+1}(t) + BF_k(t)\big)^{n-r} \tag{23}$$
$$= \sum_{r,r_1,\cdots,r_k} \binom{n-r}{r_1} \cdots \binom{n-r_{k-1}}{r_k} A^{(k+1)n-2\sum_{\ell=1}^{k-1} r_\ell - r} B^{\sum_{\ell=1}^{k} r_\ell} x^{n-r_k+r}.$$

The coefficient of x^n on the right of (23) is $\operatorname{tr}\left(\mathcal{B}_n^k[A,B]\right)$ and the coefficient of x^n on the left of (23) is

$$\sum_{r+s+u=n} \binom{r}{s}\binom{n-r}{u} \big(BF_{k-1}(t)\big)^{r-s} \big(F_k(t)\big)^s \big(BF_k(t)\big)^{n-r-u} \big(F_{k+1}(t)\big)^u$$
$$= \sum_{r+s\leq n} B^r \binom{r}{s}\binom{n-r}{s} F_{k-1}^{r-s}(t) F_k^{2s}(t) F_{k+1}^{n-r-s}(t) = c_n^k.$$

Then

$$\sum_{n=0}^{\infty} c_n^k x^n = \sum_{r,s=0}^{\infty} \binom{r}{s} B^r F_{k-1}^{r-s}(t) F_k^{2s}(t) x^{r+s} \sum_{n=r+s} \binom{n-r}{s} \big(F_{k+1}(t)x\big)^{n-r-s}$$
$$= \sum_{r,s=0}^{\infty} B^r \binom{r}{s} F_{k-1}^{r-s}(t) F_k^{2s}(t) x^{r+s} \big(1 - F_{k+1}(t)x\big)^{-s-1}$$
$$= \sum_{s=0}^{\infty} B^s F_k^{2s}(t) \big(1 - F_{k+1}(t)x\big)^{-s-1} \sum_{r\geq 0} \binom{r}{s} \big(F_{k-1}(t)x\big)^{r+s}$$
$$= \sum_{s=0}^{\infty} B^s F_k^{2s}(t) \big(1 - F_{k+1}(t)x\big)^{-s-1} \big(1 - F_{k-1}(t)x\big)^{-s-1}$$
$$= \frac{1}{\big(1 - F_{k+1}(t)x\big)\big(1 - BF_{k-1}(t)x\big)} \times \frac{1}{1 - \frac{BF_k^2(t)x^2}{\left(1-F_{k+1}(t)x\right)\left(1-BF_{k-1}(t)x\right)}}$$
$$= \frac{1}{\big(1 - F_{k+1}(t)x\big)\big(1 - BF_{k-1}(t)x\big) - BF_k^2(t)x^2}.$$

Here by the Binet formula (15), we have

$$\sum_{n=0}^{\infty} c_n^k x^n = \frac{1}{1 - \big(\alpha^k + \beta^k\big)x + (\alpha\beta)^k x^2}$$

$$= \frac{1}{\big(1-\alpha^k x\big)\big(1-\beta^k x\big)}$$
$$= \frac{1}{\alpha^k-\beta^k}\left(\frac{\alpha^k}{1-\alpha^k x}-\frac{\beta^k}{1-\beta^k x}\right).$$

It follows that

$$c_n^k = \frac{\alpha^{k(n+1)}-\beta^{k(n+1)}}{\alpha^k-\beta^k} = \frac{F_{k(n+1)}(t)}{F_k(t)}.$$

□

Theorem 4. *The eigenvalues of $\mathcal{B}_n[A,B]$ are*

$$\alpha^n, \alpha^{n-1}\beta, \cdots, \alpha\beta^{n-1}, \beta^n,$$

and the characteristic polynomial of $\mathcal{B}_n[A,B]$ is

$$\chi_n(\tau) = \prod_{i=0}^{n}\Big(\tau-\alpha^i\beta^{n-i}\Big).$$

Proof. Let $\chi_{n+1}(\tau) = \det\big(\tau I_{n+1}-\mathcal{B}_n[A,B]\big)$ and $\lambda_0, \lambda_1, \cdots, \lambda_n$ denote the eigenvalues of $\mathcal{B}_n[A,B]$. Then by Lemma 2,

$$\begin{aligned}
\frac{\chi'_{n+1}(\tau)}{\chi_{n+1}(\tau)} &= \sum_{k=0}^{n}\frac{1}{\tau-\lambda_\ell} = \sum_{k=0}^{\infty}\tau^{-k-1}\sum_{j=0}^{k}\lambda_j^k \\
&= \sum_{k=0}^{\infty}\tau^{-k-1}\mathrm{tr}\big(\mathcal{B}_n^\ell[A,B]\big)= \sum_{k=0}^{\infty}\tau^{-k-1}\frac{F_{k(n+1)}(t)}{F_k(t)} \\
&= \sum_{k=0}^{\infty}\tau^{-k-1}\frac{\alpha^{k(n+1)}-\beta^{k(n+1)}}{\alpha^k-\beta^k} = \sum_{k=0}^{\infty}\tau^{-k-1}\sum_{j=0}^{n}\alpha^{jk}\beta^{(n-j)k} \\
&= \sum_{j=0}^{n}\frac{1}{\tau-\alpha^j\beta^{n-j}}.
\end{aligned}$$

It follows that

$$\chi_{n+1}(\tau) = \prod_{j=0}^{n}\big(\tau-\alpha^j\beta^{n-j}\big).$$

□

Theorem 5. *For $a=0$ and $b\neq 0$, we have*

$$\chi_{n+1}(\tau) = \sum_{\ell=0}^{n+1}(-1)^{\frac{\ell(\ell+1)}{2}}B^{\frac{\ell(\ell-1)}{2}}\begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{F_n(t)}\tau^{n+1-\ell},$$

where $\left[\begin{smallmatrix} n+1 \\ \ell \end{smallmatrix}\right]_{F_n(t)}$ *is defined as*

$$\begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{F_n(t)} = \begin{cases} 1, & \ell = 0, n+1; \\ \frac{F_{n+1}(t)F_n(t)\cdots F_{n-\ell+2}(t)}{F_1(t)F_2(t)\cdots F_\ell(t)}, & 0 < \ell < n+1. \end{cases}$$

Proof. We use the following identity (see [20])

$$\prod_{j=0}^{n}(1 - q^j\tau) = \sum_{\ell=0}^{n+1}(-1)^\ell q^{\frac{\ell(\ell-1)}{2}} \begin{bmatrix} n+1 \\ \ell \end{bmatrix}_q \tau^\ell,$$

where $\left[\begin{smallmatrix} n+1 \\ \ell \end{smallmatrix}\right]_q$ is the q-binomial coefficient (Gaussian binomial), and is defined by

$$\begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-r+1})}{(1-q)(1-q^2)\cdots(1-q^r)},$$

where m and r are non-negative integers. If $r > m$, this evaluates to 0 and for $r = 0, m$, the value is 1.
Replacing q in the above equation by $\frac{\beta}{\alpha}$ and using the Binet formula (15), we have

$$\begin{bmatrix} n+1 \\ \ell \end{bmatrix}_q = \alpha^{\ell^2-(n+1)\ell} \begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{F_n(t)}.$$

Therefore

$$\prod_{j=0}^{n}(1 - \alpha^{-j}\beta^j\tau) = \sum_{\ell=0}^{n+1}(-1)^\ell \alpha^{\frac{\ell(\ell-1)}{2}} \beta^{\frac{\ell(\ell-1)}{2}-n\ell} \begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{F_n(t)} \tau^\ell.$$

Substituting τ by $\alpha^n\tau^{-1}$ and using $\alpha\beta = -B$, we get

$$\prod_{j=0}^{n}(\tau - \alpha^{n-j}\beta^j) = \sum_{\ell=0}^{n+1}(-1)^{\frac{\ell(\ell+1)}{2}} B^{\frac{\ell(\ell-1)}{2}} \begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{F_n(t)} \tau^{n+1-\ell},$$

which is the desired result. □

Example 4. *The characteristic polynomials of* $\chi_{n+1}(\tau)$ *for* $n = 0, 1, 2$ *are*

$$\begin{aligned} \chi_1(\tau) &= \tau - 1 \\ \chi_2(\tau) &= \tau^2 - A\tau - B \\ \chi_3(\tau) &= \tau^3 - (B + A^2)\tau^2 - (A^2B + B^2)\tau + B^3. \end{aligned}$$

3. Diagonalization of the Functional Binomial Matrix

The results of this section are for a specific case of the recurrence relation (12) with (13) for $a(t) = 0$, $b(t) = 1$ and coefficients $A(t)$ and $B(t)$ which are arbitrary functions of t.
Let $n \geq 1$ and $\mathcal{C}_n[A, B]$ be the companion matrix of the characteristic polynomial $\chi_n(\tau)$, where

$\mathcal{C}_n[A,B] = \big(c_{i,j}(A,B)\big),$

$$\begin{cases} c_{i,i+1}(A,B) = 1, & i = 0,1,\cdots,n-1; \\ c_{n,n-j}(A,B) = -(-1)^{\frac{(j+1)(j+2)}{2}} B^{\frac{j(j+1)}{2}} \left[{n+1 \atop j+1}\right]_{F_n(t)}, & j = 0,1,\cdots,n-1; \\ c_{i,j}(A,B) = 0, & \text{otherwise.} \end{cases}$$

$\mathcal{R}_n[A,B] = \big(r_{i,j}(A,B)\big)$ and $\mathcal{M}_n[A,B] = \big(m_{i,j}(A,B)\big),$

$$\begin{cases} r_{0,j}(A,B) = r_{1,j}(A,B) = \delta_{n,j}, \\ r_{i,j}(A,B) = \binom{n}{j}\big(BF_{i-1}(t)\big)^{n-j}\big(F_i(t)\big)^j, & i = 2,\cdots,n,\ j = 0,1,\cdots,n, \end{cases}$$

$$\begin{cases} m_{0,j}(A,B) = \delta_{n,j}, \\ m_{i,j}(A,B) = \binom{n}{j}\big(BF_i(t)\big)^{n-j}\big(F_{i+1}(t)\big)^j, & i = 1,\cdots,n,\ j = 0,1,\cdots,n. \end{cases}$$

Lemma 3. *For every positive integer k, we have*

$$\Big(\mathcal{B}_n^k[A,B]\Big)_{nj} = \binom{n}{j}\big(BF_k(t)\big)^{n-j}\big(F_{k+1}(t)\big)^j, \qquad j = 0,1,\cdots,n.$$

Proof. Let n be a fixed natural number. We will prove the assertion by induction on k. The above equality is valid for $k = 0$. Now assume the results is valid for $k > 0$. Then, since $\mathcal{B}_n^{k+1}[A,B] = \mathcal{B}_n^k[A,B]\mathcal{B}_n[A,B]$, we have

$$\begin{aligned} \big(\mathcal{B}_n^{k+1}[A,B]\big)_{nj} &= \sum_{i=0}^{n}\big(\mathcal{B}_n^k[A,B]\big)_{ni}\big(\mathcal{B}_n[A,B]\big)_{ij} \\ &= \sum_{i=0}^{n}\binom{n}{i}\big(BF_k(t)\big)^{n-i}\big(F_{k+1}(t)\big)^i\binom{i}{n-j}A^{i+j-n}B^{n-j} \\ &= \big(BF_{k+1}(t)\big)^{n-j}\sum_{i=0}^{n}\binom{n}{n-j}\binom{j}{i+j-n}\big(AF_{k+1}(t)\big)^{i-n+j}\big(BF_k(t)\big)^{n-i} \\ &= \binom{n}{j}\big(BF_{k+1}(t)\big)^{n-j}\sum_{i=0}^{n}\binom{j}{i+j-n}\big(AF_{k+1}(t)\big)^{i+j-n}\big(BF_k(t)\big)^{n-i} \\ &= \binom{n}{j}\big(BF_{k+1}(t)\big)^{n-j}\sum_{m=0}^{j}\binom{j}{m}\big(AF_{k+1}(t)\big)^{m}\big(BF_k(t)\big)^{j-m} \\ &= \binom{n}{j}\big(BF_{k+1}(t)\big)^{n-j}\big(AF_{k+1}(t) + BF_k(t)\big)^{j} \\ &= \binom{n}{j}\big(BF_{k+1}(t)\big)^{n-j}\big(F_{k+2}(t)\big)^{j}. \end{aligned}$$

□

Theorem 6. *Let $F_0(t) = 0$ and $F_1(t) = 1$. Then*

$$\sum_{\ell=0}^{n+1}(-1)^{\frac{\ell(\ell+1)}{2}}B^{\frac{\ell(\ell-1)}{2}}\begin{bmatrix}n+1\\ \ell\end{bmatrix}_{F_n(t)}\big(F_{n-\ell+1}(t)\big)^{n-j}\big(F_{n-\ell+2}(t)\big)^{j}=0.$$

Proof. The characteristic polynomials of $\mathcal{B}_n[A,B]$ is

$$\sum_{\ell=0}^{n+1}(-1)^{\frac{\ell(\ell+1)}{2}}B^{\frac{\ell(\ell-1)}{2}}\begin{bmatrix}n+1\\ \ell\end{bmatrix}_{F_n(t)}\tau^{n+1-\ell}=0.$$

Now by the Cayley-Hamilton Theorem [24], we get

$$\sum_{k=0}^{n+1}(-1)^{\frac{\ell(\ell+1)}{2}}B^{\frac{\ell(\ell-1)}{2}}\begin{bmatrix}n+1\\ \ell\end{bmatrix}_{F_n(t)}\mathcal{B}_n^{n-\ell+l}[A,B]=0, \tag{24}$$

where 0 denotes the $(n+1)\times(n+1)$ zero matrix. So by Lemma 3 and substituting this result into (24), we obtain

$$\sum_{\ell=0}^{n+1}(-1)^{\frac{\ell(\ell+1)}{2}}B^{\frac{\ell(\ell-1)}{2}}\begin{bmatrix}n+1\\ \ell\end{bmatrix}_{F_n(t)}\Big(\mathcal{B}_n^{n-\ell+1}[A,B]\Big)_{nj}=0.$$

Therefore

$$\sum_{\ell=0}^{n+1}(-1)^{\frac{\ell(\ell+1)}{2}}B^{\frac{\ell(\ell-1)}{2}}\begin{bmatrix}n+1\\ \ell\end{bmatrix}_{F_n(t)}\big(F_{n-\ell+1}(t)\big)^{n-j}\big(F_{n-\ell+2}(t)\big)^{j}=0.$$

□

Theorem 7. *Let $a(t) = 0$ and $b(t) = 1$. For all n, we have*

$$\mathcal{M}_n[A,B]=\mathcal{C}_n[A,B]\mathcal{R}_n[A,B]=\mathcal{R}_n[A,B]\mathcal{B}_n[A,B],$$

and so

$$\mathcal{B}_n[A,B]=\mathcal{R}_n^{-1}[A,B]\mathcal{C}_n[A,B]\mathcal{R}_n[A,B].$$

Proof. At first, we prove $\mathcal{M}_n[A,B] = \mathcal{C}_n[A,B]\mathcal{R}_n[A,B]$. In fact, multiplying the first n rows of $\mathcal{C}_n[A,B]$ by $\mathcal{R}_n[A,B]$, clearly we get the first n rows of $\mathcal{M}_n[A,B]$. For the last row, for each $0 \le j \le n$, we have

$$\begin{aligned}
&\big(\mathcal{C}_n[A,B]\mathcal{R}_n[A,B]\big)_{nj}=\\
&\quad=\sum_{k=0}^{n}\big(\mathcal{C}_n[A,B]\big)_{n,n-k}\big(\mathcal{R}_n[A,B]\big)_{n-k,j}\\
&\quad=\sum_{k=0}^{n}-(-1)^{\frac{(k+1)(k+2)}{2}}B^{\frac{k(k+1)}{2}}\begin{bmatrix}n+1\\ k+1\end{bmatrix}_{F_n(t)}\binom{n}{j}\big(BF_{n-k-1}(t)\big)^{n-j}\big(F_{n-k}(t)\big)^{j}\\
&\quad=\binom{n}{j}B^{n-j}\sum_{\ell=1}^{n+1}-(-1)^{\frac{\ell(\ell+1)}{2}}B^{\frac{\ell(\ell-1)}{2}}\begin{bmatrix}n+1\\ \ell\end{bmatrix}_{F_n(t)}\big(F_{n-\ell}(t)\big)^{n-j}\big(F_{n-\ell+1}(t)\big)^{j}
\end{aligned}$$

$$= \binom{n}{j} B^{n-j} \Bigg(\big(F_n(t)\big)^{n-j} \big(F_{n+1}(t)\big)^j + \sum_{\ell=0}^{n+1} -(-1)^{\frac{\ell(\ell+1)}{2}} B^{\frac{\ell(\ell-1)}{2}} \begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{F_n(t)} \big(F_{n-\ell}(t)\big)^{n-j} \big(F_{n-\ell+1}(t)\big)^j \Bigg)$$
$$= \binom{n}{j} \big(BF_n(t)\big)^{n-j} \big(F_{n+1}(t)\big)^j,$$

which is clearly true by Theorem 6. This proves,

$$\mathcal{M}_n[A,B] = \mathcal{C}_n[A,B]\mathcal{R}_n[A,B].$$

Since for each i, j with $0 \le i \le j \le n$, we have

$$\begin{aligned}
\big(\mathcal{R}_n[A,B]\mathcal{B}_n[A,B]\big)_{ij} &= \sum_{k=0}^{n} \big(\mathcal{R}_n[A,B]\big)_{ik} \big(\mathcal{B}_n[A,B]\big)_{kj} \\
&= \sum_{k=0}^{n} \binom{n}{k} \big(BF_{i-1}(t)\big)^{n-k} \big(F_i(t)\big)^k A^{k+j-n} B^{n-j} \binom{k}{n-j} \\
&= \binom{n}{j} \sum_{k=0}^{n} \binom{j}{n-k} A^{k+j-n} B^{2n-j-k} \big(F_{i-1}(t)\big)^{n-k} \big(F_i(t)\big)^k \\
&= \binom{n}{j} \big(BF_i(t)\big)^{n-j} \sum_{\ell=0}^{j} \binom{j}{\ell} \big(BF_{i-1}(t)\big)^{\ell} \big(AF_i(t)\big)^{n-\ell} \\
&= \binom{n}{j} \big(BF_i(t)\big)^{n-j} \big(BF_{i-1}(t) + AF_i(t)\big)^j \\
&= \binom{n}{j} \big(BF_i(t)\big)^{n-j} \big(F_{i+1}(t)\big)^j \\
&= \big(\mathcal{M}_n[A,B]\big)_{ij},
\end{aligned}$$

we get $\mathcal{M}_n[A,B] = \mathcal{R}_n[A,B]\mathcal{B}_n[A,B]$. □

Example 5.

$$\mathcal{M}_3[A,B] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ B^3 & 3B^2A & 3BA^2 & A^3 \\ A^3B^3 & 3B^2A^2(B+A^2) & 3BA(B+A^2)^2 & (B+A^2)^3 \\ B^3(B+A^2)^3 & 3B^2A(B+A^2)^2(2B+A^2) & 3BA^2(B+A^2)(2B+A^2)^2 & A^3(2B+A^2)^3 \end{pmatrix},$$

$$\mathcal{C}_3[A,B] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -B^6 & -B^3A(2B+A^2) & B(B+A^2)(2B+A^2) & (2B+A^2)A \end{pmatrix},$$

$$\mathcal{R}_3[A,B] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ B^3 & 3B^2A & 3BA^2 & A^3 \\ A^3B^3 & 3B^2A^2(B+A^2) & 3BA(B+A^2)^2 & (B+A^2)^3 \end{pmatrix},$$

and so

$$\mathcal{M}_3[A,B] = \mathcal{C}_3[A,B]\mathcal{R}_3[A,B].$$

Also,

$$\mathcal{R}_3[A,B] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ B^3 & 3B^2A & 3BA^2 & A^3 \\ A^3B^3 & 3B^2A^2(B+A^2) & 3BA(B+A^2)^2 & (B+A^2)^3 \end{pmatrix},$$

$$\mathcal{B}_3[A,B] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & B & 1 \\ 0 & B^2 & 2BA & A^2 \\ B^3 & 3B^2A & 3BA^2 & A^3 \end{pmatrix},$$

and therefore $\mathcal{M}_3[A,B] = \mathcal{R}_3[A,B]\mathcal{B}_3[A,B]$.

Let $\mathcal{V}_n$ be the Vandermonde matrix which is defined by

$$\mathcal{V}_n = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \alpha^n & \alpha^{n-1}\beta & \cdots & \alpha\beta^{n-1} & \beta^n \\ \alpha^{2n} & (\alpha^{n-1}\beta)^2 & \cdots & (\alpha\beta^{n-1})^2 & \beta^{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha^{n^2} & (\alpha^{n-1}\beta)^n & \cdots & (\alpha\beta^{n-1})^n & \beta^{n^2} \end{pmatrix}.$$

By the relation between the component matrix and the Vandermonde matrix, we can obtain Theorems 8 and 9. For this purpose, we need the following lemma.

Lemma 4 ([24], P. 4)**.** *If M be the following matrix*

$$M = \begin{pmatrix} 0 & m_1 & 0 & \cdots & 0 \\ 0 & 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & m_{n-1} \\ p_1 & p_2 & p_3 & \cdots & p_n \end{pmatrix},$$

then its eigenvalues are the roots of $p_1 + p_2\lambda + \cdots + p_n\lambda^{n-1} = \lambda^n$ and $v_1 = \big(\alpha, \alpha\lambda, , \alpha\lambda^2, \cdots, \alpha\lambda^{n-1}\big)^T$ is an eigenvector for the root λ.

Theorem 8. *Let $a(t) = 0$ and $b(t) = 1$. Eigenvectors of the matrix $\mathcal{C}_n[A, B]$ are $\mathcal{V}_n$, and also eigenvectors of the matrix $\mathcal{B}_n[A, B]$ are $E_n[A, B] = \mathcal{R}_n^{-1}[A, B]\mathcal{V}_n$.*

Proof. According to Lemma 4, columns of $\mathcal{V}_n$ are eigenvectors of $\mathcal{C}_n[A, B]$. □

Theorem 9. *For $a(t) = 0$ and $b(t) = 1$, we have*

$$\Big(\mathcal{R}_n^{-1}[A, B]\mathcal{V}_n\Big)^{-1}\mathcal{B}_n[A, B]\Big(\mathcal{R}_n^{-1}[A, B]\mathcal{V}_n\Big) = \mathrm{diag}\big(\alpha^n, \alpha^{n-1}\beta, \cdots, \alpha\beta^{n-1}, \beta^n\big).$$

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