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On approximating fixed points of a new class of generalized nonexpansive mappings in uniformly convex hyperbolic space

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ABSTRACT

In this paper, we introduce the definition of a new class of generalized nonexpansive mappings in hyperbolic space. Additionally, we construct the rewritten version of the Mann iteration process in hyperbolic space. Then, using the iterative procedure we established, we prove convergence theorems for $a-b-$generalized nonexpansive mappings in a uniformly convex hyperbolic space. Lastly, we offer a numerical example to illustrate our findings.

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1. Introduction

In order to solve practical issues in mathematics, physics, engineering, and game theory, fixed point theory is a useful area of study. Analytical solutions of fixed point problems are challenging, requiring iterative solutions. Although academics create a variety of strategies, the development on effective algorithms is still underway.

In the field of nonlinear analysis, the fixed point theory is crucial. The Picard iteration, as defined by $\varkappa_{n+1} = \Upsilon\varkappa_n$, $\forall n \in \mathbb{N}$., is one of the well-known iterative procedures. To approximate to fixed points of contraction mappings, this iteration approach has been utilized. When using nonexpansive mappings rather than contraction mappings, the Picard iterative method is unable to approach fixed points. Numerous writers have investigated new iteration processes and mapping classes in this context for the purpose of approximating fixed points.

For the class of nonexpansive self-mappings on a closed and bounded subset of a uniformly convex Banach space, Browder [6] demonstrated the existence of a fixed point. After Browder's result, researchers have developed iterative procedures to approximate fixed points in nonexpansive and nonlinear mappings, with research focusing on faster and more efficient techniques. Studies have been conducted in uniformly convex Banach spaces and CAT(0) spaces.(see [1], [7], [20], [22], [23] and the references therein)

Suzuki [19] established a new class of nonexpansive mappings and demonstrated several fixed point theorems for them. Many researchers have contributed to the literature by generalizing Suzuki's generalized non-expansive mapping (see [3], [4], [16]). More recently, Adeyemi et al. [2] introduced the generalized nonexpansive mappings and in uniformly convex hyperbolic space, they demonstrated approaching the fixed point of these mappings.

Along with the nonlinear mappings, the significance that the spaces play in the study of fixed point theory is also quite important, including Hilbert and Banach spaces. Banach spaces have convex structures, making it easy to exist fixed points. Metric spaces lack this structure, making it necessary to introduce convex structures into them. By examining the fixed points for nonexpansive mappings in convex metric spaces, Takahashi [21] was the pioneer in developing the idea of convex metric space.

Since then, many convex structures have been introduced onto metric spaces in a number of different attempts. As a result of these studies, many fixed point theorems have been obtained by applying well-known fixed point iteration processes to hyperbolic spaces (see [9], [10], [13], [17]). Hyperbolic spaces have a convex structure, with the convex structure introduced by Kohlenbach [13] being more general.

In this work, we introduce a new class of generalized nonexpansive mappings in hyperbolic space. Also, we constitute the form in hyperbolic space of the well known Mann iteration process. Then, we prove strong and Δ-convergence results for these mappings in a uniformly convex hyperbolic space using our introduced iterative process.

2. PRELIMINARIES

Firstly, Takahashi [21] proposed the idea of convex metric space in 1970 as follows:

A mapping $W : \mathfrak{S}\times\mathfrak{S}\times[0,1] \longrightarrow \mathfrak{S}$ is a convex structure in $\mathfrak{S}$ if

$$\hbar(\mathfrak{q}, W(\varkappa,\mu,\eth)) \leq (1-\eth)\hbar(\mathfrak{q},\varkappa) + \eth\hbar(\mathfrak{q},\mu),$$

for all $\varkappa$,calμ,$calfrakq$∈ $\mathfrak{S}$ and $\eth \in [0,1]$. A metric space $(\mathfrak{S},\hbar)$ together with a convex structure W defined on it is called a convex metric space. A subset $\mathcal{M}$ of a convex metric space $\mathfrak{S}$ is convex if calW($\varkappa$,calμ,$\eth$) $\in \mathcal{M}$ for all $\varkappa$,calμ∈ $\mathcal{M}$ and $\eth \in [0,1]$.

Afterwards, this idea was greatly expanded upon by numerous authors. In Kohlenbach's hyperbolic space [13], one of these convex structures is present. There are various interpretations of hyperbolic space in the literature.

A hyperbolic space $(\mathfrak{S},\hbar,W)$ (see [13]) is a metric space $(\mathfrak{S},\hbar)$ together with a mapping $W : \mathfrak{S}\times\mathfrak{S}\times[0,1] \longrightarrow \mathfrak{S}$ satisfying

(W1) $\hbar(z, W(\varkappa$,calμ,$\eth)) \leq (1-\eth)\hbar(z,\varkappa) + \eth\hbar(z$,cal$\mu)$,

(W2) $\hbar(W(\varkappa$,calμ,$\eth_1), W(\varkappa$,calμ,$\eth_2)) = |\eth_1 - \eth_2|\,\hbar(\varkappa$,cal$\mu)$,

(W3) $W(\varkappa$,calμ,$\eth) = W($calμ,$\varkappa$, $(1-\eth))$,

(W4) $\hbar(W(\varkappa, z, \eth), W(cal\mu$,W,$\eth)) \leq (1 - \eth)\hbar(\varkappa$,cal$\mu)$+$\eth\hbar(z, W)$,

for all $\varkappa$,calμ,z,W∈ $\mathfrak{S}$ and $\eth, \eth_1, \eth_2 \in [0,1]$.

A hyperbolic space $(\mathfrak{S},\hbar,W)$ is said to be uniformly convex [18] if for all calfrakq,$\varkappa$,calμ∈ $\mathfrak{S}$, $r > 0$ and $\varepsilon \in (0,2]$, there exists a $\delta \in (0,1]$ such that $\hbar\left(z, W\left(\varkappa,\mu,\frac{1}{2}\right),\mathfrak{q}\right) \leq (1-\delta)r$ whenever $\hbar(\varkappa$,calfrakq) $\leq r$, $\hbar$(calμ,$calfrakq$)$\leq r$ and $\hbar(\varkappa$,calμ)$\geq \varepsilon r$.

A mapping $\eta : (0,\infty)\times(0,2] \longrightarrow (0,1]$ providing such $\delta = \eta(r,\varepsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$ is called modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε).

We now compile some fundamental definitions of asymptotic centers and radiuses.

Let $\mathfrak{S}$ be a hyperbolic space and $\{\varkappa_n\}$ be a bounded sequence in $\mathfrak{S}$. For $\varkappa \in \mathfrak{S}$, define a continuous functional $r(., \{\varkappa_n\}) : \mathfrak{S} \longrightarrow [0,\infty)$ by

$$r(\varkappa, \{\varkappa_n\}) = \lim_{n\longrightarrow\infty} \sup \hbar(\varkappa, \varkappa_n).$$

The asymptotic radius $r(\{\varkappa_n\})$ of $\{\varkappa_n\}$ is given by

$$r(\{\varkappa_n\}) = \inf\{r(\varkappa,\{\varkappa_n\}) : \varkappa \in \mathcal{X}\}.$$

The asymptotic radius $r_{\mathcal{M}}(\{\varkappa_n\})$ of $\{\varkappa_n\}$ with respect to a subset $\mathcal{M}$ of $\mathfrak{S}$ is given by

$$r_{\mathcal{M}}(\{\varkappa_n\}) = \inf\{r(\varkappa,\{\varkappa_n\}) : \varkappa \in \mathcal{M}\}.$$

The asymptotic center $A(\{\varkappa_n\})$ of $\{\varkappa_n\}$ is the set

$$A(\{\varkappa_n\}) = \{\varkappa \in \mathcal{X} : r(\varkappa,\{\varkappa_n\}) = r(\{\varkappa_n\})\}.$$

The asymptotic center $A_{\mathcal{M}}(\{\varkappa_n\})$ of $\{\varkappa_n\}$ with respect to a subset $\mathcal{M}$ of $\mathfrak{S}$ is the set

$$A(\{\varkappa_n\}) = \{\varkappa \in \mathcal{X} : r(\varkappa,\{\varkappa_n\}) = r_{\mathcal{M}}(\{\varkappa_n\})\}.$$

Lim [15] proceeded by thinking about how Δ-convergence was defined in a metric space in 1976, and Dhompongsa and Panyanak [8] have studied its analogue in CAT(0) spaces. Khan et al. resumed their examination of Δ-convergence in the overall structure of hyperbolic spaces in [11].

Now, we recall the notion of Δ-convergent.

[12]A sequence $\{\varkappa_n\}$ in $\mathfrak{S}$ is said to be Δ-convergent to $\varkappa \in \mathfrak{S}$, if, for every subsequence $\{\mathfrak{k}_n\}$ of $\{\varkappa_n\}$, $\varkappa$ is the unique asymptotic center of $\{\mathfrak{k}_n\}$. In this case, $\varkappa$ is called as Δ-limit of $\{\varkappa_n\}$ and we write $\Delta - \lim_{n\longrightarrow\infty} \varkappa_n = \varkappa$.

The generalized nonexpansive mapping in uniformly convex hyperbolic space was first developed in 2021 by Adeyemi et al. [2] as follows: Let $\mathcal{M}$ be a nonempty subset of a hyperbolic space $\mathfrak{S}$. A mapping $\Upsilon : \mathcal{M} \longrightarrow \mathcal{M}$ is called generalized nonexpansive mapping if there exist $\alpha,\beta,\gamma \in [0,1)$, with $\gamma+\beta < 1$ such that for all $\varkappa$,$cal\mu \in \mathcal{M}$,

$$(1-\alpha)\hbar(\Upsilon\varkappa,\varkappa) \leq \hbar(\varkappa,\mu)$$
$$\Longrightarrow \hbar(\Upsilon\varkappa,\Upsilon\mu) \leq$$
$$\beta\hbar(\mu,\Upsilon\varkappa) + \gamma\hbar(\varkappa,\Upsilon\mu) + [1-(\gamma+\beta)]\,\hbar(\varkappa,\mu).$$

We will require the following outcomes for the follow-up:

[14]Let $(\mathfrak{S},\hbar,W)$ be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{\varkappa_n\}$ in $\mathfrak{S}$ has a unique

asymptotic center with respect to any nonempty closed convex subset $\mathcal{M}$ of $\mathfrak{S}$.

[11]Let $(\mathfrak{S}, \hbar, W)$ be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $\varkappa \in \mathfrak{S}$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{\varkappa_n\}$ and $\{\mu_n\}$ are sequences in $\mathfrak{S}$ such that

$$\lim_{n\longrightarrow\infty} \sup \hbar(\varkappa_n, \varkappa) \leq r, \quad \lim_{n\longrightarrow\infty} \sup \hbar(\mu_n, \varkappa) \leq r,$$

$$\lim_{n\longrightarrow\infty} \hbar(W(\varkappa_n, \mu_n, \alpha_n), \varkappa) = r$$

for some $r \geq 0$, then

$$\lim_{n\longrightarrow\infty} \hbar(\varkappa_n, \mu_n) = 0.$$

3. MAIN RESULTS

In the context of uniformly convex hyperbolic space, we develop a new class of generalized nonexpansive mapping and establish the version in hyperbolic space of the Mann iteration process. Also, we give Δ-convergence and strong convergence results for these mappings we introduced in uniformly convex hyperbolic space using the new form of the Mann iteration process.

Let $\mathcal{M}$ be a nonempty subset of a hyperbolic pace $\mathfrak{S}$. A mapping $\Upsilon : \mathcal{M} \longrightarrow \mathcal{M}$ is called $a - b-$generalized nonexpansive mapping if there exist $a, b \in [0, \frac{1}{2})$ and $\alpha \in [0, 1)$ with $2a + 2b < 1$ such that for all $\varkappa$,cal$\mu \in \mathcal{M}$,

$$(1 - \alpha)\hbar(\Upsilon\varkappa, \varkappa) \leq \hbar(\varkappa, \mu)$$

$$\Longrightarrow \hbar(\Upsilon\varkappa, \Upsilon\mu) \leq a\left[\hbar(\mu, \Upsilon\varkappa) + \hbar(\varkappa, \Upsilon\mu)\right]$$

$$+b\left[\hbar(\varkappa, \Upsilon\varkappa) + \hbar(\mu, \Upsilon\mu)\right] + \left[1 - (2a + 2b)\right]\hbar(\varkappa, \mu).$$

The Mann iteration process has been extensively studied for approximating fixed points of nonexpansive mappings. With Υ being a self-mapping on a subset of a Banach space, the Mann iteration process is defined as follows:

$$\begin{cases} \varkappa_1 \in \mathcal{M} \\ \varkappa_{n+1} = (1 - \alpha_n)\varkappa_n + \alpha_n\Upsilon\varkappa_n, \ n \geq 1, \end{cases} \tag{0}$$

where $\{\alpha_n\}$ is real sequences in $[0, 1]$.

A conversion of the Mann iteration process (0) from Banach space to hyperbolic space is seen in the iteration process that follows:

$$\begin{aligned} \varkappa_1 &\in \mathcal{M}, \\ \varkappa_{n+1} &= W(\varkappa_n, \Upsilon\varkappa_n, \alpha_n), \ n \geq 1. \end{aligned} \tag{1}$$

Assume that $\mathcal{M}$ is a nonempty subset of a metric space $(\mathfrak{S}, \hbar)$. Then Υ, a self-mapping on $\mathcal{M}$, is nonexpansive if $\hbar(\Upsilon\varkappa, \Upsilon$cal$\mu) \leq \hbar(\varkappa$,cal$\mu) for all \varkappa$,cal$\mu \in \mathcal{M}$. From this point forward, the term F will refer to the collection of all common fixed points for nonexpansive mappings on $\mathcal{M}$.

Within this part, for nonexpansive mappings in uniformly convex hyperbolic spaces, we demonstrate a few convergence theorems.

We start by outlining the crucial lemmas below.

Let $\mathcal{M}$ be a nonempty, closed and convex subset of a hyperbolic space $\mathfrak{S}$ and Υ be an $a - b-$generalized nonexpansive self mappings on $\mathcal{M}$ with $F \neq \emptyset$. Then for the sequence $\{\varkappa_n\}$ defined in (1), we have $\lim_{n\longrightarrow\infty} \hbar(\varkappa_n, \varpi)$ exists for each $\varpi \in F$.

Proof For any $\varpi \in F$, it follows from (1) that

$$\begin{aligned} \hbar(\varkappa_{n+1}, \varpi) &= \hbar(W(\varkappa_n, \Upsilon\varkappa_n, \alpha_n), \varpi) \\ &\leq (1 - \alpha_n)\hbar(\varkappa_n, \varpi) + \alpha_n\hbar(\Upsilon\varkappa_n, \varpi) \\ &\leq (1 - \alpha_n)\hbar(\varkappa_n, \varpi) \\ &+\alpha_n a\left[\hbar(\varpi, \Upsilon\varkappa_n) + \hbar(\varkappa_n, \Upsilon\varpi)\right] \\ &+\alpha_n b\left[\hbar(\varkappa_n, \Upsilon\varkappa_n) + \hbar(\varpi, \Upsilon\varpi)\right] \\ &+\alpha_n\left[1 - (2a + 2b)\right]\hbar(\varkappa_n, \varpi) \\ &\leq \left[1 - \alpha_n(a + b)\right]\hbar(\varkappa_n, \varpi) \\ &+\alpha_n(a + b)\hbar(\varpi, \Upsilon\varkappa_n). \end{aligned}$$

and

$$\begin{aligned} \hbar(\varpi, \Upsilon\varkappa_n) &= \hbar(\Upsilon\varpi, \Upsilon\varkappa_n) \\ &\leq a\left[\hbar(\varkappa_n, \Upsilon\varpi) + \hbar(\varpi, \Upsilon\varkappa_n)\right] \\ &+b\left[\hbar(\varpi, \Upsilon\varpi) + \hbar(\varkappa_n, \Upsilon\varkappa_n)\right] \\ &+\left[1 - (2a + 2b)\right]\hbar(\varpi, \varkappa_n) \\ &\Longrightarrow \left[1 - (a + b)\right]\hbar(\varpi, \Upsilon\varkappa_n) \\ &\leq \left[1 - (a + b)\right]\hbar(\varpi, \varkappa_n) \\ &\Longrightarrow \hbar(\varpi, \Upsilon\varkappa_n) \leq \hbar(\varpi, \varkappa_n) \end{aligned}$$

Writing (3) in (3), we have

$$\begin{aligned} \hbar(\varkappa_{n+1}, \varpi) &\leq \left[1 - \alpha_n(a + b)\right]\hbar(\varkappa_n, \varpi) \\ &+\alpha_n(a + b)\hbar(\varpi, \varkappa_n) \\ &= \hbar(\varkappa_n, \varpi). \end{aligned}$$

Hence $\lim_{n\longrightarrow\infty} \hbar(\varkappa_n, \varpi)$ exists for each $\varpi \in F$. □

Let us consider a subset $\mathcal{M}$ of a uniformly convex hyperbolic space $\mathfrak{S}$ with monotone modulus of uniform convexity η . Let the set $\mathcal{M}$ be a nonempty, closed and convex and Υ be an $a - b-$generalized nonexpansive self mappings on $\mathcal{M}$ with $F \neq \emptyset$. Assume that the sequence $\{\varkappa_n\}$ is defined by (1). Then

$$\lim_{n\longrightarrow\infty} \hbar(\varkappa_n, \Upsilon\varkappa_n) = 0.$$

Proof Let $\varpi \in F$. By Lemma 3, it follows that $\lim_{n\longrightarrow\infty} \hbar(\varkappa_n, \varpi)$ exists. We may assume that

$$\lim_{n\longrightarrow\infty} \hbar(\varkappa_n, \varpi) = r.$$

(i) Let $r = 0$. By (3), we have

$$\hbar(\varkappa_n, \Upsilon\varkappa_n) \leq \hbar(\varkappa_n, \varpi) + \hbar(\varpi, \Upsilon\varkappa_n) \leq 2\hbar(\varkappa_n, \varpi).$$

Taking limit for $n \longrightarrow \infty$, we obtain $\lim_{n\longrightarrow\infty} \hbar(\varkappa_n, \Upsilon\varkappa_n) = 0$.

(ii) Let $r > 0$. By (3), we get

$$\lim_{n\longrightarrow\infty} \sup \hbar(\Upsilon\varkappa_n, \varpi) \leq \lim_{n\longrightarrow\infty} \sup \hbar(\varkappa_n, \varpi) = r$$

and since

$$\lim_{n\longrightarrow\infty} \hbar(\varkappa_{n+1}, \varpi) = \lim_{n\longrightarrow\infty} \hbar(W(\varkappa_n, \Upsilon\varkappa_n, \alpha_n), \varpi) = r,$$

thus, from Lemma 2, we conclude that

$$\lim_{n\longrightarrow\infty} \hbar(\varkappa_n, \Upsilon\varkappa_n) = 0.$$

□

We now demonstrate the result about the Δ-convergence of the iteration process specified by (1) in a uniformly convex hyperbolic space.

Let $\mathcal{M}, \mathfrak{S}, \Upsilon$ and $\{\varkappa_n\}$ be the same as in Lemma 3. Then the sequence $\{\varkappa_n\}$ Δ-converges to a point in F.

Proof From proof of Lemma 3, it is easily seen that the sequence $\{\varkappa_n\}$ is bounded. According to Lemma 2, $\{\varkappa_n\}$ has a unique asymptotic center, which is $A_{\mathcal{M}}(\{\varkappa_n\}) = \{\varkappa\}$. Suppose that $\{\mathfrak{k}_n\}$ is any subsequence of $\{\varkappa_n\}$ such that $A_{\mathcal{M}}(\{\mathfrak{k}_n\}) = \{\mathfrak{k}\}$. By Lemma 3, we have

$$\lim_{n\longrightarrow\infty} \hbar(\mathfrak{q}_n, \Upsilon\mathfrak{q}_n) = 0. \quad (1)$$

We claim that $calfrakq \in F$. So, we calculate

$$\begin{aligned}
\hbar(\Upsilon\mathfrak{q}, \mathfrak{q}_n) &= \hbar(\Upsilon\mathfrak{q}, \Upsilon\mathfrak{q}_n) + \hbar(\Upsilon\mathfrak{q}_n, \mathfrak{q}_n) \\
&\leq a\,[\hbar(\mathfrak{q}_n, \Upsilon\mathfrak{q}) + \hbar(\mathfrak{q}, \Upsilon\mathfrak{q}_n)] \\
&\quad + b\,[\hbar(\mathfrak{q}, \Upsilon\mathfrak{q}) + \hbar(\mathfrak{q}_n, \Upsilon\mathfrak{q}_n)] \\
&\quad + [1 - (2a + 2b)]\,\hbar(\mathfrak{q}, \mathfrak{q}_n) + \hbar(\Upsilon\mathfrak{q}_n, \mathfrak{q}_n) \\
&\leq a\,[\hbar(\mathfrak{q}_n, \Upsilon\mathfrak{q}) + \hbar(\mathfrak{q}, \mathfrak{q}_n) + \hbar(\mathfrak{q}_n, \Upsilon\mathfrak{q}_n)] \\
&\quad + b\,[\hbar(\mathfrak{q}, \mathfrak{q}_n) + \hbar(\mathfrak{q}_n, \Upsilon\mathfrak{q}) + \hbar(\mathfrak{q}_n, \Upsilon\mathfrak{q}_n)] \\
&\quad + [1 - (2a + 2b)]\,\hbar(\mathfrak{q}, \mathfrak{q}_n) + \hbar(\Upsilon\mathfrak{q}_n, \mathfrak{q}_n) \\
\Longrightarrow \hbar(\Upsilon\mathfrak{q}, \mathfrak{q}_n) &\leq \hbar(\mathfrak{q}, \mathfrak{q}_n) + \left[\frac{1 + a + b}{1 - (a + b)}\right] \hbar(\Upsilon\mathfrak{q}_n, \mathfrak{q}_n).
\end{aligned}$$

Taking lim sup on both sides of the last inequality and using (1), we obtain

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \sup \hbar(\Upsilon\mathfrak{q}, \mathfrak{q}_n) &\leq \lim_{n\longrightarrow\infty} \sup \hbar(\mathfrak{q}, \mathfrak{q}_n) \\
&\Longrightarrow r(\Upsilon\mathfrak{q}, \{\mathfrak{q}_n\}) \leq r(\mathfrak{q}, \{\mathfrak{q}_n\}).
\end{aligned}$$

The fact that the asymptotic center is unique suggests that $\Upsilon calfrakq = calfrakq$. This means that $calfrakq \in F$. Since $\lim_{n\longrightarrow\infty} \hbar(\varkappa_n, calfrakq)$ exists, and taking into account the uniqueness of the asymptotic center, we get

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \sup \hbar(\mathfrak{q}_n, \mathfrak{q}) &< \lim_{n\longrightarrow\infty} \sup \hbar(\mathfrak{q}_n, \varkappa) \\
&\leq \lim_{n\longrightarrow\infty} \sup \hbar(\varkappa_n, \varkappa) \\
&< \lim_{n\longrightarrow\infty} \sup \hbar(\varkappa_n, \mathfrak{q}) = \lim_{n\longrightarrow\infty} \sup \hbar(\mathfrak{q}_n, \mathfrak{q})
\end{aligned}$$

which is a contradiction. Hence $\varkappa = calfrakq$. Thus $A(\{\mathfrak{k}_n\}) = \{\mathfrak{k}\}$ for all subsequences $\{\mathfrak{k}_n\}$ of $\{\varkappa_n\}$, that is, $\{\varkappa_n\}$ Δ-converges to $\varkappa \in F$. □

Let $\mathcal{M}$ be a subset of a metric space $\mathfrak{S}$. A sequence $\{\varkappa_n\}$ in $\mathfrak{S}$ is called as Fejér monotone with respect to $\mathcal{M}$. if $\hbar(\varkappa_{n+1}, \varpi) \leq \hbar(\varkappa_n, \varpi)$ for all $\varpi \in \mathcal{M}$ and $n \in \mathbb{N}$.

The following result is required for the sake of proving the main theorem:

[5]Let $(\mathfrak{S}, \hbar)$ be a complete metric space and $\mathcal{M}$ be a nonempty closed subset of $\mathfrak{S}$. Consider the sequence $\{\varkappa_n\}$ in $\mathcal{M}$ and suppose that $\{\varkappa_n\}$ is Fejér monotone with respect to $\mathcal{M}$. Then $\{\varkappa_n\}$ converges to some $\varpi \in \mathcal{M}$ if and only if $\lim_{n\longrightarrow\infty} \hbar(\varkappa_n, \mathcal{M}) = 0$.

Following that, we establish the strong convergence for the iteration process defined by (1).

Let $\mathcal{M}, \mathfrak{S}, \Upsilon$ and $\{\varkappa_n\}$ be the same as in Lemma 3. Then $\{\varkappa_n\}$ converges strongly to some $\varpi \in F$ if and only if $\liminf_{n\longrightarrow\infty} \hbar(\varkappa_n, F) = 0$ where $\hbar(\varkappa, F) = \inf\{\hbar(\varkappa, \varpi) : \varpi \in F\}$.

Proof If $\{\varkappa_n\}$ converges to $\varpi \in F$, then $\lim_{n\longrightarrow\infty} \hbar(\varkappa_n, \varpi) = 0$. Since $0 \leq \hbar(\varkappa_n, F) \leq \hbar(\varkappa_n, \varpi)$, we have $\liminf_{n\longrightarrow\infty} \hbar(\varkappa_n, F) = 0$.

On the contrary, presume that $\liminf_{n\longrightarrow\infty} \hbar(\varkappa_n, F) = 0$. As a result of Lemma 3 that $\lim_{n\longrightarrow\infty} \hbar(\varkappa_n, F)$ exists. Thus by hypothesis, $\lim_{n\longrightarrow\infty} \hbar(\varkappa_n, F) = 0$. From Lemma 3, it is said that $\{\varkappa_n\}$ is Fejér monotone with respect to F. Therefore, Lemma 3 indicate that $\{\varkappa_n\}$ converges strongly to a point ϖ in F. □

In Theorem 3, the condition $\limsup_{n\longrightarrow\infty} \hbar(\varkappa_n, F) = 0$ can be used in place of the condition $\liminf_{n\longrightarrow\infty} \hbar(\varkappa_n, F) = 0$.

Let $\mathfrak{S} = \mathbb{R}$ with metric defined by $\hbar(\varkappa, \mu) = |\varkappa - \mu|$ and $\mathcal{M} = \left[-\frac{1}{3}, \frac{1}{3}\right]$. Choose a mapping $\Upsilon : \mathcal{M} \longrightarrow \mathcal{M}$ as

$$\Upsilon\varkappa = \begin{cases} \frac{1-x}{2}, & \text{if } \varkappa \neq \frac{1}{3}, \\ 0, & \text{if } \varkappa = \frac{1}{3}. \end{cases}$$

Selecting $a = \frac{1}{6}$ and $b = \frac{1}{6}$ and for $\alpha \in [0, 1)$, then Υ is an $a - b-$generalized nonexpansive mapping. To prove that, we consider three different cases as follows:

Case 1 For $\varkappa = \frac{1}{3}$ and $\mu = \frac{1}{3}$, since $\hbar(\Upsilon\varkappa, \Upsilon\mu) = |\Upsilon\varkappa - \mp\mu| = 0$, we have

$$a\left[\hbar(\mu, \Upsilon\varkappa) + \hbar(\varkappa, \Upsilon\mu)\right] + b\left[\hbar(\varkappa, \Upsilon\varkappa) + \hbar(\mu, \Upsilon\mu)\right]$$

$$+ \left[1 - (2a + 2b)\right]\hbar(\varkappa, \mu) \geq 0 = \hbar(\Upsilon\varkappa, \Upsilon\mu).$$

Case 2 For $\varkappa = \frac{1}{3}$ and $\mu \neq \frac{1}{3}$, we have $\hbar(\Upsilon\varkappa, \Upsilon\mu) = \frac{1}{2}|\mu - 1|$ and

$$a\left[\hbar(\mu, \Upsilon\varkappa) + \hbar(\varkappa, \Upsilon\mu)\right]$$

$$+ b\left[\hbar(\varkappa, \Upsilon\varkappa) + \hbar(\mu, \Upsilon\mu)\right] + \left[1 - (2a + 2b)\right]\hbar(\varkappa, \mu)$$

$$= \frac{1}{6}\left[|\mu| + \left|\frac{1}{3} - \frac{1-\mu}{2}\right|\right] + \frac{1}{6}\left[\frac{1}{3} + \left|\mu - \frac{1-\mu}{2}\right|\right]$$

$$+ \left[1 - (\frac{1}{3} + \frac{1}{3})\right]\left|\frac{1}{3} - \mu\right| \geq \frac{1}{6}|\mu| + \frac{7}{36}|3\mu - 1|$$

$$> \frac{1}{2}|\mu - 1| = \hbar(\Upsilon\varkappa, \Upsilon\mu).$$

Case 3 For $\varkappa \neq \frac{1}{3}$ and $\mu \neq \frac{1}{3}$, we have $\hbar(\Upsilon\varkappa, \Upsilon\mu) = |\Upsilon\varkappa - \mp\mu| = \left|\frac{1-x}{2} - \frac{1-\mu}{2}\right| = \frac{1}{2}|\varkappa - \mu|$ and

$$a\left[\hbar(\mu, \Upsilon\varkappa) + \hbar(\varkappa, \Upsilon\mu)\right] + b\left[\hbar(\varkappa, \Upsilon\varkappa)\right.$$

$$\left.+ \hbar(\mu, \Upsilon\mu)\right) + \left[1 - (2a + 2b)\right]\hbar(\varkappa, \mu)$$

$$= \frac{1}{6}\left[\left|\mu - \frac{1-x}{2}\right| + \left|x - \frac{1-\mu}{2}\right|\right]$$

$$+ \frac{1}{6}\left[\left|x - \frac{1-x}{2}\right| + \left|\mu - \frac{1-\mu}{2}\right|\right]$$

$$+ \left[1 - (\frac{1}{3} + \frac{1}{3})\right]|x - \mu|$$

$$= \frac{1}{12}\left[|2\mu + \varkappa - 1| + |2\varkappa + \mu - 1|\right]$$

$$+ \frac{1}{12}\left[|3x - 1| + |3\mu - 1|\right] + \frac{1}{3}|x - \mu| \geq \frac{1}{12}|\mu - x|$$

$$+ \frac{1}{12}|3\varkappa - 3\mu| + \frac{1}{3}|x - \mu| = \frac{2}{3}|x - \mu|$$

$$\geq \frac{1}{2}|\varkappa - \mu| = \hbar(\Upsilon\varkappa, \Upsilon\mu).$$

In the all above cases we have $\hbar(\Upsilon\varkappa, \Upsilon\mu) \leq a\left[\hbar(\mu, \Upsilon\varkappa) + \hbar(\varkappa, \Upsilon\mu)\right] + b\left[\hbar(\varkappa, \Upsilon\varkappa) + \hbar(\mu, \Upsilon\mu)\right] + \left[1 - (2a + 2b)\right]\hbar(\varkappa, \mu)$, therefore Υ become an $a - b-$generalized nonexpansive mapping.

Now, we show that iteration process (1) converges strongly and Δ-converges to fixed point $\varpi = \frac{1}{3}$.

Choosing $\alpha_n = \frac{n}{2n+3}$ for all $n \geq 1$, the hypotheses of Lemma 3 are verified. Since the conditions of Theorem 3 and Theorem3, iteration process (1) converges strongly and Δ-converges to fixed point $\varpi = \frac{1}{3}$.

In this study, firstly, a new class of generalized nonexpansive mapping called $a - b-$generalized nonexpansive mapping has been introduced in hyperbolic space. Additionally, we construct the form in hyperbolic space of the Mann iteration process, which is well-known in the literature. Finally, we prove convergence theorems for $a - b-$generalized nonexpansive mappins in a uniformly convex hyperbolic space using the modified form of the Mann iteration process.

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