Turk. J. Math. Comput. Sci.
17(1)(2025) 47–58
© MatDer
DOI : 10.47000/tjmcs.1362700

Some Tensorial and Hadamard Product Inequalities for Convex Functions of Selfadjoint Operators in Hilbert Spaces

Silvestru Sever Dragomir [1,2]

[1] *Applied Mathematics Research Group, ISILC, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.*
[2] *School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa.*

Received: 19-09-2023 • Accepted: 25-06-2024

Abstract. Let H be a Hilbert space. In this paper we show among others that, if f is continuous differentiable convex on the open interval I and A, B are selfadjoint operators in $B(H)$ with spectra $\mathrm{Sp}(A), \mathrm{Sp}(B) \subset I$, then we have the tensorial inequality

$$\begin{aligned}(f'(A) \otimes 1)(A \otimes 1 - 1 \otimes B) &\geq f(A) \otimes 1 - 1 \otimes f(B) \\ &\geq (A \otimes 1 - 1 \otimes B)(1 \otimes f'(B))\end{aligned}$$

and the inequality for Hadamard product

$$\begin{aligned}(f'(A) A) \circ 1 - f'(A) \circ B &\geq [f(A) - f(B)] \circ 1 \\ &\geq A \circ f'(B) - (f'(B) B) \circ 1.\end{aligned}$$

.

2020 AMS Classification: 47A63, 47A99

Keywords: Tensorial product, Hadamard product, selfadjoint operators, convex functions.

1. Introduction

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_k)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2], we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_k) dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k)$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

Email address: sever.dragomir@vu.edu.au (S.S. Dragomir)

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [7] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [5, p. 173]

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A,\ B \geq 0. \tag{1.1}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st) \, dE(t) \otimes dF(s)$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive invertible operators $A, B > 0$

$$A \#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t \in [0, 1]$ and

$$A \# B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of # and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B) \otimes (B\#A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [8] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned}(A\#B) \otimes (A\#B) &\leq \frac{1}{2} \left[(A\#_\alpha B) \otimes (A\#_{1-\alpha} B) + (A\#_{1-\alpha} B) \otimes (A\#_\alpha B)\right] \\ &\leq \frac{1}{2} (A \otimes B + B \otimes A)\end{aligned}$$

for $A, B > 0$ and $\alpha \in [0, 1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying

$$\left\langle (A \circ B) e_j, e_j \right\rangle = \left\langle Ae_j, e_j \right\rangle \left\langle Be_j, e_j \right\rangle$$

for all $j \in \mathbb{N}$, where $\left\{e_j\right\}_{j \in \mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [4], we have the representation

$$A \circ B = \mathcal{U}^* (A \otimes B) \mathcal{U}, \tag{1.2}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U} e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If *f is super-multiplicative and operator convex (sub-multiplicative and operator concave)* on $[0, \infty)$, then also [5, p. 173]

$$f(A \circ B) \geq (\leq) f(A) \circ f(B) \text{ for all } A,\ B \geq 0.$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2} \circ B^{1/2} \leq \left(\frac{A + B}{2}\right) \circ 1 \text{ for } A,\ B \geq 0$$

and Fiedler inequality

$$A \circ A^{-1} \geq 1 \text{ for } A > 0.$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A \circ B \leq \left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2} \text{ for } A,\ B \geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A \circ B \leq \left(A^2 \circ B^2\right)^{1/2} \text{ for } A,\ B \geq 0.$$

It has been shown in [6] that $\left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2}$ and $\left(A^2 \circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Motivated by the above results, in this paper we show among others that, if f is continuous differentiable convex on the open interval I and A, B are selfadjoint operators in $B(H)$ with spectra $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, then we have the tensorial inequality

$$\begin{aligned}(f'(A) \otimes 1)(A \otimes 1 - 1 \otimes B) &\geq f(A) \otimes 1 - 1 \otimes f(B) \\ &\geq (A \otimes 1 - 1 \otimes B)(1 \otimes f'(B))\end{aligned}$$

and the inequality for Hadamard product

$$\begin{aligned}(f'(A) A) \circ 1 - f'(A) \circ B &\geq (f(A) - f(B)) \circ 1 \\ &\geq A \circ f'(B) - (f'(B) B) \circ 1.\end{aligned}$$

2. Main Results

We start to the following result that is related to super/sub-multiplicative tensorial inequalities in (1.1):

Theorem 2.1. *Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. Assume that $0 \leq r < R$ and $0 \leq A, B \leq 1$, then*

$$h(r)\, h(rA \otimes B) \geq h(rA) \otimes h(rB). \tag{2.1}$$

If $R = \infty$, then the inequality (2.1) also holds for A, $B \geq 1$. In this case for R, if either $0 \leq A \leq 1$ and $B \geq 1$ or $A \geq 1$ and $0 \leq B \leq 1$, then the reverse inequality in (2.1) holds as well.

Proof. We use the Čebyšev inequality for synchronous (the same monotonicity) sequences $(c_i)_{i\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}}$ and nonnegative weights $(p_i)_{i\in\mathbb{N}}$:

$$\sum_{i=0}^{n} p_i \sum_{i=0}^{n} p_i c_i b_i \geq \sum_{i=0}^{n} p_i c_i \sum_{i=0}^{n} p_i b_i,$$

for any $n \in \mathbb{N}$.

Assume that $0 < r < R$. Let t, $s \in (0,1)$ and define the sequences $c_i := t^i$, $b_i := s^i$. These sequences are decreasing and if we apply Čebyšev's inequality for these sequences and the weights $p_i := a_i r^i \geq 0$ we get

$$\sum_{i=0}^{n} a_i r^i \sum_{i=0}^{n} a_i (rts)^i \geq \sum_{i=0}^{n} a_i (rt)^i \sum_{i=0}^{n} a_i (rs)^i \tag{2.2}$$

for any $n \in \mathbb{N}$.

Since the series

$$\sum_{i=0}^{\infty} a_i r^i,\ \sum_{i=0}^{\infty} a_i (rts)^i,\ \sum_{i=0}^{\infty} a_i (rt)^i \text{ and } \sum_{i=0}^{\infty} a_i (rs)^i$$

are convergent, then by letting $n \to \infty$ in (2.2) we get

$$h(r)\, h(rts) \geq h(rt)\, h(rs) \tag{2.3}$$

for all $0 < r < R$ and t, $s \in [0,1]$.

Consider the function

$$h_r(t) = \frac{h(rt)}{h(r)},\ t \in [0,1].$$

We observe that, by (2.3), the function h_r is super-multiplicative on $[0,1]$ and by making use of (1.1) we derive the desired result (2.1).

The other parts of the theorem follow in a similar way, we omit the details. □

Corollary 2.2. *With the assumptions of Theorem 2.1 and if h is operator concave on $[0, R)$, then*

$$h(r)\,h(rA \circ B) \geq h(rA) \circ h(rB) \tag{2.4}$$

for either $0 \leq A, B \leq 1$ or $A, B \geq 1$ in the case when $R = \infty$. In this last case for R, if h is operator convex on $[0, \infty)$ and either $0 \leq A \leq 1$ and $B \geq 1$ or $A \geq 1$ and $0 \leq B \leq 1$ then the reverse inequality in (2.4) holds as well.

Proof. As in [5, p. 173], by using Davis-Choi-Jensen's inequality we have

$$\begin{aligned} h(r)\,h(rA \circ B) &= h(r)\,h(r\mathcal{U}^* (A \otimes B)\,\mathcal{U}) \geq h(r)\,\mathcal{U}^* h(rA \otimes B)\,\mathcal{U} \\ &\geq \mathcal{U}^* (h(rA) \otimes h(rB))\,\mathcal{U} = h(rA) \circ h(rB). \end{aligned}$$

and the inequality (2.4) is proved. □

We also have the following double inequality for tensorial product of operators:

Theorem 2.3. *Assume that f is continuous differentiable convex on the open interval I and A, B are selfadjoint operators in $B(H)$ with spectra $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, then*

$$\begin{aligned} (f'(A) \otimes 1)(A \otimes 1 - 1 \otimes B) &\geq f(A) \otimes 1 - 1 \otimes f(B) \\ &\geq (A \otimes 1 - 1 \otimes B)(1 \otimes f'(B)). \end{aligned} \tag{2.5}$$

Proof. Using the gradient inequality for the differentiable convex f on I we have

$$f'(t)(t-s) \geq f(t) - f(s) \geq f'(s)(t-s)$$

for all $t, s \in I$.

Assume that

$$A = \int_I t dE(t) \text{ and } B = \int_I s dF(s)$$

are the spectral resolutions of A and B.

These imply that

$$\begin{aligned} \int_I \int_I f'(t)(t-s)\,dE(t) \otimes dF(s) &\geq \int_I \int_I (f(t) - f(s))\,dE(t) \otimes dF(s) \\ &\geq \int_I \int_I f'(s)(t-s)\,dE(t) \otimes dF(s). \end{aligned}$$

Observe that

$$\begin{aligned} \int_I \int_I f'(t)(t-s)\,dE(t) \otimes dF(s) &= \int_I \int_I (f'(t)\,t - f'(t)\,s)\,dE(t) \otimes dF(s) \\ &= \int_I \int_I f'(t)\,t dE(t) \otimes dF(s) - \int_I \int_I f'(t)\,s dE(t) \otimes dF(s) \\ &= (f'(A)\,A) \otimes 1 - f'(A) \otimes B, \end{aligned} \tag{2.6}$$

$$\int_I \int_I (f(t) - f(s))\,dE(t) \otimes dF(s) = f(A) \otimes 1 - 1 \otimes f(B)$$

and

$$\begin{aligned} \int_I \int_I f'(s)(t-s)\,dE(t) \otimes dF(s) &= \int_I \int_I (tf'(s) - f'(s)\,s)\,dE(t) \otimes dF(s) \\ &= \int_I \int_I tf'(s)\,dE(t) \otimes dF(s) - \int_I \int_I f'(s)\,s dE(t) \otimes dF(s) \\ &= A \otimes f'(B) - 1 \otimes (f'(B)\,B) \end{aligned}$$

and by (2.6) we derive the inequality of interest:

$$\begin{aligned} (f'(A)\,A) \otimes 1 - f'(A) \otimes B &\geq f(A) \otimes 1 - 1 \otimes f(B) \\ &\geq A \otimes f'(B) - 1 \otimes (f'(B)\,B). \end{aligned} \tag{2.7}$$

Now, by utilizing the tensorial property

$$(XU)\otimes(YV)=(X\otimes Y)(U\otimes V),$$

for any $X, U, Y, V \in B(H)$, we have

$$(f'(A)A)\otimes 1=(f'(A)\otimes 1)(A\otimes 1),$$

$$f'(A)\otimes B=(f'(A)\otimes 1)(1\otimes B),$$

$$A\otimes f'(B)=(A\otimes 1)(1\otimes f'(B))$$

and

$$1\otimes(f'(B)B)=1\otimes(Bf'(B))=(1\otimes B)(1\otimes f'(B)).$$

Therefore,

$$\begin{aligned}(f'(A)A)\otimes 1-f'(A)\otimes B&=(f'(A)\otimes 1)(A\otimes 1)-(f'(A)\otimes 1)(1\otimes B)\\&=(f'(A)\otimes 1)(A\otimes 1-1\otimes B)\end{aligned}$$

and

$$\begin{aligned}A\otimes f'(B)-1\otimes(f'(B)B)&=(A\otimes 1)(1\otimes f'(B))-(1\otimes B)(1\otimes f'(B))\\&=(A\otimes 1-1\otimes B)(1\otimes f'(B))\end{aligned}$$

and by (2.7) we derive (2.5). □

Corollary 2.4. *With the assumptions of Theorem 2.3 and if $A_j \in B(H)$ with spectra $\operatorname{Sp}(A_j)\subset I$, $p_j\geq 0$ for $j\in\{1,...,n\}$ with $\sum_{j=1}^{n}p_j=1$, then*

$$\begin{aligned}&\left(\sum_{j=1}^{n}p_jf'(A_j)A_j\right)\otimes 1-\left(\sum_{j=1}^{n}p_jf'(A_j)\right)\otimes B\\&\geq\left(\sum_{j=1}^{n}p_jf(A_j)\right)\otimes 1-1\otimes f(B)\\&\geq\left(\left(\sum_{j=1}^{n}p_jA_j\right)\otimes 1-1\otimes B\right)(1\otimes f'(B)).\end{aligned}\tag{2.8}$$

In particular, we have

$$\begin{aligned}&\left(\sum_{j=1}^{n}p_jf'(A_j)A_j\right)\otimes 1-\left(\sum_{j=1}^{n}p_jf'(A_j)\right)\otimes\left(\sum_{j=1}^{n}p_jA_j\right)\\&\geq\left(\sum_{j=1}^{n}p_jf(A_j)\right)\otimes 1-1\otimes f\left(\sum_{j=1}^{n}p_jA_j\right)\\&\geq\left(\left(\sum_{j=1}^{n}p_jA_j\right)\otimes 1-1\otimes\left(\sum_{j=1}^{n}p_jA_j\right)\right)\left(1\otimes f'\left(\sum_{j=1}^{n}p_jA_j\right)\right).\end{aligned}\tag{2.9}$$

Proof. From Theorem 2.3 we have

$$\begin{aligned}(f'(A_j)A_j)\otimes 1-f'(A_j)\otimes B&\geq f(A_j)\otimes 1-1\otimes f(B)\\&\geq(A_j\otimes 1-1\otimes B)(1\otimes f'(B))\end{aligned}$$

for $j\in\{1,...,n\}$.

If we multiply by $p_j \geq 0$, $j \in \{1, ..., n\}$ and then sum from 1 to n, then we get

$$\begin{aligned}
&\sum_{j=1}^{n} p_j \left(f'\left(A_j\right) A_j\right) \otimes 1 - \sum_{j=1}^{n} p_j f'\left(A_j\right) \otimes B \\
&\geq \sum_{j=1}^{n} p_j f\left(A_j\right) \otimes 1 - \sum_{j=1}^{n} p_j \left(1 \otimes f(B)\right) \\
&\geq \sum_{j=1}^{n} p_j \left(A_j \otimes 1 - 1 \otimes B\right) \left(1 \otimes f'(B)\right) \\
&= \left(\left(\sum_{j=1}^{n} p_j A_j\right) \otimes 1 - 1 \otimes B\right) \left(1 \otimes f'(B)\right),
\end{aligned}$$

for a selfadjoint operator B with $\operatorname{Sp}(B) \subset I$, which gives (2.8).

Since $\operatorname{Sp}\left(A_j\right) \subset I$ and $p_j \geq 0$ for $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$, hence $\operatorname{Sp} Sp\left(\sum_{j=1}^{n} p_j A_j\right) \subset I$ and by taking $B = \sum_{j=1}^{n} p_j A_j$ in (2.8), we get (2.9). □

Remark 2.5. With the assumptions of Corollary 2.4 and if

$$\left(\sum_{j=1}^{n} p_j A_j\right) \otimes 1 = 1 \otimes \left(\sum_{j=1}^{n} p_j A_j\right), \tag{2.10}$$

then

$$\begin{aligned}
&\left(\sum_{j=1}^{n} p_j f'\left(A_j\right) A_j\right) \otimes 1 - \left(\sum_{j=1}^{n} p_j f'\left(A_j\right)\right) \otimes \left(\sum_{j=1}^{n} p_j A_j\right) \\
&\geq \left(\sum_{j=1}^{n} p_j f\left(A_j\right)\right) \otimes 1 - 1 \otimes f\left(\sum_{j=1}^{n} p_j A_j\right) \geq 0.
\end{aligned}$$

Theorem 2.6. *Assume that f is continuous differentiable convex on the open interval I and A, B are selfadjoint operators in $B(H)$ with spectra $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, then*

$$\begin{aligned}
\left(f'(A) A\right) \circ 1 - f'(A) \circ B &\geq \left(f(A) - f(B)\right) \circ 1 \\
&\geq A \circ f'(B) - \left(f'(B) B\right) \circ 1.
\end{aligned} \tag{2.11}$$

Proof. If we multiply the inequality (2.7) to the left with $\mathcal{U}^*$ and at the right with $\mathcal{U}$, we get

$$\begin{aligned}
&\mathcal{U}^* \left[\left(f'(A) A\right) \otimes 1 - f'(A) \otimes B\right] \mathcal{U} \\
&\geq \mathcal{U}^* \left[f(A) \otimes 1 - 1 \otimes f(B)\right] \mathcal{U} \\
&\geq \mathcal{U}^* \left[A \otimes f'(B) - 1 \otimes \left(f'(B) B\right)\right] \mathcal{U},
\end{aligned}$$

namely

$$\begin{aligned}
&\mathcal{U}^* \left(\left(f'(A) A\right) \otimes 1\right) \mathcal{U} - \mathcal{U}^* \left(f'(A) \otimes B\right) \mathcal{U} \\
&\geq \mathcal{U}^* \left(f(A) \otimes 1\right) \mathcal{U} - \mathcal{U}^* \left(1 \otimes f(B)\right) \mathcal{U} \\
&\geq \mathcal{U}^* \left(A \otimes f'(B)\right) \mathcal{U} - \mathcal{U}^* \left(1 \otimes \left(f'(B) B\right)\right) \mathcal{U}.
\end{aligned}$$

Using representation (1.2) we get

$$\begin{aligned}
\left(f'(A) A\right) \circ 1 - f'(A) \circ B &\geq f(A) \circ 1 - 1 \circ f(B) \\
&\geq A \circ f'(B) - 1 \circ \left(f'(B) B\right),
\end{aligned}$$

which gives (2.11). □

Remark 2.7. If $\left\{e_j\right\}_{j\in\mathbb{N}}$ is an orthonormal basis for the separable Hilbert space H, then, under the assumptions of Theorem 2.6, we have

$$\begin{aligned}&\left\langle f'(A)Ae_j, e_j\right\rangle - \left\langle f'(A)e_j, e_j\right\rangle\left\langle Be_j, e_j\right\rangle \\ &\geq \left\langle f(A)e_j, e_j\right\rangle - \left\langle f(B)e_j, e_j\right\rangle \\ &\geq \left\langle Ae_j, e_j\right\rangle\left\langle f'(B)e_j, e_j\right\rangle - \left\langle f'(B)Be_j, e_j\right\rangle,\end{aligned}$$

for all $j \in \mathbb{N}$.

Corollary 2.8. *With the assumptions of Theorem 2.6 and if $A_j \in B(H)$ with spectra $\operatorname{Sp}\left(A_j\right) \subset I$, $p_j \geq 0$ for $j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$, then*

$$\begin{aligned}&\left(\sum_{j=1}^n p_j f'\left(A_j\right)A_j\right)\circ 1 - \left(\sum_{j=1}^n p_j f'\left(A_j\right)\right)\circ B \\ &\geq \left(\sum_{j=1}^n p_j f\left(A_j\right) - f(B)\right)\circ 1 \\ &\geq \left(\sum_{j=1}^n p_j A_j\right)\circ f'(B) - \left(f'(B)B\right)\circ 1.\end{aligned} \tag{2.12}$$

In particular,

$$\begin{aligned}&\left(\sum_{j=1}^n p_j f'\left(A_j\right)A_j\right)\circ 1 - \left(\sum_{j=1}^n p_j f'\left(A_j\right)\right)\circ\left(\sum_{j=1}^n p_j A_j\right) \\ &\geq \left(\sum_{j=1}^n p_j f\left(A_j\right)\right)\circ 1 - f\left(\sum_{j=1}^n p_j A_j\right)\circ 1 \\ &\geq \left(\sum_{j=1}^n p_j A_j\right)\circ f'\left(\sum_{j=1}^n p_j A_j\right) - \left(f'\left(\sum_{j=1}^n p_j A_j\right)\sum_{j=1}^n p_j A_j\right)\circ 1.\end{aligned} \tag{2.13}$$

Proof. If we replace in (2.11) $B = A_j$, multiply by p_j and sum over j from 1 to n, then we get (2.12).

The inequality (2.13) follows by taking $B = \sum_{j=1}^n p_j A_j$ in (2.12). □

3. Some Examples

Let $h(z) = \sum_{n=0}^\infty a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. We have the following examples

$$\begin{aligned}h(z) &= \sum_{n=1}^\infty \frac{1}{n}z^n = \ln\frac{1}{1-z},\ z \in D(0,1); \\ h(z) &= \sum_{n=0}^\infty \frac{1}{(2n)!}z^{2n} = \cosh z,\ z \in \mathbb{C}; \\ h(z) &= \sum_{n=0}^\infty \frac{1}{(2n+1)!}z^{2n+1} = \sinh z,\ z \in \mathbb{C}; \\ h(z) &= \sum_{n=0}^\infty z^n = \frac{1}{1-z},\ z \in D(0,1).\end{aligned}$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$h(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = \exp(z) \qquad z \in \mathbb{C},$$
$$h(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1} = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right), \qquad z \in D(0,1);$$
$$h(z) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!} z^{2n+1} = \sin^{-1}(z), \qquad z \in D(0,1);$$

and

$$h(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1} = \tanh^{-1}(z), \qquad z \in D(0,1)$$
$$h(z) = {}_2F_1(\alpha,\beta,\gamma,z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \alpha,\beta,\gamma > 0,$$
$$z \in D(0,1);$$

where Γ is *Gamma function*.

Assume that $0 < r < 1$ and $0 \le A, B \le 1$, then by (2.1) for $h(z) = (1-z)^{-1}$ we get

$$(1-r)^{-1}(1-rA \otimes B)^{-1} \ge (1-rA)^{-1} \otimes (1-rB)^{-1},$$

for $h(z) = \ln(1-z)^{-1}$ we obtain

$$\ln(1-r)^{-1} \ln(1-rA \otimes B)^{-1} \ge \ln(1-rA)^{-1} \otimes \ln(1-rB)^{-1},$$

while for $h(z) = \sin^{-1}(z)$ we derive

$$\sin^{-1}(r)\sin^{-1}(rA \otimes B) \ge \sin^{-1}(rA) \otimes \sin^{-1}(rB).$$

If $r > 0$ and either $0 \le A, B \le 1$ or $A, B \ge 1$, then by (2.1) for $h(z) = \exp z$ we get

$$\exp(r(1 + A \otimes B)) \ge \exp(rA) \otimes \exp(rB). \tag{3.1}$$

If either $0 \le A \le 1$ and $B \ge 1$ or $A \ge 1$ and $0 \le B \le 1$ then the reverse inequality in (3.1) holds as well.

By (2.1) for $h(z) = \cosh z$ or $\sinh z$ we get

$$\cosh(r)\cosh(rA \otimes B) \ge \cosh(rA) \otimes \cosh(rB) \tag{3.2}$$

or

$$\sinh(r)\sinh(rA \otimes B) \ge \sinh(rA) \otimes \sinh(rB) \tag{3.3}$$

for either $0 \le A, B \le 1$ or $A, B \ge 1$.

If either $0 \le A \le 1$ and $B \ge 1$ or $A \ge 1$ and $0 \le B \le 1$, then the reverse inequality in (3.2) or (3.3) holds as well.

If we take the convex function $f(t) = -\ln t$, $t > 0$, then from (2.7) for $A, B > 0$ we get

$$1 - A^{-1} \otimes B \le (\ln A) \otimes 1 - 1 \otimes (\ln B) \le A \otimes B^{-1} - 1.$$

From (2.9) we get

$$\begin{aligned}
&\left(\sum_{j=1}^{n} p_j A_j^{-1}\right) \otimes \left(\sum_{j=1}^{n} p_j A_j\right) - 1 \\
&\ge 1 \otimes \ln\left(\sum_{j=1}^{n} p_j A_j\right) - \left(\sum_{j=1}^{n} p_j \ln A_j\right) \otimes 1 \\
&\ge \left(1 \otimes \left(\sum_{j=1}^{n} p_j A_j\right) - \left(\sum_{j=1}^{n} p_j A_j\right) \otimes 1\right)\left(1 \otimes \left(\sum_{j=1}^{n} p_j A_j\right)^{-1}\right),
\end{aligned}$$

where $A_j > 0$ and $p_j \ge 0$ for $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$.

Moreover, if the condition (2.10) is satisfied, then

$$\left(\sum_{j=1}^{n} p_j A_j^{-1}\right) \otimes \left(\sum_{j=1}^{n} p_j A_j\right) - 1$$
$$\geq 1 \otimes \ln\left(\sum_{j=1}^{n} p_j A_j\right) - \left(\sum_{j=1}^{n} p_j \ln A_j\right) \otimes 1 \geq 0.$$

From (2.11) we get

$$A^{-1} \circ B - 1 \geq (\ln B - \ln A) \circ 1 \geq 1 - A \circ B^{-1}$$

for $A, B > 0$.

If $A_j > 0$ and $p_j \geq 0$ for $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$ then by (2.13) we derive

$$\left(\sum_{j=1}^{n} p_j A_j^{-1}\right) \circ \left(\sum_{j=1}^{n} p_j A_j\right) - 1$$
$$\geq \left(\ln\left(\sum_{j=1}^{n} p_j A_j\right) - \sum_{j=1}^{n} p_j \ln A_j\right) \circ 1$$
$$\geq \left(\sum_{j=1}^{n} p_j A_j\right) \circ \left(\sum_{j=1}^{n} p_j A_j\right)^{-1} - 1 \geq 0.$$

The last inequality follows by Fiedler inequality $B \circ B^{-1} \geq 1$, see for instance [5, p. 176].

If we take the convex function $f(t) = t \ln t$, $t > 0$, then from (2.7) for $A, B > 0$ we get

$$((\ln A) \otimes 1 + 1)(A \otimes 1 - 1 \otimes B) \geq (A \ln A) \otimes 1 - 1 \otimes (B \ln B)$$
$$\geq (A \otimes 1 - 1 \otimes B)(1 \otimes \ln B + 1).$$

From (2.9) we get

$$\left(\sum_{j=1}^{n} p_j A_j \ln A_j + \sum_{j=1}^{n} p_j A_j\right) \otimes 1$$
$$- \left(\sum_{j=1}^{n} p_j \ln\left(A_j\right) + 1\right) \otimes \left(\sum_{j=1}^{n} p_j A_j\right)$$

$$\geq \left(\sum_{j=1}^{n} p_j A_j \ln A_j\right) \otimes 1 - 1 \otimes \left[\left(\sum_{j=1}^{n} p_j A_j\right) \ln\left(\sum_{j=1}^{n} p_j A_j\right)\right]$$
$$\geq \left(\left(\sum_{j=1}^{n} p_j A_j\right) \otimes 1 - 1 \otimes \left(\sum_{j=1}^{n} p_j A_j\right)\right)$$
$$\times \left(1 \otimes \ln\left(\sum_{j=1}^{n} p_j A_j\right) + 1\right),$$

where $A_j > 0$ and $p_j \geq 0$ for $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$.

From (2.11) we get

$$(A \ln A + A) \circ 1 - (\ln A + 1) \circ B \geq (A \ln A - B \ln B) \circ 1$$
$$\geq A \circ (\ln B + 1) - (B \ln B + B) \circ 1$$

for $A, B > 0$.

From (2.13) we get

$$\begin{aligned}
&\left(\sum_{j=1}^{n} p_j A_j \ln\left(A_j\right)\right) \circ 1 - \left(\sum_{j=1}^{n} p_j \ln\left(A_j\right)\right) \circ \left(\sum_{j=1}^{n} p_j A_j\right) \\
&\geq \left(\sum_{j=1}^{n} p_j A_j \ln A_j - \left(\sum_{j=1}^{n} p_j A_j\right) \ln\left(\sum_{j=1}^{n} p_j A_j\right)\right) \circ 1 \\
&\geq \left(\sum_{j=1}^{n} p_j A_j\right) \circ \ln\left(\sum_{j=1}^{n} p_j A_j\right) - \left[\ln\left(\sum_{j=1}^{n} p_j A_j\right) \sum_{j=1}^{n} p_j A_j\right] \circ 1,
\end{aligned}$$

where $A_j > 0$ and $p_j \geq 0$ for $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$.

If we write the inequality (2.5) for the convex function $f(t) = t^r$, $r \in (-\infty, 0) \cup [1, \infty)$, then we get

$$\begin{aligned}
r\left(A^{r-1} \otimes 1\right)(A \otimes 1 - 1 \otimes B) &\geq A^r \otimes 1 - 1 \otimes B^r \\
&\geq r(A \otimes 1 - 1 \otimes B)\left(1 \otimes B^{r-1}\right),
\end{aligned}$$

for $A, B > 0$.

For $r = 2$, we get

$$\begin{aligned}
2(A \otimes 1)(A \otimes 1 - 1 \otimes B) &\geq A^2 \otimes 1 - 1 \otimes B^2 \\
&\geq 2(A \otimes 1 - 1 \otimes B)(1 \otimes B),
\end{aligned}$$

while for $r = -1$ we get

$$\begin{aligned}
\left(A^{-2} \otimes 1\right)(1 \otimes B - A \otimes 1) &\geq A^{-1} \otimes 1 - 1 \otimes B^{-1} \\
&\geq (1 \otimes B - A \otimes 1)\left(1 \otimes B^{-2}\right),
\end{aligned}$$

for $A, B > 0$.

From (2.9) we derive

$$\begin{aligned}
&r\left[\left(\sum_{j=1}^{n} p_j A_j^r\right) \otimes 1 - \left(\sum_{j=1}^{n} p_j A_j^{r-1}\right) \otimes \left(\sum_{j=1}^{n} p_j A_j\right)\right] \\
&\geq \left(\sum_{j=1}^{n} p_j A_j^r\right) \otimes 1 - 1 \otimes \left(\sum_{j=1}^{n} p_j A_j\right)^r \\
&\geq r\left(\left(\sum_{j=1}^{n} p_j A_j\right) \otimes 1 - 1 \otimes \left(\sum_{j=1}^{n} p_j A_j\right)\right)\left(1 \otimes \left(\sum_{j=1}^{n} p_j A_j\right)^{r-1}\right),
\end{aligned}$$

where $A_j > 0$ and $p_j \geq 0$ for $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$.

For $r = 2$ we get

$$\begin{aligned}
&2\left[\left(\sum_{j=1}^{n} p_j A_j^2\right) \otimes 1 - \left(\sum_{j=1}^{n} p_j A_j\right) \otimes \left(\sum_{j=1}^{n} p_j A_j\right)\right] \\
&\geq \left(\sum_{j=1}^{n} p_j A_j^2\right) \otimes 1 - 1 \otimes \left(\sum_{j=1}^{n} p_j A_j\right)^2 \\
&\geq 2\left(\left(\sum_{j=1}^{n} p_j A_j\right) \otimes 1 - 1 \otimes \left(\sum_{j=1}^{n} p_j A_j\right)\right)\left(1 \otimes \left(\sum_{j=1}^{n} p_j A_j\right)\right),
\end{aligned}$$

while for $r=-1$, we get

$$\begin{aligned}
&\left(\sum_{j=1}^{n} p_j A_j^{-2}\right) \otimes \left(\sum_{j=1}^{n} p_j A_j\right) - \left(\sum_{j=1}^{n} p_j A_j^{-1}\right) \otimes 1 \\
&\geq \left(\sum_{j=1}^{n} p_j A_j^{-1}\right) \otimes 1 - 1 \otimes \left(\sum_{j=1}^{n} p_j A_j\right)^{-1} \\
&\geq \left(1 \otimes \left(\sum_{j=1}^{n} p_j A_j\right) - \left(\sum_{j=1}^{n} p_j A_j\right) \otimes 1\right)\left(1 \otimes \left(\sum_{j=1}^{n} p_j A_j\right)^{-2}\right).
\end{aligned}$$

From (2.11) written for the convex function $f(t)=t^r$, $r\in(-\infty,0)\cup[1,\infty)$, we get

$$r\left(A^r \circ 1 - A^{r-1} \circ B\right) \geq (A^r - B^r) \circ 1 \geq r\left(A \circ B^{r-1} - B^r \circ 1\right),$$

for $A, B > 0$.

For $r=2$ we get

$$2\left(A^2 \circ 1 - A \circ B\right) \geq \left(A^2 - B^2\right) \circ 1 \geq 2\left(A \circ B - B^2 \circ 1\right),$$

while for $r=-1$, we get

$$A^{-2} \circ B - A^{-1} \circ 1 \geq \left(A^{-1} - B^{-1}\right) \circ 1 \geq B^{-1} \circ 1 - A \circ B^{-2},$$

for $A, B > 0$.

If $A_j > 0$ and $p_j \geq 0$ for $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$, then by (2.13)

$$\begin{aligned}
&r\left[\left(\sum_{j=1}^{n} p_j A_j^{r}\right) \circ 1 - \left(\sum_{j=1}^{n} p_j A_j^{r-1}\right) \circ \left(\sum_{j=1}^{n} p_j A_j\right)\right] \\
&\geq \left(\sum_{j=1}^{n} p_j A_j^{r} - \left(\sum_{j=1}^{n} p_j A_j\right)^{r}\right) \circ 1 \\
&\geq r\left[\left(\sum_{j=1}^{n} p_j A_j\right) \circ \left(\sum_{j=1}^{n} p_j A_j\right)^{r-1} - \left(\sum_{j=1}^{n} p_j A_j\right)^{r} \circ 1\right].
\end{aligned}$$

For $r=2$, then we get

$$\begin{aligned}
&2\left[\left(\sum_{j=1}^{n} p_j A_j^{2}\right) \circ 1 - \left(\sum_{j=1}^{n} p_j A_j\right) \circ \left(\sum_{j=1}^{n} p_j A_j\right)\right] \\
&\geq \left(\sum_{j=1}^{n} p_j A_j^{2} - \left(\sum_{j=1}^{n} p_j A_j\right)^{2}\right) \circ 1 \\
&\geq 2\left[\left(\sum_{j=1}^{n} p_j A_j\right) \circ \left(\sum_{j=1}^{n} p_j A_j\right) - \left(\sum_{j=1}^{n} p_j A_j\right)^{2} \circ 1\right],
\end{aligned}$$

while for $r=-1$ we get

$$\begin{aligned}
&\left(\sum_{j=1}^{n} p_j A_j^{-2}\right) \circ \left(\sum_{j=1}^{n} p_j A_j\right) - \left(\sum_{j=1}^{n} p_j A_j^{-1}\right) \circ 1 \\
&\geq \left(\sum_{j=1}^{n} p_j A_j^{-1} - \left(\sum_{j=1}^{n} p_j A_j\right)^{-1}\right) \circ 1 \\
&\geq \left(\sum_{j=1}^{n} p_j A_j\right)^{-1} \circ 1 - \left(\sum_{j=1}^{n} p_j A_j\right) \circ \left(\sum_{j=1}^{n} p_j A_j\right)^{-2}.
\end{aligned}$$

Acknowledgement

The author would like to thank the anonymous referees for valuable suggestions that have been implemented in the final version of the manuscript.

Authors Contribution Statement

The author has read and agreed the published verison of the manuscript.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

References

[1] Ando, T., *Concavity of certain maps on positive definite matrices and applications to Hadamard products*, Lin. Alg. & Appl., **26**(1979), 203–241.

[2] Araki, H., Hansen, F., *Jensen's operator inequality for functions of several variables*, Proc. Amer. Math. Soc., **128**(7)(2000), 2075–2084.

[3] Aujila, J.S., Vasudeva, H.L., *Inequalities involving Hadamard product and operator means*, Math. Japon., **42**(1995), 265–272.

[4] Fujii, J.I., *The Marcus-Khan theorem for Hilbert space operators*, Math. Jpn., **41**(1995), 531–535.

[5] Furuta, T., Mićić Hot, J., Pečarić, J., Seo, Y., Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space, Element, Zagreb, 2005.

[6] Kitamura, K., Seo, Y., *Operator inequalities on Hadamard product associated with Kadison's Schwarz inequalities*, Scient. Math.,**1**(2)(1998), 237–241.

[7] Korányi, A., *On some classes of analytic functions of several variables*, Trans. Amer. Math. Soc., **101**(1961), 520–554.

[8] Wada, S., *On some refinement of the Cauchy-Schwarz Inequality*, Lin. Alg. & Appl., **420**(2007), 433–440.