



Tensorial and Hadamard Product Integral Inequalities for Convex Functions of Continuous Fields of Operators in Hilbert Spaces

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Abstract

Let H be a Hilbert space and Ω a locally compact Hausdorff space endowed with a Radon measure μ with $\int_{\Omega} 1 d\mu(t) = 1$. In this paper we show among others that, if f is continuous differentiable convex on the open interval I , $(A_{\tau})_{\tau \in \Omega}$ is a continuous field of positive operators in $B(H)$ with spectra in I for each $\tau \in \Omega$ and B an operator with spectrum in I , then we have

$$\begin{aligned} & \int_{\Omega} (f'(A_{\tau}) A_{\tau}) d\mu(\tau) \otimes 1 - \int_{\Omega} f'(A_{\tau}) d\mu(\tau) \otimes B \\ & \geq \int_{\Omega} f(A_{\tau}) d\mu(\tau) \otimes 1 - 1 \otimes f(B) \\ & \geq \left(\int_{\Omega} A_{\tau} d\mu(\tau) \otimes 1 - (1 \otimes B) \right) (1 \otimes f'(B)) \end{aligned}$$

and the Hadamard product inequality

$$\begin{aligned} & \int_{\Omega} (f'(A_{\tau}) A_{\tau}) d\mu(\tau) \circ 1 - \int_{\Omega} f'(A_{\tau}) d\mu(\tau) \circ B \\ & \geq \int_{\Omega} f(A_{\tau}) d\mu(\tau) \circ 1 - 1 \circ f(B) \\ & \geq \int_{\Omega} A_{\tau} d\mu(\tau) \circ f'(B) - 1 \circ (f'(B) B). \end{aligned}$$

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1. Introduction

Let I_1, \dots, I_k be intervals from \mathbb{R} and let $f : I_1 \times \dots \times I_k \rightarrow \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, \dots, A_n)$ be a k -tuple of bounded selfadjoint operators on Hilbert spaces H_1, \dots, H_k such that the spectrum of A_i is contained in I_i for $i = 1, \dots, k$. We say that such a k -tuple is in the domain of f . If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, \dots, k$; by following [2], we define

$$f(A_1, \dots, A_k) := \int_{I_1} \dots \int_{I_k} f(\lambda_1, \dots, \lambda_k) dE_1(\lambda_1) \otimes \dots \otimes dE_k(\lambda_k) \quad (1.1)$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes \dots \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [7] for functions of two variables and have the property that

$$f(A_1, \dots, A_k) = f_1(A_1) \otimes \dots \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, \dots, t_k) = f_1(t_1) \dots f_k(t_k)$ of k functions each depending on only one variable.

It is known that, if f is super-multiplicative (sub-multiplicative) on $[0, \infty)$, namely

$$f(st) \geq (\leq) f(s)f(t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [5, p. 173]

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A, B \geq 0. \quad (1.2)$$

This follows by observing that, if

$$A = \int_{[0, \infty)} t dE(t) \text{ and } B = \int_{[0, \infty)} s dF(s)$$

are the spectral resolutions of A and B , then

$$f(A \otimes B) = \int_{[0, \infty)} \int_{[0, \infty)} f(st) dE(t) \otimes dF(s) \quad (1.3)$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A, B > 0$

$$A \#_t B := A^{1/2} (A^{-1/2} B A^{-1/2})^t A^{1/2}$$

where $t \in [0, 1]$ and

$$A \# B := A^{1/2} (A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and \otimes we have

$$A \# B = B \# A \text{ and } (A \# B) \otimes (B \# A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [8] obtained the following *Caltebaut type inequalities* for tensorial product

$$\begin{aligned} (A \# B) \otimes (A \# B) &\leq \frac{1}{2} [(A \#_\alpha B) \otimes (A \#_{1-\alpha} B) + (A \#_{1-\alpha} B) \otimes (A \#_\alpha B)] \\ &\leq \frac{1}{2} (A \otimes B + B \otimes A) \end{aligned} \quad (1.4)$$

for $A, B > 0$ and $\alpha \in [0, 1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying

$$\langle (A \circ B) e_j, e_j \rangle = \langle A e_j, e_j \rangle \langle B e_j, e_j \rangle$$

for all $j \in \mathbb{N}$, where $\{e_j\}_{j \in \mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H .

It is known that, see [4], we have the representation

$$A \circ B = \mathcal{U}^* (A \otimes B) \mathcal{U} \quad (1.5)$$

where $\mathcal{U} : H \rightarrow H \otimes H$ is the isometry defined by $\mathcal{U}e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If f is *super-multiplicative and operator concave (sub-multiplicative and operator convex)* on $[0, \infty)$, then also [5, p. 173]

$$f(A \circ B) \geq (\leq) f(A) \circ f(B) \text{ for all } A, B \geq 0. \quad (1.6)$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2} \circ B^{1/2} \leq \left(\frac{A+B}{2} \right) \circ 1 \text{ for } A, B \geq 0$$

and Fiedler inequality

$$A \circ A^{-1} \geq 1 \text{ for } A > 0.$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A \circ B \leq (A^2 \circ 1)^{1/2} (B^2 \circ 1)^{1/2} \text{ for } A, B \geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A \circ B \leq (A^2 \circ B^2)^{1/2} \text{ for } A, B \geq 0.$$

It has been shown in [6] that $(A^2 \circ 1)^{1/2} (B^2 \circ 1)^{1/2}$ and $(A^2 \circ B^2)^{1/2}$ are incomparable for 2-square positive definite matrices A and B .

Let Ω be a locally compact Hausdorff space endowed with a Radon measure μ . A field $(A_t)_{t \in \Omega}$ of operators in $B(H)$ is called a continuous field of operators if the parametrization $t \mapsto A_t$ is norm continuous on $B(H)$. If, in addition, the norm function $t \mapsto \|A_t\|$ is Lebesgue integrable on Ω , we can form the Bochner integral $\int_{\Omega} A_t d\mu(t)$, which is the unique operator in $B(H)$ such that $\varphi(\int_{\Omega} A_t d\mu(t)) = \int_{\Omega} \varphi(A_t) d\mu(t)$ for every bounded linear functional φ on $B(H)$. Assume also that, $\int_{\Omega} 1 d\mu(t) = 1$.

Motivated by the above results, in this paper we show among others that, if f is continuous differentiable convex on the open interval I , $(A_{\tau})_{\tau \in \Omega}$ is a continuous field of positive operators in $B(H)$ with spectra in I for each $\tau \in \Omega$ and B an operator with spectrum in I , then we have

$$\begin{aligned} & \int_{\Omega} (f'(A_{\tau}) A_{\tau}) d\mu(\tau) \otimes 1 - \int_{\Omega} f'(A_{\tau}) d\mu(\tau) \otimes B \\ & \geq \int_{\Omega} f(A_{\tau}) d\mu(\tau) \otimes 1 - 1 \otimes f(B) \\ & \geq \left(\int_{\Omega} A_{\tau} d\mu(\tau) \otimes 1 - (1 \otimes B) \right) (1 \otimes f'(B)) \end{aligned}$$

and the Hadamard product inequality

$$\begin{aligned} & \int_{\Omega} (f'(A_{\tau}) A_{\tau}) d\mu(\tau) \circ 1 - \int_{\Omega} f'(A_{\tau}) d\mu(\tau) \circ B \\ & \geq \int_{\Omega} f(A_{\tau}) d\mu(\tau) \circ 1 - 1 \circ f(B) \\ & \geq \int_{\Omega} A_{\tau} d\mu(\tau) \circ f'(B) - 1 \circ (f'(B) B). \end{aligned}$$

2. Main Results

We also have the following double inequality for tensorial product of operators:

Lemma 2.1. *Assume that f is continuous differentiable convex on the open interval I and A, B are selfadjoint operators in $B(H)$ with spectra in I , then*

$$\begin{aligned} (f'(A) \otimes 1)(A \otimes 1 - 1 \otimes B) &\geq f(A) \otimes 1 - 1 \otimes f(B) \\ &\geq (A \otimes 1 - 1 \otimes B)(1 \otimes f'(B)). \end{aligned} \quad (2.1)$$

Proof. Using the gradient inequality for the differentiable convex f on I we have

$$f'(t)(t-s) \geq f(t) - f(s) \geq f'(s)(t-s)$$

for all $t, s \in I$.

Assume that

$$A = \int_I t dE(t) \quad \text{and} \quad B = \int_I s dF(s)$$

are the spectral resolutions of A and B .

These imply that

$$\begin{aligned} \int_I \int_I f'(t)(t-s) dE(t) \otimes dF(s) &\geq \int_I \int_I (f(t) - f(s)) dE(t) \otimes dF(s) \\ &\geq \int_I \int_I f'(s)(t-s) dE(t) \otimes dF(s). \end{aligned} \quad (2.2)$$

Observe that

$$\begin{aligned} &\int_I \int_I f'(t)(t-s) dE(t) \otimes dF(s) \\ &= \int_I \int_I (f'(t)t - f'(t)s) dE(t) \otimes dF(s) \\ &= \int_I \int_I f'(t)t dE(t) \otimes dF(s) - \int_I \int_I f'(t)s dE(t) \otimes dF(s) \\ &= (f'(A)A) \otimes 1 - f'(A) \otimes B, \end{aligned} \quad (2.3)$$

$$\int_I \int_I (f(t) - f(s)) dE(t) \otimes dF(s) = f(A) \otimes 1 - 1 \otimes f(B)$$

and

$$\begin{aligned} &\int_I \int_I f'(s)(t-s) dE(t) \otimes dF(s) \\ &= \int_I \int_I (tf'(s) - f'(s)s) dE(t) \otimes dF(s) \\ &= \int_I \int_I tf'(s) dE(t) \otimes dF(s) - \int_I \int_I f'(s)s dE(t) \otimes dF(s) \\ &= A \otimes f'(B) - 1 \otimes (f'(B)B) \end{aligned}$$

and by (2.3) we derive the inequality of interest:

$$\begin{aligned} (f'(A)A) \otimes 1 - f'(A) \otimes B &\geq f(A) \otimes 1 - 1 \otimes f(B) \\ &\geq A \otimes f'(B) - 1 \otimes (f'(B)B). \end{aligned} \quad (2.4)$$

Now, by utilizing the tensorial property

$$(XU) \otimes (YV) = (X \otimes Y)(U \otimes V),$$

for any $X, U, Y, V \in B(H)$, we have

$$\begin{aligned} (f'(A)A) \otimes 1 &= (f'(A) \otimes 1)(A \otimes 1), \\ f'(A) \otimes B &= (f'(A) \otimes 1)(1 \otimes B), \end{aligned}$$

$$A \otimes f'(B) = (A \otimes 1)(1 \otimes f'(B))$$

and

$$1 \otimes (f'(B)B) = 1 \otimes (Bf'(B)) = (1 \otimes B)(1 \otimes f'(B)).$$

Therefore

$$\begin{aligned} (f'(A)A) \otimes 1 - f'(A) \otimes B &= (f'(A) \otimes 1)(A \otimes 1) - (f'(A) \otimes 1)(1 \otimes B) \\ &= (f'(A) \otimes 1)(A \otimes 1 - 1 \otimes B) \end{aligned}$$

and

$$\begin{aligned} A \otimes f'(B) - 1 \otimes (f'(B)B) &= (A \otimes 1)(1 \otimes f'(B)) - (1 \otimes B)(1 \otimes f'(B)) \\ &= (A \otimes 1 - 1 \otimes B)(1 \otimes f'(B)) \end{aligned}$$

and by (2.4) we derive (2.1). \square

Corollary 2.2. *Assume that f is continuous differentiable convex on the open interval I and A, B are selfadjoint operators in $B(H)$ with spectra in I , then*

$$\begin{aligned} (f'(A)A) \circ 1 - f'(A) \circ B &\geq (f(A) - f(B)) \circ 1 \\ &\geq A \circ f'(B) - (f'(B)B) \circ 1. \end{aligned} \quad (2.5)$$

Proof. If we multiply the inequality (2.4) to the left with \mathcal{U}^* and at the right with \mathcal{U} , we get

$$\begin{aligned} &\mathcal{U}^* [(f'(A)A) \otimes 1 - f'(A) \otimes B] \mathcal{U} \\ &\geq \mathcal{U}^* [f(A) \otimes 1 - 1 \otimes f(B)] \mathcal{U} \\ &\geq \mathcal{U}^* [A \otimes f'(B) - 1 \otimes (f'(B)B)] \mathcal{U}, \end{aligned}$$

namely

$$\begin{aligned} &\mathcal{U}^* ((f'(A)A) \otimes 1) \mathcal{U} - \mathcal{U}^* (f'(A) \otimes B) \mathcal{U} \\ &\geq \mathcal{U}^* (f(A) \otimes 1) \mathcal{U} - \mathcal{U}^* (1 \otimes f(B)) \mathcal{U} \\ &\geq \mathcal{U}^* (A \otimes f'(B)) \mathcal{U} - \mathcal{U}^* (1 \otimes (f'(B)B)) \mathcal{U}. \end{aligned}$$

Using representation (1.5) we get

$$\begin{aligned} (f'(A)A) \circ 1 - f'(A) \circ B &\geq f(A) \circ 1 - 1 \circ f(B) \\ &\geq A \circ f'(B) - 1 \circ (f'(B)B), \end{aligned} \quad (2.6)$$

which gives (2.5). \square

In what follows, we assume that, $\int_{\Omega} 1 d\mu(t) = 1$.

Theorem 2.3. *Assume that f is continuous differentiable convex on the open interval I . Let $(A_{\tau})_{\tau \in \Omega}$ and $(B_{\tau})_{\tau \in \Omega}$ be continuous fields of positive operators in $B(H)$ with spectra in I for each $\tau \in \Omega$. Then we have*

$$\begin{aligned} &\int_{\Omega} (f'(A_{\tau})A_{\tau}) d\mu(\tau) \otimes 1 - \int_{\Omega} f'(A_{\tau}) d\mu(\tau) \otimes \int_{\Omega} B_{\tau} d\mu(\tau) \\ &\geq \int_{\Omega} f(A_{\tau}) d\mu(\tau) \otimes 1 - 1 \otimes \int_{\Omega} f(B_{\tau}) d\mu(\tau) \\ &\geq \int_{\Omega} A_{\tau} d\mu(\tau) \otimes \int_{\Omega} f'(B_{\tau}) d\mu(\tau) - 1 \otimes \int_{\Omega} f'(B_{\tau}) B_{\tau} d\mu(\tau) \end{aligned} \quad (2.7)$$

and the Hadamard product inequality

$$\begin{aligned}
& \int_{\Omega} (f'(A_{\tau}) A_{\tau}) d\mu(\tau) \circ 1 - \int_{\Omega} f'(A_{\tau}) d\mu(\tau) \circ \int_{\Omega} B_{\tau} d\mu(\tau) \\
& \geq \int_{\Omega} f(A_{\tau}) d\mu(\tau) \circ 1 - 1 \circ \int_{\Omega} f(B_{\tau}) d\mu(\tau) \\
& \geq \int_{\Omega} A_{\tau} d\mu(\tau) \circ \int_{\Omega} f'(B_{\tau}) d\mu(\tau) - 1 \circ \int_{\Omega} f'(B_{\tau}) B_{\tau} d\mu(\tau).
\end{aligned} \tag{2.8}$$

Proof. From Lemma 2.1 we have

$$\begin{aligned}
(f'(A_{\tau}) A_{\tau}) \otimes 1 - f'(A_{\tau}) \otimes B_{\gamma} & \geq f(A_{\tau}) \otimes 1 - 1 \otimes f(B_{\gamma}) \\
& \geq A_{\tau} \otimes f'(B_{\gamma}) - 1 \otimes (f'(B_{\gamma}) B_{\gamma}).
\end{aligned} \tag{2.9}$$

for all $\tau, \gamma \in \Omega$.

If we take the integral \int_{Ω} over $d\mu(\tau)$ in (2.9), then we get

$$\begin{aligned}
& \int_{\Omega} [(f'(A_{\tau}) A_{\tau}) \otimes 1 - f'(A_{\tau}) \otimes B_{\gamma}] d\mu(\tau) \\
& \geq \int_{\Omega} [f(A_{\tau}) \otimes 1 - 1 \otimes f(B_{\gamma})] d\mu(\tau) \\
& \geq \int_{\Omega} [A_{\tau} \otimes f'(B_{\gamma}) - 1 \otimes (f'(B_{\gamma}) B_{\gamma})] d\mu(\tau)
\end{aligned} \tag{2.10}$$

for all $\gamma \in \Omega$.

By using the properties of integral and tensorial product, we derive that

$$\begin{aligned}
& \int_{\Omega} [(f'(A_{\tau}) A_{\tau}) \otimes 1 - f'(A_{\tau}) \otimes B_{\gamma}] d\mu(\tau) \\
& = \int_{\Omega} (f'(A_{\tau}) A_{\tau}) d\mu(\tau) \otimes 1 - \int_{\Omega} f'(A_{\tau}) d\mu(\tau) \otimes B_{\gamma}, \\
& \int_{\Omega} [f(A_{\tau}) \otimes 1 - 1 \otimes f(B_{\gamma})] d\mu(\tau) \\
& = \int_{\Omega} f(A_{\tau}) d\mu(\tau) \otimes 1 - 1 \otimes f(B_{\gamma})
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\Omega} [A_{\tau} \otimes f'(B_{\gamma}) - 1 \otimes (f'(B_{\gamma}) B_{\gamma})] d\mu(\tau) \\
& = \int_{\Omega} A_{\tau} d\mu(\tau) \otimes f'(B_{\gamma}) - 1 \otimes (f'(B_{\gamma}) B_{\gamma}).
\end{aligned}$$

By utilizing (2.10) we derive

$$\begin{aligned}
& \int_{\Omega} (f'(A_{\tau}) A_{\tau}) d\mu(\tau) \otimes 1 - \int_{\Omega} f'(A_{\tau}) d\mu(\tau) \otimes B_{\gamma} \\
& \geq \int_{\Omega} f(A_{\tau}) d\mu(\tau) \otimes 1 - 1 \otimes f(B_{\gamma}) \\
& \geq \int_{\Omega} A_{\tau} d\mu(\tau) \otimes f'(B_{\gamma}) - 1 \otimes (f'(B_{\gamma}) B_{\gamma})
\end{aligned} \tag{2.11}$$

for all $\gamma \in \Omega$.

If we take the integral \int_{Ω} over $d\mu(\gamma)$ in (2.11) and use the properties of the integral and tensorial product, we derive (2.7).

If we multiply the inequality (2.7) to the left with \mathcal{U}^* and at the right with \mathcal{U} , use the properties of the integral, then we also get the inequality (2.8). \square

Corollary 2.4. *Assume that f is continuous differentiable convex on the open interval I . Let $(A_\tau)_{\tau \in \Omega}$ be a continuous field of positive operators in $B(H)$ with spectra in I for each $\tau \in \Omega$ and B an operator with spectrum in I . Then we have*

$$\begin{aligned} & \int_{\Omega} (f'(A_\tau) A_\tau) d\mu(\tau) \otimes 1 - \int_{\Omega} f'(A_\tau) d\mu(\tau) \otimes B \\ & \geq \int_{\Omega} f(A_\tau) d\mu(\tau) \otimes 1 - 1 \otimes f(B) \\ & \geq \int_{\Omega} A_\tau d\mu(\tau) \otimes f'(B) - 1 \otimes (f'(B) B) \end{aligned} \quad (2.12)$$

and the Hadamard product inequality

$$\begin{aligned} & \int_{\Omega} (f'(A_\tau) A_\tau) d\mu(\tau) \circ 1 - \int_{\Omega} f'(A_\tau) d\mu(\tau) \circ B \\ & \geq \int_{\Omega} f(A_\tau) d\mu(\tau) \circ 1 - 1 \circ f(B) \\ & \geq \int_{\Omega} A_\tau d\mu(\tau) \circ f'(B) - 1 \circ (f'(B) B). \end{aligned} \quad (2.13)$$

The proof follows by Theorem 2.3 for $B_\tau = B$ for $\tau \in \Omega$.

We observe that

$$\int_{\Omega} A_\tau d\mu(\tau) \otimes f'(B) = \left(\int_{\Omega} A_\tau d\mu(\tau) \otimes 1 \right) (1 \otimes f'(B))$$

and

$$1 \otimes (f'(B) B) = 1 \otimes (B f'(B)) = (1 \otimes B) (1 \otimes f'(B)),$$

therefore

$$\begin{aligned} & \int_{\Omega} A_\tau d\mu(\tau) \otimes f'(B) - 1 \otimes (f'(B) B) \\ & = \left(\int_{\Omega} A_\tau d\mu(\tau) \otimes 1 \right) (1 \otimes f'(B)) - (1 \otimes B) (1 \otimes f'(B)) \\ & = \left(\int_{\Omega} A_\tau d\mu(\tau) \otimes 1 - (1 \otimes B) \right) (1 \otimes f'(B)) \end{aligned}$$

and from (2.12) we get

$$\begin{aligned} & \int_{\Omega} f(A_\tau) d\mu(\tau) \otimes 1 - 1 \otimes f(B) \\ & \geq \left(\int_{\Omega} A_\tau d\mu(\tau) \otimes 1 - (1 \otimes B) \right) (1 \otimes f'(B)). \end{aligned} \quad (2.14)$$

Remark 2.5. With the assumptions of Corollary 2.4 and if we take $B = \int_{\Omega} A_\tau d\mu(\tau)$, for which we have the spectrum in I , then we have the following Jensen's type tensorial inequalities

$$\begin{aligned} & \int_{\Omega} (f'(A_\tau) A_\tau) d\mu(\tau) \otimes 1 - \int_{\Omega} f'(A_\tau) d\mu(\tau) \otimes \int_{\Omega} A_\tau d\mu(\tau) \\ & \geq \int_{\Omega} f(A_\tau) d\mu(\tau) \otimes 1 - 1 \otimes f\left(\int_{\Omega} A_\tau d\mu(\tau)\right) \\ & \geq \left(\int_{\Omega} A_\tau d\mu(\tau) \otimes 1 - \left(1 \otimes \int_{\Omega} A_\tau d\mu(\tau)\right) \right) \left(1 \otimes f\left(\int_{\Omega} A_\tau d\mu(\tau)\right)\right) \end{aligned} \quad (2.15)$$

and the Hadamard product inequalities

$$\begin{aligned}
& \int_{\Omega} (f'(A_{\tau}) A_{\tau}) d\mu(\tau) \circ 1 - \int_{\Omega} f'(A_{\tau}) d\mu(\tau) \circ \int_{\Omega} A_{\tau} d\mu(\tau) \\
& \geq \int_{\Omega} f(A_{\tau}) d\mu(\tau) \circ 1 - 1 \circ f\left(\int_{\Omega} A_{\tau} d\mu(\tau)\right) \\
& \geq \int_{\Omega} A_{\tau} d\mu(\tau) \circ f'\left(\int_{\Omega} A_{\tau} d\mu(\tau)\right) \\
& - 1 \circ \left(f'\left(\int_{\Omega} A_{\tau} d\mu(\tau)\right) \int_{\Omega} A_{\tau} d\mu(\tau)\right).
\end{aligned} \tag{2.16}$$

3. Some Examples

Assume that A, B have the spectra in I , then by (2.15) and (2.16) we get

$$\begin{aligned}
& \int_0^1 f'((1-t)A + tB) ((1-t)A + tB) dt \otimes 1 \\
& - \int_0^1 f'((1-t)A + tB) dt \otimes \frac{A+B}{2} \\
& \geq \int_0^1 f((1-t)A + tB) dt \otimes 1 - 1 \otimes f\left(\frac{A+B}{2}\right) \\
& \geq \left(\frac{A+B}{2} \otimes 1 - 1 \otimes \frac{A+B}{2}\right) \left(1 \otimes f'\left(\frac{A+B}{2}\right)\right)
\end{aligned} \tag{3.1}$$

and the Hadamard product inequalities

$$\begin{aligned}
& \int_0^1 f'((1-t)A + tB) ((1-t)A + tB) dt \circ 1 \\
& - \int_0^1 f'((1-t)A + tB) dt \circ \frac{A+B}{2} \\
& \geq \int_0^1 f((1-t)A + tB) dt \circ 1 - 1 \circ f\left(\frac{A+B}{2}\right) \\
& \geq \frac{A+B}{2} \circ f'\left(\frac{A+B}{2}\right) - 1 \circ \left(f'\left(\frac{A+B}{2}\right) \frac{A+B}{2}\right).
\end{aligned} \tag{3.2}$$

For $f(x) = \exp x$, $x \in \mathbb{R}$ and from (3.1) and (3.2) we derive the exponential inequalities

$$\begin{aligned}
& \int_0^1 \exp((1-t)A + tB) ((1-t)A + tB) dt \otimes 1 \\
& - \int_0^1 \exp((1-t)A + tB) dt \otimes \frac{A+B}{2} \\
& \geq \int_0^1 \exp((1-t)A + tB) dt \otimes 1 - 1 \otimes \exp\left(\frac{A+B}{2}\right) \\
& \geq \left(\frac{A+B}{2} \otimes 1 - 1 \otimes \frac{A+B}{2}\right) \left(1 \otimes \exp\left(\frac{A+B}{2}\right)\right)
\end{aligned} \tag{3.3}$$

and the Hadamard product inequalities

$$\begin{aligned} & \int_0^1 \exp((1-t)A + tB) ((1-t)A + tB) dt \circ 1 \\ & - \int_0^1 \exp((1-t)A + tB) dt \circ \frac{A+B}{2} \\ & \geq \int_0^1 \exp((1-t)A + tB) dt \circ 1 - 1 \circ f\left(\frac{A+B}{2}\right) \\ & \geq \frac{A+B}{2} \circ \exp\left(\frac{A+B}{2}\right) - 1 \circ \left(\exp\left(\frac{A+B}{2}\right) \frac{A+B}{2}\right). \end{aligned} \tag{3.4}$$

It is known that if A and B are commuting, i.e. $AB = BA$, then the exponential function satisfies the property

$$\exp(A) \exp(B) = \exp(B) \exp(A) = \exp(A+B).$$

Also, if A is invertible and $a, b \in \mathbb{R}$ with $a < b$ then

$$\int_a^b \exp(tA) dt = A^{-1} [\exp(bA) - \exp(aA)].$$

Moreover, if A and B are commuting and $B - A$ is invertible, then

$$\begin{aligned} \int_0^1 \exp((1-s)A + sB) ds &= \int_0^1 \exp(s(B-A)) \exp(A) ds \\ &= \left(\int_0^1 \exp(s(B-A)) ds \right) \exp(A) \\ &= (B-A)^{-1} [\exp(B-A) - I] \exp(A) \\ &= (B-A)^{-1} [\exp(B) - \exp(A)]. \end{aligned}$$

So, if A and B are commuting and $B - A$ is invertible, then by (3.3) and (3.4) we get

$$\begin{aligned} & \int_0^1 \exp((1-t)A + tB) ((1-t)A + tB) dt \otimes 1 \\ & - (B-A)^{-1} [\exp(B) - \exp(A)] \otimes \frac{A+B}{2} \\ & \geq (B-A)^{-1} [\exp(B) - \exp(A)] \otimes 1 - 1 \otimes \exp\left(\frac{A+B}{2}\right) \\ & \geq \left(\frac{A+B}{2} \otimes 1 - 1 \otimes \frac{A+B}{2}\right) \left(1 \otimes \exp\left(\frac{A+B}{2}\right)\right) \end{aligned} \tag{3.5}$$

and the Hadamard product inequalities

$$\begin{aligned} & \int_0^1 \exp((1-t)A + tB) ((1-t)A + tB) dt \circ 1 \\ & - (B-A)^{-1} [\exp(B) - \exp(A)] \circ \frac{A+B}{2} \\ & \geq (B-A)^{-1} [\exp(B) - \exp(A)] \circ 1 - 1 \circ \exp\left(\frac{A+B}{2}\right) \\ & \geq \frac{A+B}{2} \circ \exp\left(\frac{A+B}{2}\right) - 1 \circ \left(\exp\left(\frac{A+B}{2}\right) \frac{A+B}{2}\right). \end{aligned} \tag{3.6}$$

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