Erzincan Üniversitesi
Fen Bilimleri Enstitüsü Dergisi
2024,17(1), 259-267
ISSN: 1307-9085, e-ISSN: 2149-4584
Araştırma Makalesi

Erzincan University
Journal of Science and Technology
2024,17(1), 259-267
DOI: 10.18185/erzifbed.1362369
Research Article

Reconstruction of a Ruled Surface in 3-dimensional Euclidean Space

Mustafa Dede[1*], Cumali Ekici[2*], Mahmut Koçak[2*]

[1]Kilis 7 Aralık University, Department of Mathematics, Turkey

[2]Eskişehir Osmangazi University, Department of Mathematics and Computer Science, Turkey

Received: 18/09/2023, **Revised:** 15/11/2023, **Accepted:** 15/11/2023, **Published:** 28/03/2024

Abstract
In this paper, firstly, a summary of certain results related to the differential geometry of ruled surfaces is provided. Subsequently, the signature curve for ruled surfaces in Euclidean 3-space is introduced. Additionally, a simple algorithm for the reconstruction of a ruled surface, which is both efficient and entirely local, requiring only the initial motion direction and starting point, is presented. Finally, the efficiency and accuracy of the algorithm are demonstrated through several examples.

Keywords: Ruled surface, Signature curve, Curvature, Reconstruction

3-boyutlu Öklid Uzayında bir Regle Yüzeyin Yeniden Yapılandırılması

Öz

Bu makalede, öncelikle regle yüzeylerin diferesiyel geometrisi ile ilgili bazı sonuçların bir özeti sunulmuştur. Daha sonra, Öklid 3-uzayında regle yüzeyler için işaret eğrisi tanıtılmıştır. Ek olarak, bir regle yüzeyin yeniden yapılandırılması için hem verimli hem de tamamen yerel olan, yalnızca ilk hareket yönü ve başlangıç noktası gerektiren basit bir algoritma sunulmuştur. Son olarak, algoritmanın verimliliği ve doğruluğu birkaç örnekle gösterilmiştir.

Anahtar Kelimeler: Regle yüzey, İşaret eğrisi, Eğrilik, Yeniden yapılanma

*Corresponding Author: cekici@ogu.edu.tr
Mustafa DEDE, https://orcid.org/0000-0003-2652-637X
Cumali EKİCİ, https://orcid.org/0000-0002-3247-5727
Mahmut KOÇAK, https://orcid.org/0000-0001-7774-0144

1. Introduction

In the Euclidean plane, the parametric representation of a curve $\alpha(t)$ yields its signature curve $S(t) = \left(\kappa(t), \kappa_s(t)\right)$, where $\kappa(t)$ signifies the curvature and $\kappa_s(t)$ is the rate of change with respect to the arc length t. Calabi et al. used the signature curves to recognize invariant properties of visual objects in the Euclidean plane [1,2]. They also introduced a new method for invariant recognition of visual objects that involves numerical approximation of $\kappa(t)$ and $\kappa_s(t)$ to a differential invariant signature curve. Surazhsky and Elber explored the process of uniquely deriving a planar curve's curvature signature and, more importantly, the methodology for reconstructing the curve from its curvature signature [3]. Numerous geometers have examined these inquiries and extended the scope. In 2012, Hickman demonstrated a method for generating all planar curves that exhibit a specific signature curve [4]. In 2000, Boutin introduced a 3-dimensional adaptation of the Euclidean signature curve, encompassing curvature, torsion, and their respective derivatives concerning arc length [5]. In more recent times, Wu and Li introduced an algorithm for replicating a motion trajectory through the utilization of the signature curve within the Euclidean space [6,7,8]. The exploration of ruled surfaces represents an intriguing research domain within surface theory, with applications spanning various areas in CAD and CAGD [9]. Peternell delved into the reconstruction of developable surfaces from scattered data points, while Ryuh et al. harnessed ruled surfaces for robot motion planning [10, 11]. Pottmann et al. explored offsets of rational ruled surfaces [12]. Furthermore, the geometry of ruled surfaces plays a crucial role in the study of kinematic and positional mechanisms, as demonstrated by Kühnel, Abdel Baky, Ekici and Çöken, Zhang, Ünlütürk et al. and Ekici et al. [13,14,15,16,17,18]. Similar to the fundamental theorem of curves, which facilitates the determination of a curve based on curvature and torsion, a similar structure can be recognized in the field of reconstruction of a ruled surface. In this context, the signature curve plays a crucial role. The basic rationale behind the inclusion of a signature curve in a ruled surface reconstruction algorithm stems from the inherent independence of signature curves with respect to the choice of coordinate systems. As a result, the freedom to add segments of constant curvature at critical points of the curve is allowed. However, this freedom, together with an Euclidean transformation, constitutes the only freedom available to preserve the integrity of the signature curve. Moreover, these curves provide an accurate representation of the shape of the ruled surface and rigorously capture all its critical geometric features. The organization of this paper proceeds as follows: Section 2 describes the basic properties inherent to ruled surfaces. The following section briefly summarises the algorithmic concept governing the reconstruction of ruled surfaces. Finally, Section 4 analyses the pragmatic applications of the proposed algorithm, providing several illustrative examples for extensive discussion.

2. Ruled Surfaces

This section is devoted to the so-called ruled surfaces. A ruled surface in Euclidean 3-space $\mathbb{R}^3$ is a (smooth or discrete) one-parameter family of straight lines [9]. We briefly recall some basic facts on ruled surfaces. Recall that, using standard parameters u and v, a ruled surface can be parameterized by

$$\varphi(u,v) = \alpha(u) + vX(u) \tag{1}$$

where $\|X(u)\| = \|X'(u)\| = 1$ and $\langle \alpha'(u), X'(u)\rangle = 0$.

Here $\alpha(u)$ is the striction line, the parameter u is the arc length of the spherical curve $X(u)$.

A ruled surface is, up to Euclidean motions, uniquely determined by the following quantities:

$$\begin{aligned} F &= \langle \alpha', X\rangle \\ Q &= \det(\alpha', X, X') \\ J &= \det(X, X', X'') \end{aligned} \tag{2}$$

each of which is a function of u. Conversely, every choice of these quantities uniquely determines a ruled surface.

Furthermore, the derivative of the striction line is completely determined by F and Q using the equation

$$\alpha' = FX + QX \wedge X' \tag{3}$$

where Q is the parameter of the distribution.

In the moving frame, $\{X, X', X \wedge X'\}$ we have the Frenet type matrix

$$\frac{d}{du}\begin{bmatrix} X \\ X' \\ X \wedge X' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & J \\ 0 & -J & 0 \end{bmatrix}\begin{bmatrix} X \\ X' \\ X \wedge X' \end{bmatrix}. \tag{4}$$

3. Reconstruction of a Ruled Surface from its Signature

As in the basic known theorem for curves; given curvature and torsion, the curve can be found by the difference in position. Similar to this structure, the reconstructed regular surface can be found using the signature curve. Recalling the fact that the ruled surfaces can be completely determined by F, Q and J we have the following definition.

Definition 2.1 Let φ be ruled surface in Euclidean space $\mathbb{R}^3$. The signature curve of the ruled surface is defined by

$$S(u) = \{F(u), Q(u), J(u)\}. \tag{5}$$

We can easily see that the signature curve of ruled surface is invariant under translations and rotation.

This paper aims to formulate an algorithm for the reproduction of the ruled surface based on its signature, denoted as *S*. Initially, the reproduction process involves the generation of the ruling and the striction line. Then, the reproduced ruling and the striction line are combined to reconstruct the ruled surface.

Firstly, to reproduce the moving frame of ruled surface we use the so called finite difference. Let u_i and $u_i + \Delta u_i$ be two consecutive points. The derivatives of moving frame vectors $M = \{X, X', X \wedge X'\}$ can be obtained by

$$\begin{aligned}\frac{d}{du}X(u_i) &= \frac{X(u_i+\Delta u_i)-X(u_i)}{\Delta u_i}\\ \frac{d}{du}X'(u_i) &= \frac{X'(u_i+\Delta u_i)-X'(u_i)}{\Delta u_i}\\ \frac{d}{du}(X\wedge X')(u_i) &= \frac{(X\wedge X')(u_i+\Delta u_i)-(X\wedge X')(u_i)}{\Delta u_i}.\end{aligned} \tag{6}$$

From Eq. (4) and (6) we can derive the following iteration equation expressed in matrix form as

$$\begin{bmatrix} X(u_i+\Delta u_i)\\ X'(u_i+\Delta u_i)\\ (X\wedge X')(u_i+\Delta u_i)\end{bmatrix} = \begin{bmatrix} 1 & \Delta u_i & 0\\ -\Delta u_i & 1 & J\Delta u_i\\ 0 & -J\Delta u_i & 1\end{bmatrix}\begin{bmatrix} X(u_i)\\ X'(u_i)\\ (X\wedge X')(u_i)\end{bmatrix}. \tag{7}$$

We use this matrix to compute new vectors from old ones.

Since the moving frame vectors are unit vectors, we need to normalize the vectors, then we have the reproduced moving frame vectors obtained by

$$\begin{aligned}\tilde{X}(u_i+\Delta u_i) &= \frac{X(u_i+\Delta u_i)}{\|X(u_i+\Delta u_i)\|}\\ \widetilde{X'}(u_i+\Delta u_i) &= \frac{X'(u_i+\Delta u_i)}{\|X'(u_i+\Delta u_i)\|}\\ (\widetilde{X\wedge X'})(u_i+\Delta u_i) &= \frac{(X\wedge X')(u_i+\Delta u_i)}{\|(X\wedge X')(u_i+\Delta u_i)\|}\end{aligned} \tag{8}$$

where $\|.\|$ indicates the Euclidean norm of a vector.

The reproduced moving frame $M(u_i+\Delta u_i)$ at the point $u_i+\Delta u_i$ from $M(u_i)$ denoted by

$$M(u_i+\Delta u_i) = \{\tilde{X}(u_i+\Delta u_i), \widetilde{X'}(u_i+\Delta u_i), (\widetilde{X\wedge X'})(u_i+\Delta u_i)\}. \tag{9}$$

To the reproduce the moving frame vectors we need the initial motion direction given by

$$\Theta(u_0) = \{\tilde{X}(u_0), \widetilde{X'}(u_0), (\widetilde{X\wedge X'})(u_0)\}$$

where we set the initial point as u_0. Starting from the initial motion direction $\Theta(u_0)$ and the provided signature S of the ruled surface, by iteratively applying equations (7) and (8), we can compute the moving frame $M(u_i+\Delta u_i)$ for all points on the reconstructed ruled surface.

It is no surprise that the reproduced vector $X(u_i+\Delta u_i)$ can be considered to be the ruling of the reproduced ruled surface.

Next, we turn our attention to the reproduction of the striction line. To reproduce the striction line we use again finite difference method. Then, the derivative of the striction line of the ruled surface is obtained by

$$\frac{d}{du}\alpha(u_i) = \frac{\alpha(u_i+\Delta u_i)-\alpha(u_i)}{\Delta u_i}. \tag{10}$$

Substituting (3) into (10) gives

$$\alpha(u_i+\Delta u_i) = \alpha(u_i) + \Delta u_i[F(u_i)X(u_i) + Q(u_i)(X\wedge X')(u_i)]. \tag{11}$$

Using the initial starting point $\alpha(u_0)$ and (11) we can easily reproduce all the points of the striction line.

As the final step, combining iteration equations (8) and (11), the reproduced ruled surface $\varphi(u_i+\Delta u_i, v)$ can be easily constructed in the following form:

$$\varphi(u_i+\Delta u_i, v) = \alpha(u_i+\Delta u_i) + v\tilde{X}(u_i+\Delta u_i)\,. \tag{12}$$

To summarize, once the signature data is available, the above equation can be used to reproduce all the points of the ruled surface. Therefore, given the initial starting point $\alpha(u_0)$ of the striction line, the initial motion direction $\Theta(u_0)$ of the moving frame, and iterate equation (12), we can reproduce all the points of the ruled surface.

4. Examples

In this section, we tested the performance of our algorithm. We give some examples as follows:

Example 3.1 Let us consider the well-known ruled surface helicoid parametrized by

$$\varphi(u,v) = (vcosu, vsinu, u). \tag{13}$$

We have the moving frame as

$$\begin{aligned} X(u) &= (cosu, sinu, 0) \\ X'(u) &= (-sinu, cosu, 0) \\ (X \wedge X')(u) &= (0,0,1). \end{aligned} \tag{14}$$

F, J and Q are obtained by

$$F = J = 0, \quad Q = 1. \tag{15}$$

In this example, we can simply set the initial motion direction $\Theta(1)$ in the following form:

$$\begin{aligned} X(1) &= (1,0,0) \\ X'(1) &= (0,1,0) \\ (X \wedge X')(1) &= (0,0,1). \end{aligned} \tag{16}$$

The initial point of origin is selected as

$$\alpha(1) = (10,5,0). \tag{17}$$

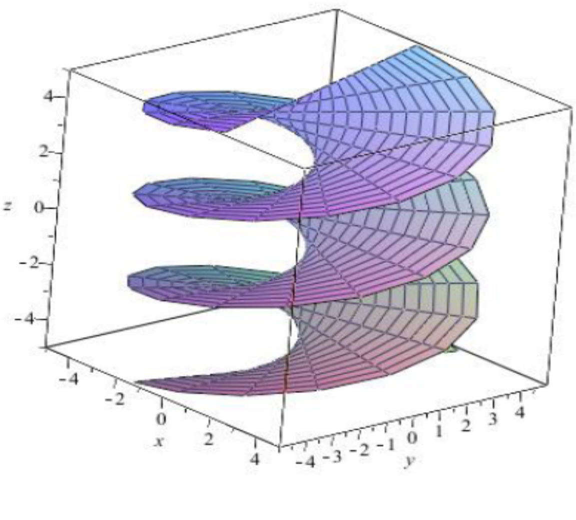

Figure 1. (a) The original ruled surfaces

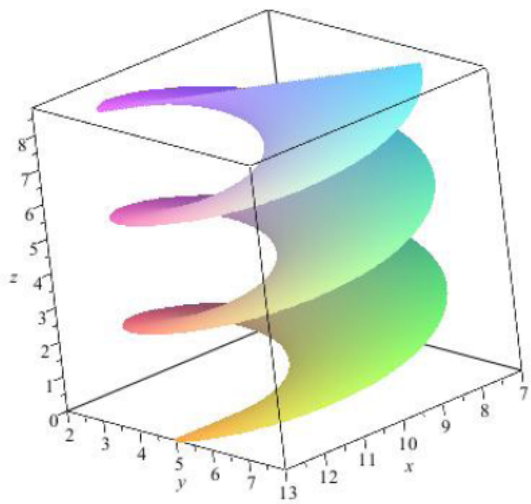

Figure 1. (b) The reproduced ruled surfaces

Now, consider the reconstruction of the ruled surface. At the initial starting point $\alpha(1)$ of the striction line, the initial motion direction $\Theta(1)$ of the moving frame of the ruled surface and signature curve $S(u)$ respectively, are firstly given as input data. Then, we reproduced the ruled surface with points $v \in (-3,3)$ and $u_i \in (1,10)$ with $\Delta u = u_{i+1} - \Delta u_i$ for $\Delta u = 0.01$. The ruled surface, which was produced by the signature of the original ruled surface, is shown in Figure 1(b). The original ruled surface is illustrated in Figure

1(a). Figure 1(a) and Figure 1(b) show that the original and reproduced ruled surfaces are similar. They have the same shape (signature curve) but they are not identical in space position.

Example 3.2 The Wallis' conical edge represents a ruled surface defined by the parametric equation:

$$\varphi(u,v)=\left(vcosu, vsinu, \sqrt{4-3cos^2u}\right).$$

The derived calculations yield the expression $S(u)$ as follows:

$$S(u)=\left(0,\frac{3cosusinu}{\sqrt{4-3cos^2u}},0\right).$$

Assuming that the initial direction of motion $\Theta(-3)$ and the initial starting point $\alpha(-3)$ are given by:

$$\begin{aligned}\tilde{X}(-3)&=\ (-0.989,-0.141,0)\\ \widetilde{X'}(-3)&=\ (-0.141,-0.989,0)\\ (\widetilde{X\wedge X'})(-3)&=\ (0,0,1)\end{aligned}$$

and

$$\alpha(-3)=(0,0,1.02) \tag{18}$$

respectively.

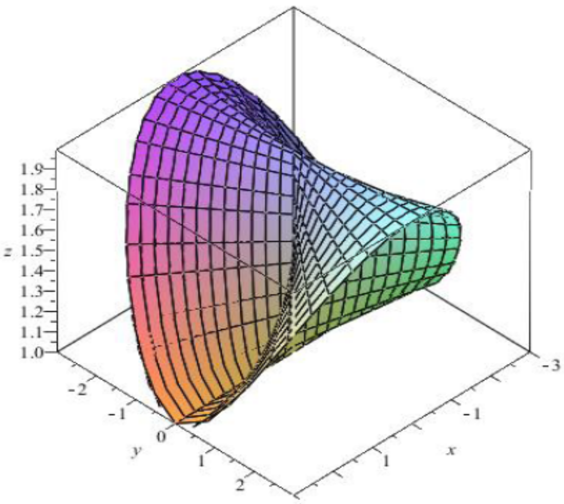

Figure 2. (a) The original ruled surfaces

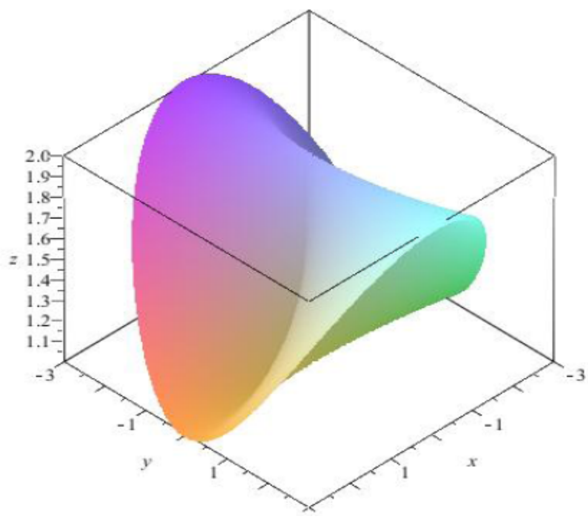

Figure 2. (b) The reproduced ruled surfaces

The ruled surface is then reproduced with points $v\in(-3,3)$ and $u_i\in(1,10)$ such that $\Delta u=u_{i+1}-\Delta u_i$ for $\Delta u=0.01$. Figure 2(a) and Figure 2(b) show that if we choose the initial motion direction and the initial starting point identical to the original ruled surface then the original and reproduced ruled surfaces are completely identical.

Example 3.3 Assume that the signature curve is given as input data in the following form:

$$S(u)=\left(-2u,0,\frac{u}{2}\right).$$

We start at an arbitrary initial motion direction $\Theta(-4)$ given by

$$\begin{aligned} \tilde{X}(-4) &= \quad (1,0,0) \\ \tilde{X}'(-4) &= \quad (0,1,0) \\ (\widetilde{X \wedge X'})(-4) &= \quad (0,0,1). \end{aligned} \tag{19}$$

In addition, assume we are given the initial starting point of the striction line of the ruled surface in the following form:

$$\alpha(-4) = (5,4,3). \tag{20}$$

Now using $\alpha(-4)$, $\Theta(-4)$, and $S(u)$, we can produce a ruled surface, illustrated in Figure 3. In Figure 3(a) and Figure 3(b), the appearance of the reconstructed ruled surface is plotted by rotating it in two different directions. We, here, computed the ruled surface with points $v \in (-3,3)$, $u_i \in (-4,4)$, and $\Delta u = u_{i+1} - \Delta u_i$ for $\Delta u = 0.01$.

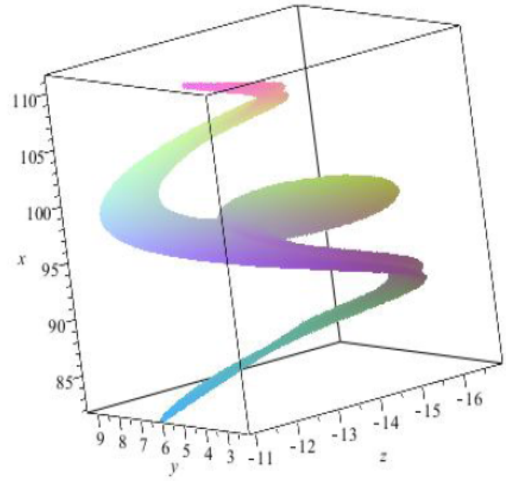

Figure 3. (a) The produced ruled surface

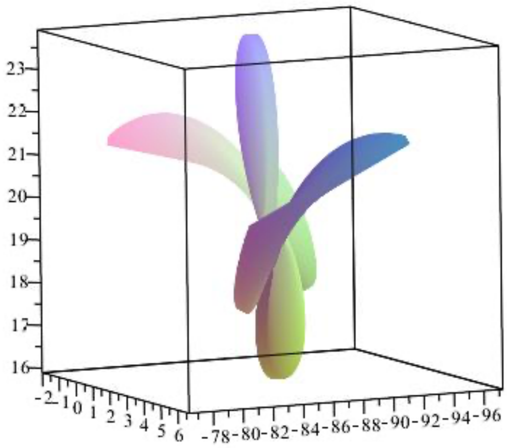

Figure 3. (b) The reproduced ruled surfaces

Conclusion

This article is devoted to the reproduction of ruled surfaces in Euclidean space. Examples are reported to show the reproduction algorithm for ruled surfaces is flexible and easy. The validity and effectiveness of the formulation are checked through several examples. According to our experiments, the accuracy of the ruled surface reproduction will be sufficient in most applications. If necessary, it can be improved by reducing the mesh size Δu. For future research our algorithm can be applied to Lorentz space as well. Furthermore, we will apply this method for surface reconstruction.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

Author Contributions

The authors contributed equally.

Acknowledgments

We would like to thank the referees for guiding us to provide important this publication and for the stimulating questions.

References

[1] Calabi, E., Olver, P. J., Shakiban, C., Tannenbaum, A., Haker, S. (1998). Differential and numerically invariant signature curves applied to object recognition, Int. J. Computer Vision, 26, 107-135.

[2] Calabi, E., Olver, P. J., Tannenbaum, A. (1996). Affine geometry, curve flows, and invariant numerical approximations, Adv. Math., 124, 154-196.

[3] Surazhsky, T., Elber, G. (2002). Metamorphosis of planar parametric curves via curvature interpolation, International Journal of Shape Modeling, 8, 201-216.

[4] Hickman, M. S. (2012). Euclidean signature curves, J. Math. Imaging. Vis., 43, 206-213.

[5] Boutin, M. (2000). Numerically invarint signature curves , Int. J. Comput. Vision, 40(3), 235-248.

[6] Wu, S., Li, Y. F., Zhang, J.W. (2007). Signature descriptor for free form trajectory modeling, in Proc. IEEE International Conference on Integration Technology, Shenzhen, China, 167-172.

[7] Wu, S., Li, Y. F. (2010). Motion trajectory reproduction from generalized signature description, Pattern Recognition, 43, 204-221.

[8] Wu, S., Li, Y. F. (2008). On signature invariants for effective motion trajectory recognition, The International Journal of Robotics Research, 27, 895-917.

[9] Chen, H. Y., Pottmann H. (1974). Approximation by ruled surfaces. J. Comput. Appl. Math. 102, 143-156.

[10] Ryuh, B. S., Pennock, G.R. (1988). Accurate motion of a robot end-effector using the curvature theory of ruled surfaces, Journal of Mechanisms, Transmissions, and Automation in Design, 110, 383-388.

[11] Peternell, M. (2004). Developable surface fitting to point clouds. Comp. Aided Geom. Design, 21, 785-803.

[12] Pottmann, H., Lüa, W., Ravani, B. (1996). Rational ruled surfaces and their Offsets, Graphical Models and Image Processing, 58, 544-552.

[13] Kühnel, W. (1994). Ruled W-surfaces, Arch. Math., 62, 475-480.

[14] Ekici, C., Çöken, A. C. (2012). The integral invariants of parallel timelike ruled surfaces, J. Math. Anal. Appl. 393, 97-107.

[15] Zhang, X.M., Zhu, L.M., Ding, H., Xiong, Y.L. (2012). Kinematic generation of ruled surface based on rational motion of point-line, Science China Technological Sciences, 55, 62-71.

[16] Abdel Baky, R. A. (2003). On the Blaschke approach of ruled surfaces, Tamkang J. Math., 34, 107-116.

[17] Ünlütürk, Y., Çimdiker, M., Ekici, C. (2016). Characteristic properties of the parallel ruled surfaces with Darboux frame in Euclidean 3-space, Communication in Mathematical Modeling and Applications. 1(1), 26-43.

[18] Ekici, C., Kaymanlı U.,G., Okur, S. (2021). A new characterization of ruled surfaces according to q-frame vectors in Euclidean 3-space, International Journal of Mathematical Combinatorics, 3, 20-31.