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Research Article

Tensorial and Hadamard product integral inequalities for convex functions of continuous fields of operators in Hilbert spaces

Silvestru Sever Dragomir[1,2]

[1] *Applied Mathematics Research Group, ISILC Victoria University, PO Box 14428 Melbourne City, MC 8001, Australia*

[2] *School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa*

Abstract

Let H be a Hilbert space and Ω a locally compact Hausdorff space endowed with a Radon measure μ with $\int_\Omega 1d\mu(t) = 1$. In this paper we show among others that, if f is continuous differentiable convex on the open interval I, $(A_\tau)_{\tau\in\Omega}$ is a continuous field of positive operators in $B(H)$ with spectra in I for each $\tau \in \Omega$ and B an operator with spectrum in I, then we have

$$\begin{aligned}&\int_\Omega (f'(A_\tau)A_\tau)\,d\mu(\tau)\otimes 1 - \int_\Omega f'(A_\tau)\,d\mu(\tau)\otimes B\\ &\geq \int_\Omega f(A_\tau)\,d\mu(\tau)\otimes 1 - 1\otimes f(B)\\ &\geq \left(\int_\Omega A_\tau d\mu(\tau)\otimes 1 - (1\otimes B)\right)(1\otimes f'(B))\end{aligned}$$

and the Hadamard product inequality

$$\begin{aligned}&\int_\Omega (f'(A_\tau)A_\tau)\,d\mu(\tau)\circ 1 - \int_\Omega f'(A_\tau)\,d\mu(\tau)\circ B\\ &\geq \int_\Omega f(A_\tau)\,d\mu(\tau)\circ 1 - 1\circ f(B)\\ &\geq \int_\Omega A_\tau d\mu(\tau)\circ f'(B) - 1\circ(f'(B)B).\end{aligned}$$

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Email address: sever.dragomir@ajmaa.org (S.S. Dragomir)
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1. Introduction

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i (\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2], we define

$$f (A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f (\lambda_1, ..., \lambda_k) dE_1 (\lambda_1) \otimes ... \otimes dE_k (\lambda_k) \tag{1.1}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [7] for functions of two variables and have the property that

$$f (A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f (st) \geq (\leq) f (s) f (t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [5, p. 173]

$$f (A \otimes B) \geq (\leq) f (A) \otimes f (B) \text{ for all } A,\ B \geq 0. \tag{1.2}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} tdE (t) \text{ and } B = \int_{[0,\infty)} sdF (s)$$

are the spectral resolutions of A and B, then

$$f (A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f (st) dE (t) \otimes dF (s) \tag{1.3}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A, B > 0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$$

where $t \in [0, 1]$ and

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B) \otimes (B\#A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [8] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned}(A\#B) \otimes (A\#B) &\leq \frac{1}{2} [(A\#_\alpha B) \otimes (A\#_{1-\alpha} B) + (A\#_{1-\alpha} B) \otimes (A\#_\alpha B)] \\ &\leq \frac{1}{2} (A \otimes B + B \otimes A)\end{aligned} \tag{1.4}$$

for $A, B > 0$ and $\alpha \in [0, 1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying

$$\langle (A \circ B) e_j, e_j\rangle = \langle Ae_j, e_j\rangle \langle Be_j, e_j\rangle$$

for all $j \in \mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [4], we have the representation

$$A \circ B = \mathcal{U}^* (A \otimes B) \mathcal{U} \tag{1.5}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U} e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If f *is super-multiplicative and operator concave (sub-multiplicative and operator convex)* on $[0, \infty)$, then also [5, p. 173]

$$f (A \circ B) \geq (\leq) f (A) \circ f (B) \text{ for all } A,\ B \geq 0. \tag{1.6}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2} \circ B^{1/2} \leq \left(\frac{A+B}{2}\right) \circ 1 \text{ for } A,\ B \geq 0$$

and Fiedler inequality

$$A \circ A^{-1} \geq 1 \text{ for } A > 0.$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A \circ B \leq \left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2} \text{ for } A,\ B \geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A \circ B \leq \left(A^2 \circ B^2\right)^{1/2} \text{ for } A,\ B \geq 0.$$

It has been shown in [6] that $\left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2}$ and $\left(A^2 \circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Let Ω be a locally compact Hausdorff space endowed with a Radon measure μ. A field $(A_t)_{t\in\Omega}$ of operators in $B(H)$ is called a continuous field of operators if the parametrization $t \longmapsto A_t$ is norm continuous on $B(H)$. If, in addition, the norm function $t \longmapsto \|A_t\|$ is Lebesgue integrable on Ω, we can form the Bochner integral $\int_\Omega A_t d\mu(t)$, which is the unique operator in $B(H)$ such that $\varphi\left(\int_\Omega A_t d\mu(t)\right) = \int_\Omega \varphi(A_t) d\mu(t)$ for every bounded linear functional φ on $B(H)$. Assume also that, $\int_\Omega 1 d\mu(t) = 1$.

Motivated by the above results, in this paper we show among others that, if f is continuous differentiable convex on the open interval I, $(A_\tau)_{\tau\in\Omega}$ is a continuous field of positive operators in $B(H)$ with spectra in I for each $\tau \in \Omega$ and B an operator with spectrum in I, then we have

$$\begin{aligned}
&\int_\Omega (f'(A_\tau) A_\tau) d\mu(\tau) \otimes 1 - \int_\Omega f'(A_\tau) d\mu(\tau) \otimes B \\
&\geq \int_\Omega f(A_\tau) d\mu(\tau) \otimes 1 - 1 \otimes f(B) \\
&\geq \left(\int_\Omega A_\tau d\mu(\tau) \otimes 1 - (1 \otimes B)\right) (1 \otimes f'(B))
\end{aligned}$$

and the Hadamard product inequality

$$\begin{aligned}
&\int_\Omega (f'(A_\tau) A_\tau) d\mu(\tau) \circ 1 - \int_\Omega f'(A_\tau) d\mu(\tau) \circ B \\
&\geq \int_\Omega f(A_\tau) d\mu(\tau) \circ 1 - 1 \circ f(B) \\
&\geq \int_\Omega A_\tau d\mu(\tau) \circ f'(B) - 1 \circ (f'(B) B).
\end{aligned}$$

2. Main Results

We also have the following double inequality for tensorial product of operators:

Lemma 2.1. *Assume that f is continuous differentiable convex on the open interval I and A, B are selfadjoint operators in $B(H)$ with spectra in I, then*

$$\begin{aligned}(f'(A)\otimes 1)(A\otimes 1-1\otimes B) &\geq f(A)\otimes 1-1\otimes f(B) \\ &\geq (A\otimes 1-1\otimes B)(1\otimes f'(B)).\end{aligned} \tag{2.1}$$

Proof. Using the gradient inequality for the differentiable convex f on I we have

$$f'(t)(t-s)\geq f(t)-f(s)\geq f'(s)(t-s)$$

for all t, $s\in I$.

Assume that

$$A=\int_I tdE(t) \text{ and } B=\int_I sdF(s)$$

are the spectral resolutions of A and B.

These imply that

$$\begin{aligned}\int_I\int_I f'(t)(t-s)\,dE(t)\otimes dF(s) &\geq \int_I\int_I (f(t)-f(s))\,dE(t)\otimes dF(s) \\ &\geq \int_I\int_I f'(s)(t-s)\,dE(t)\otimes dF(s).\end{aligned} \tag{2.2}$$

Observe that

$$\begin{aligned}&\int_I\int_I f'(t)(t-s)\,dE(t)\otimes dF(s) \\ &=\int_I\int_I (f'(t)t-f'(t)s)\,dE(t)\otimes dF(s) \\ &=\int_I\int_I f'(t)tdE(t)\otimes dF(s)-\int_I\int_I f'(t)sdE(t)\otimes dF(s) \\ &=(f'(A)A)\otimes 1-f'(A)\otimes B,\end{aligned} \tag{2.3}$$

$$\int_I\int_I (f(t)-f(s))\,dE(t)\otimes dF(s)=f(A)\otimes 1-1\otimes f(B)$$

and

$$\begin{aligned}&\int_I\int_I f'(s)(t-s)\,dE(t)\otimes dF(s) \\ &=\int_I\int_I (tf'(s)-f'(s)s)\,dE(t)\otimes dF(s) \\ &=\int_I\int_I tf'(s)\,dE(t)\otimes dF(s)-\int_I\int_I f'(s)sdE(t)\otimes dF(s) \\ &=A\otimes f'(B)-1\otimes(f'(B)B)\end{aligned}$$

and by (2.3) we derive the inequality of interest:

$$\begin{aligned}(f'(A)A)\otimes 1-f'(A)\otimes B &\geq f(A)\otimes 1-1\otimes f(B) \\ &\geq A\otimes f'(B)-1\otimes(f'(B)B).\end{aligned} \tag{2.4}$$

Now, by utilizing the tensorial property

$$(XU)\otimes(YV)=(X\otimes Y)(U\otimes V),$$

for any X, U, Y, $V\in B(H)$, we have

$$(f'(A)A)\otimes 1=(f'(A)\otimes 1)(A\otimes 1),$$

$$f'(A)\otimes B=(f'(A)\otimes 1)(1\otimes B),$$

$$A \otimes f'(B) = (A \otimes 1)(1 \otimes f'(B))$$

and

$$1 \otimes (f'(B)B) = 1 \otimes (Bf'(B)) = (1 \otimes B)(1 \otimes f'(B)).$$

Therefore

$$\begin{aligned}(f'(A)A) \otimes 1 - f'(A) \otimes B &= (f'(A) \otimes 1)(A \otimes 1) - (f'(A) \otimes 1)(1 \otimes B) \\ &= (f'(A) \otimes 1)(A \otimes 1 - 1 \otimes B)\end{aligned}$$

and

$$\begin{aligned}A \otimes f'(B) - 1 \otimes (f'(B)B) &= (A \otimes 1)(1 \otimes f'(B)) - (1 \otimes B)(1 \otimes f'(B)) \\ &= (A \otimes 1 - 1 \otimes B)(1 \otimes f'(B))\end{aligned}$$

and by (2.4) we derive (2.1). □

Corollary 2.2. *Assume that f is continuous differentiable convex on the open interval I and A, B are selfadjoint operators in $B(H)$ with spectra in I, then*

$$\begin{aligned}(f'(A)A) \circ 1 - f'(A) \circ B &\geq (f(A) - f(B)) \circ 1 \\ &\geq A \circ f'(B) - (f'(B)B) \circ 1.\end{aligned} \tag{2.5}$$

Proof. If we multiply the inequality (2.4) to the left with $\mathcal{U}^*$ and at the right with $\mathcal{U}$, we get

$$\begin{aligned}&\mathcal{U}^*[(f'(A)A) \otimes 1 - f'(A) \otimes B]\mathcal{U} \\ &\geq \mathcal{U}^*[f(A) \otimes 1 - 1 \otimes f(B)]\mathcal{U} \\ &\geq \mathcal{U}^*[A \otimes f'(B) - 1 \otimes (f'(B)B)]\mathcal{U},\end{aligned}$$

namely

$$\begin{aligned}&\mathcal{U}^*((f'(A)A) \otimes 1)\mathcal{U} - \mathcal{U}^*(f'(A) \otimes B)\mathcal{U} \\ &\geq \mathcal{U}^*(f(A) \otimes 1)\mathcal{U} - \mathcal{U}^*(1 \otimes f(B))\mathcal{U} \\ &\geq \mathcal{U}^*(A \otimes f'(B))\mathcal{U} - \mathcal{U}^*(1 \otimes (f'(B)B))\mathcal{U}.\end{aligned}$$

Using representation (1.5) we get

$$\begin{aligned}(f'(A)A) \circ 1 - f'(A) \circ B &\geq f(A) \circ 1 - 1 \circ f(B) \\ &\geq A \circ f'(B) - 1 \circ (f'(B)B),\end{aligned} \tag{2.6}$$

which gives (2.5). □

In what follows, we assume that, $\int_\Omega 1 d\mu(t) = 1$.

Theorem 2.3. *Assume that f is continuous differentiable convex on the open interval I. Let $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ be continuous fields of positive operators in $B(H)$ with spectra in I for each $\tau \in \Omega$. Then we have*

$$\begin{aligned}&\int_\Omega (f'(A_\tau)A_\tau)\,d\mu(\tau) \otimes 1 - \int_\Omega f'(A_\tau)\,d\mu(\tau) \otimes \int_\Omega B_\tau d\mu(\tau) \\ &\geq \int_\Omega f(A_\tau)\,d\mu(\tau) \otimes 1 - 1 \otimes \int_\Omega f(B_\tau)\,d\mu(\tau) \\ &\geq \int_\Omega A_\tau d\mu(\tau) \otimes \int_\Omega f'(B_\tau)\,d\mu(\tau) - 1 \otimes \int_\Omega f'(B_\tau)B_\tau d\mu(\tau)\end{aligned} \tag{2.7}$$

and the Hadamard product inequality

$$\begin{aligned}
&\int_{\Omega}\left(f'\left(A_{\tau}\right)A_{\tau}\right)d\mu\left(\tau\right)\circ 1-\int_{\Omega}f'\left(A_{\tau}\right)d\mu\left(\tau\right)\circ\int_{\Omega}B_{\tau}d\mu\left(\tau\right) \qquad (2.8)\\
&\geq\int_{\Omega}f\left(A_{\tau}\right)d\mu\left(\tau\right)\circ 1-1\circ\int_{\Omega}f\left(B_{\tau}\right)d\mu\left(\tau\right)\\
&\geq\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\circ\int_{\Omega}f'\left(B_{\tau}\right)d\mu\left(\tau\right)-1\circ\int_{\Omega}f'\left(B_{\tau}\right)B_{\tau}d\mu\left(\tau\right).
\end{aligned}$$

Proof. From Lemma 2.1 we have

$$\begin{aligned}
\left(f'\left(A_{\tau}\right)A_{\tau}\right)\otimes 1-f'\left(A_{\tau}\right)\otimes B_{\gamma}&\geq f\left(A_{\tau}\right)\otimes 1-1\otimes f\left(B_{\gamma}\right) \qquad (2.9)\\
&\geq A_{\tau}\otimes f'\left(B_{\gamma}\right)-1\otimes\left(f'\left(B_{\gamma}\right)B_{\gamma}\right).
\end{aligned}$$

for all τ, $\gamma\in\Omega$.

If we take the integral $\int_{\Omega}$ over $d\mu(\tau)$ in (2.9), then we get

$$\begin{aligned}
&\int_{\Omega}\left[\left(f'\left(A_{\tau}\right)A_{\tau}\right)\otimes 1-f'\left(A_{\tau}\right)\otimes B_{\gamma}\right]d\mu\left(\tau\right) \qquad (2.10)\\
&\geq\int_{\Omega}\left[f\left(A_{\tau}\right)\otimes 1-1\otimes f\left(B_{\gamma}\right)\right]d\mu\left(\tau\right)\\
&\geq\int_{\Omega}\left[A_{\tau}\otimes f'\left(B_{\gamma}\right)-1\otimes\left(f'\left(B_{\gamma}\right)B_{\gamma}\right)\right]d\mu\left(\tau\right)
\end{aligned}$$

for all $\gamma\in\Omega$.

By using the properties of integral and tensorial product, we derive that

$$\begin{aligned}
&\int_{\Omega}\left[\left(f'\left(A_{\tau}\right)A_{\tau}\right)\otimes 1-f'\left(A_{\tau}\right)\otimes B_{\gamma}\right]d\mu\left(\tau\right)\\
&=\int_{\Omega}\left(f'\left(A_{\tau}\right)A_{\tau}\right)d\mu\left(\tau\right)\otimes 1-\int_{\Omega}f'\left(A_{\tau}\right)d\mu\left(\tau\right)\otimes B_{\gamma},
\end{aligned}$$

$$\begin{aligned}
&\int_{\Omega}\left[f\left(A_{\tau}\right)\otimes 1-1\otimes f\left(B_{\gamma}\right)\right]d\mu\left(\tau\right)\\
&=\int_{\Omega}f\left(A_{\tau}\right)d\mu\left(\tau\right)\otimes 1-1\otimes f\left(B_{\gamma}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Omega}\left[A_{\tau}\otimes f'\left(B_{\gamma}\right)-1\otimes\left(f'\left(B_{\gamma}\right)B_{\gamma}\right)\right]d\mu\left(\tau\right)\\
&=\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\otimes f'\left(B_{\gamma}\right)-1\otimes\left(f'\left(B_{\gamma}\right)B_{\gamma}\right).
\end{aligned}$$

By utilizing (2.10) we derive

$$\begin{aligned}
&\int_{\Omega}\left(f'\left(A_{\tau}\right)A_{\tau}\right)d\mu\left(\tau\right)\otimes 1-\int_{\Omega}f'\left(A_{\tau}\right)d\mu\left(\tau\right)\otimes B_{\gamma} \qquad (2.11)\\
&\geq\int_{\Omega}f\left(A_{\tau}\right)d\mu\left(\tau\right)\otimes 1-1\otimes f\left(B_{\gamma}\right)\\
&\geq\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\otimes f'\left(B_{\gamma}\right)-1\otimes\left(f'\left(B_{\gamma}\right)B_{\gamma}\right)
\end{aligned}$$

for all $\gamma\in\Omega$.

If we take the integral $\int_{\Omega}$ over $d\mu(\gamma)$ in (2.11) and use the properties of the integral and tensorial product, we derive (2.7).

If we multiply the inequality (2.7) to the left with $\mathcal{U}^*$ and at the right with $\mathcal{U}$, use the properties of the integral, the we also get the inequality (2.8). □

Corollary 2.4. *Assume that f is continuous differentiable convex on the open interval I. Let $(A_\tau)_{\tau\in\Omega}$ be a continuous field of positive operators in $B(H)$ with spactra in I for each $\tau\in\Omega$ and B an operatorwith spectrum in I. Then we have*

$$\begin{aligned}&\int_\Omega (f'(A_\tau)A_\tau)\,d\mu(\tau)\otimes 1-\int_\Omega f'(A_\tau)\,d\mu(\tau)\otimes B \\ &\geq \int_\Omega f(A_\tau)\,d\mu(\tau)\otimes 1-1\otimes f(B) \\ &\geq \int_\Omega A_\tau d\mu(\tau)\otimes f'(B)-1\otimes(f'(B)B)\end{aligned} \tag{2.12}$$

and the Hadamard product inequality

$$\begin{aligned}&\int_\Omega (f'(A_\tau)A_\tau)\,d\mu(\tau)\circ 1-\int_\Omega f'(A_\tau)\,d\mu(\tau)\circ B \\ &\geq \int_\Omega f(A_\tau)\,d\mu(\tau)\circ 1-1\circ f(B) \\ &\geq \int_\Omega A_\tau d\mu(\tau)\circ f'(B)-1\circ(f'(B)B).\end{aligned} \tag{2.13}$$

The proof follows by Theorem 2.3 for $B_\tau=B$ for $\tau\in\Omega$.

We observe that

$$\int_\Omega A_\tau d\mu(\tau)\otimes f'(B)=\left(\int_\Omega A_\tau d\mu(\tau)\otimes 1\right)(1\otimes f'(B))$$

and

$$1\otimes(f'(B)B)=1\otimes(Bf'(B))=(1\otimes B)(1\otimes f'(B)),$$

therefore

$$\begin{aligned}&\int_\Omega A_\tau d\mu(\tau)\otimes f'(B)-1\otimes(f'(B)B) \\ &=\left(\int_\Omega A_\tau d\mu(\tau)\otimes 1\right)(1\otimes f'(B))-(1\otimes B)(1\otimes f'(B)) \\ &=\left(\int_\Omega A_\tau d\mu(\tau)\otimes 1-(1\otimes B)\right)(1\otimes f'(B))\end{aligned}$$

and from (2.12) we get

$$\begin{aligned}&\int_\Omega f(A_\tau)\,d\mu(\tau)\otimes 1-1\otimes f(B) \\ &\geq\left(\int_\Omega A_\tau d\mu(\tau)\otimes 1-(1\otimes B)\right)(1\otimes f'(B)).\end{aligned} \tag{2.14}$$

Remark 2.5. With the assumptions of Corollary 2.4 and if we take $B=\int_\Omega A_\tau d\mu(\tau)$, for which have the spectrum in I, then we have the following Jensen's type tensorial inequalities

$$\begin{aligned}&\int_\Omega (f'(A_\tau)A_\tau)\,d\mu(\tau)\otimes 1-\int_\Omega f'(A_\tau)\,d\mu(\tau)\otimes\int_\Omega A_\tau d\mu(\tau) \\ &\geq \int_\Omega f(A_\tau)\,d\mu(\tau)\otimes 1-1\otimes f\left(\int_\Omega A_\tau d\mu(\tau)\right) \\ &\geq\left(\int_\Omega A_\tau d\mu(\tau)\otimes 1-\left(1\otimes\int_\Omega A_\tau d\mu(\tau)\right)\right)\left(1\otimes f'\left(\int_\Omega A_\tau d\mu(\tau)\right)\right)\end{aligned} \tag{2.15}$$

and the Hadamard product inequalities

$$\begin{aligned}
&\int_{\Omega}\left(f'\left(A_{\tau}\right)A_{\tau}\right)d\mu\left(\tau\right)\circ 1-\int_{\Omega}f'\left(A_{\tau}\right)d\mu\left(\tau\right)\circ\int_{\Omega}A_{\tau}d\mu\left(\tau\right) \qquad (2.16)\\
&\geq\int_{\Omega}f\left(A_{\tau}\right)d\mu\left(\tau\right)\circ 1-1\circ f\left(\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\right)\\
&\geq\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\circ f'\left(\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\right)\\
&-1\circ\left(f'\left(\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\right)\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\right).
\end{aligned}$$

3. Some Examples

Assume that A, B have the spectra in I, then by (2.15) and (2.16) we get

$$\begin{aligned}
&\int_0^1 f'\left(\left(1-t\right)A+tB\right)\left(\left(1-t\right)A+tB\right)dt\otimes 1 \qquad (3.1)\\
&-\int_0^1 f'\left(\left(1-t\right)A+tB\right)dt\otimes\frac{A+B}{2}\\
&\geq\int_0^1 f\left(\left(1-t\right)A+tB\right)dt\otimes 1-1\otimes f\left(\frac{A+B}{2}\right)\\
&\geq\left(\frac{A+B}{2}\otimes 1-1\otimes\frac{A+B}{2}\right)\left(1\otimes f'\left(\frac{A+B}{2}\right)\right)
\end{aligned}$$

and the Hadamard product inequalities

$$\begin{aligned}
&\int_0^1 f'\left(\left(1-t\right)A+tB\right)\left(\left(1-t\right)A+tB\right)dt\circ 1 \qquad (3.2)\\
&-\int_0^1 f'\left(\left(1-t\right)A+tB\right)dt\circ\frac{A+B}{2}\\
&\geq\int_0^1 f\left(\left(1-t\right)A+tB\right)dt\circ 1-1\circ f\left(\frac{A+B}{2}\right)\\
&\geq\frac{A+B}{2}\circ f'\left(\frac{A+B}{2}\right)-1\circ\left(f'\left(\frac{A+B}{2}\right)\frac{A+B}{2}\right).
\end{aligned}$$

For $f\left(x\right)=\exp x$, $x\in\mathbb{R}$ and from (3.1) and (3.2) we derive the exponential inequalities

$$\begin{aligned}
&\int_0^1 \exp\left(\left(1-t\right)A+tB\right)\left(\left(1-t\right)A+tB\right)dt\otimes 1 \qquad (3.3)\\
&-\int_0^1 \exp\left(\left(1-t\right)A+tB\right)dt\otimes\frac{A+B}{2}\\
&\geq\int_0^1 \exp\left(\left(1-t\right)A+tB\right)dt\otimes 1-1\otimes f\left(\frac{A+B}{2}\right)\\
&\geq\left(\frac{A+B}{2}\otimes 1-1\otimes\frac{A+B}{2}\right)\left(1\otimes\exp\left(\frac{A+B}{2}\right)\right)
\end{aligned}$$

and the Hadamard product inequalities

$$\int_0^1 \exp\left(\left(1-t\right)A+tB\right)\left(\left(1-t\right)A+tB\right)dt\circ 1 \tag{3.4}$$
$$-\int_0^1 \exp\left(\left(1-t\right)A+tB\right)dt\circ\frac{A+B}{2}$$
$$\geq \int_0^1 \exp\left(\left(1-t\right)A+tB\right)dt\circ 1-1\circ f\left(\frac{A+B}{2}\right)$$
$$\geq \frac{A+B}{2}\circ\exp\left(\frac{A+B}{2}\right)-1\circ\left(\exp\left(\frac{A+B}{2}\right)\frac{A+B}{2}\right).$$

It is known that if A and B are commuting, i.e. $AB=BA$, then the exponential function satisfies the property

$$\exp\left(A\right)\exp\left(B\right)=\exp\left(B\right)\exp\left(A\right)=\exp\left(A+B\right).$$

Also, if A is invertible and $a,b\in\mathbb{R}$ with $a<b$ then

$$\int_a^b \exp\left(tA\right)dt=A^{-1}\left[\exp\left(bA\right)-\exp\left(aA\right)\right].$$

Moreover, if A and B are commuting and $B-A$ is invertible, then

$$\begin{aligned}\int_0^1 \exp\left(\left(1-s\right)A+sB\right)ds &= \int_0^1 \exp\left(s\left(B-A\right)\right)\exp\left(A\right)ds\\ &=\left(\int_0^1 \exp\left(s\left(B-A\right)\right)ds\right)\exp\left(A\right)\\ &=\left(B-A\right)^{-1}\left[\exp\left(B-A\right)-I\right]\exp\left(A\right)\\ &=\left(B-A\right)^{-1}\left[\exp\left(B\right)-\exp\left(A\right)\right].\end{aligned}$$

So, if A and B are commuting and $B-A$ is invertible, then by (3.3) and (3.4) we get

$$\int_0^1 \exp\left(\left(1-t\right)A+tB\right)\left(\left(1-t\right)A+tB\right)dt\otimes 1 \tag{3.5}$$
$$-\left(B-A\right)^{-1}\left[\exp\left(B\right)-\exp\left(A\right)\right]\otimes\frac{A+B}{2}$$
$$\geq\left(B-A\right)^{-1}\left[\exp\left(B\right)-\exp\left(A\right)\right]\otimes 1-1\otimes\exp\left(\frac{A+B}{2}\right)$$
$$\geq\left(\frac{A+B}{2}\otimes 1-1\otimes\frac{A+B}{2}\right)\left(1\otimes\exp\left(\frac{A+B}{2}\right)\right)$$

and the Hadamard product inequalities

$$\int_0^1 \exp\left(\left(1-t\right)A+tB\right)\left(\left(1-t\right)A+tB\right)dt\circ 1 \tag{3.6}$$
$$-\left(B-A\right)^{-1}\left[\exp\left(B\right)-\exp\left(A\right)\right]\circ\frac{A+B}{2}$$
$$\geq\left(B-A\right)^{-1}\left[\exp\left(B\right)-\exp\left(A\right)\right]\circ 1-1\circ\exp\left(\frac{A+B}{2}\right)$$
$$\geq \frac{A+B}{2}\circ\exp\left(\frac{A+B}{2}\right)-1\circ\left(\exp\left(\frac{A+B}{2}\right)\frac{A+B}{2}\right).$$

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