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AN OSTROWSKI TYPE TENSORIAL NORM INEQUALITY FOR CONTINUOUS FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Let H be a Hilbert space. Assume that f is continuously differentiable on I with $\|f'\|_{I,\infty} := \sup_{t\in I} |f'(t)| < \infty$ and A, B are selfadjoint operators with $Sp(A), Sp(B) \subset I$, then

$$\left\| f\left((1-\lambda) A \otimes 1 + \lambda 1 \otimes B\right) - \int_0^1 f\left((1-u) A \otimes 1 + u 1 \otimes B\right) du \right\| \leq \|f'\|_{I,\infty} \left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right] \|1 \otimes B - A \otimes 1\|$$

for $\lambda \in [0,1]$. In particular, we have the midpoint inequality

$$\left\| f\left(\frac{A \otimes 1 + 1 \otimes B}{2}\right) - \int_0^1 f\left((1-u) A \otimes 1 + u 1 \otimes B\right) du \right\| \leq \frac{1}{4} \|f'\|_{I,\infty} \|1 \otimes B - A \otimes 1\|.$$

1. Introduction

In 1938, A. Ostrowski [13], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x \in [a,b]$.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right] \|f'\|_\infty (b-a), \tag{1.1}$$

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for all $x \in [a,b]$ *and the constant* $\frac{1}{4}$ *is the best possible.*

If we take $x = \frac{a+b}{2}$, we get the *midpoint inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{4}\left\| f' \right\|_\infty (b-a),$$

with $\frac{1}{4}$ as best possible constant.

In order to extend this result for tensorial products of selfadjoint operators and norms, we need the following preparations.

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2], we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_k)\, dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k) \tag{1.2}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [7] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if* f *is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f(st) \ge (\le)\, f(s)\, f(t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [10, p. 173]

$$f(A \otimes B) \ge (\le)\, f(A) \otimes f(B) \text{ for all } A,\ B \ge 0. \tag{1.3}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st)\, dE(t) \otimes dF(s) \tag{1.4}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B > 0$

$$A \#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t \in [0, 1]$ and

$$A \# B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B)\otimes(B\#A) = (A\otimes B)\#(B\otimes A).$$

In 2007, S. Wada [14] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned}(A\#B)\otimes(A\#B) &\leq \frac{1}{2}\left[(A\#_{\alpha}B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_{\alpha}B)\right] \\ &\leq \frac{1}{2}(A\otimes B+B\otimes A)\end{aligned} \tag{1.5}$$

for A, $B>0$ and $\alpha\in[0,1]$. For other similar results, see [1], [3] and [8]-[11].

Motivated by the above results, if f is continuously differentiable on I with $\|f'\|_{I,\infty}:=\sup_{t\in I}|f'(t)|<\infty$ and A, B are selfadjoint operators with $Sp(A)$, $Sp(B)\subset I$, then

$$\begin{aligned}&\left\|f((1-\lambda)A\otimes 1+\lambda 1\otimes B)-\int_0^1 f((1-u)A\otimes 1+u1\otimes B)\,du\right\| \\ &\leq \|f'\|_{I,\infty}\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|1\otimes B-A\otimes 1\|\end{aligned}$$

for $\lambda\in[0,1]$.

In particular, we have the midpoint inequality

$$\begin{aligned}&\left\|f\left(\frac{A\otimes 1+1\otimes B}{2}\right)-\int_0^1 f((1-u)A\otimes 1+u1\otimes B)\,du\right\| \\ &\leq \frac{1}{4}\|f'\|_{I,\infty}\|1\otimes B-A\otimes 1\|.\end{aligned}$$

2. Main Results

Recall the following property of the tensorial product

$$(AC)\otimes(BD) = (A\otimes B)(C\otimes D) \tag{2.1}$$

that holds for any $A,B,C,D\in B(H)$.

If we take $C=A$ and $D=B$, then we get

$$A^2\otimes B^2 = (A\otimes B)^2.$$

By induction and using (2.1) we derive that

$$A^n\otimes B^n = (A\otimes B)^n \text{ for natural } n\geq 0. \tag{2.2}$$

In particular

$$A^n\otimes 1 = (A\otimes 1)^n \text{ and } 1\otimes B^n = (1\otimes B)^n \tag{2.3}$$

for all $n\geq 0$.

We also observe that, by (2.1), the operators $A\otimes 1$ and $1\otimes B$ are commutative and

$$(A\otimes 1)(1\otimes B) = (1\otimes B)(A\otimes 1) = A\otimes B. \tag{2.4}$$

Moreover, for two natural numbers m, n we have

$$(A\otimes 1)^m(1\otimes B)^n = (1\otimes B)^n(A\otimes 1)^m = A^m\otimes B^n. \tag{2.5}$$

We have the following representation results for continuous functions:

Lemma 2.1. *Assume A and B are selfadjoint operators with $Sp(A) \subset I$ and $Sp(B) \subset J$. Let f, h be continuous on I, g, k continuous on J and φ continuous on an interval K that contains the sum of the intervals $h(I) + k(J)$, then*

$$\begin{aligned}&(f(A) \otimes 1 + 1 \otimes g(B)) \varphi(h(A) \otimes 1 + 1 \otimes k(B)) \\ &= \int_I \int_J (f(t) + g(s)) \varphi(h(t) + k(s)) dE_t \otimes dF_s,\end{aligned} \tag{2.6}$$

where A and B have the spectral resolutions

$$A = \int_I t dE(t) \text{ and } B = \int_J s dF(s). \tag{2.7}$$

Proof. By Stone-Weierstrass theorem, any continuous function can be approximated by a sequence of polynomials, therefore it suffices to prove the equality for the power function $\varphi(t) = t^n$ with n any natural number.

For natural number $n \geq 1$ we have

$$\begin{aligned}\mathcal{K} &:= \int_I \int_J (f(t) + g(s)) (h(t) + k(s))^n dE_t \otimes dF_s \\ &= \int_I \int_J (f(t) + g(s)) \sum_{m=0}^{n} C_n^m [h(t)]^m [k(s)]^{n-m} dE_t \otimes dF_s \\ &= \sum_{m=0}^{n} C_n^m \int_I \int_J (f(t) + g(s)) [h(t)]^m [k(s)]^{n-m} dE_t \otimes dF_s \\ &= \sum_{m=0}^{n} C_n^m \left[\int_I \int_J f(t) [h(t)]^m [k(s)]^{n-m} dE_t \otimes dF_s \right. \\ &\left. + \int_I \int_J [h(t)]^m g(s) [k(s)]^{n-m} dE_t \otimes dF_s\right].\end{aligned} \tag{2.8}$$

Observe that

$$\begin{aligned}&\int_I \int_J f(t) [h(t)]^m [k(s)]^{n-m} dE_t \otimes dF_s \\ &= f(A) [h(A)]^m \otimes [k(B)]^{n-m} = (f(A) \otimes 1) \left([h(A)]^m \otimes [k(B)]^{n-m}\right) \\ &= (f(A) \otimes 1) ([h(A)]^m \otimes 1) \left(1 \otimes [k(B)]^{n-m}\right) \\ &= (f(A) \otimes 1) (h(A) \otimes 1)^m (1 \otimes k(B))^{n-m}\end{aligned}$$

and

$$\begin{aligned}&\int_I \int_J [h(t)]^m g(s) [k(s)]^{n-m} dE_t \otimes dF_s \\ &= [h(A)]^m \otimes \left(g(B) [k(B)]^{n-m}\right) = (1 \otimes g(B)) \left([h(A)]^m \otimes [k(B)]^{n-m}\right) \\ &= (1 \otimes g(B)) ([h(A)]^m \otimes 1) \left(1 \otimes [k(B)]^{n-m}\right) \\ &= (1 \otimes g(B)) (h(A) \otimes 1)^m (1 \otimes k(B))^{n-m},\end{aligned}$$

with $h(A) \otimes 1$ and $1 \otimes k(B)$ commutative.

Therefore

$$\begin{aligned}\mathcal{K} &= (f(A)\otimes 1+1\otimes g(B))\sum_{m=0}^{n} C_n^m (h(A)\otimes 1)^m (1\otimes k(B))^{n-m} \\ &= (f(A)\otimes 1+1\otimes g(B))(h(A)\otimes 1+1\otimes k(B))^n ,\end{aligned}$$

for which the commutativity of $h(A)\otimes 1$ and $1\otimes k(B)$ has been employed. □

Theorem 2.2. *Assume that f is continuously differentiable on I, A and B are selfadjoint operators with $Sp(A), Sp(B)\subset I$, then*

$$\begin{aligned}& f((1-\lambda)A\otimes 1+\lambda 1\otimes B)-\int_0^1 f((1-u)A\otimes 1+u1\otimes B)\,du \qquad (2.9)\\ &=\lambda^2(1\otimes B-A\otimes 1)\\ &\times\int_0^1 uf'((1-u\lambda)A\otimes 1+u\lambda 1\otimes B)\,du\\ &-(1-\lambda)^2(1\otimes B-A\otimes 1)\\ &\times\int_0^1 uf'(u(1-\lambda)A\otimes 1+(1-(1-\lambda)u)1\otimes B)\,du,\end{aligned}$$

for all $\lambda\in[0,1]$. In particular, for $\lambda=\frac{1}{2}$, we have the midpoint identity

$$\begin{aligned}& f\left(\frac{A\otimes 1+1\otimes B}{2}\right)-\int_0^1 f((1-u)A\otimes 1+u1\otimes B)\,du \qquad (2.10)\\ &=\frac{1}{4}(1\otimes B-A\otimes 1)\int_0^1 u\left[f'\left(\left(1-\frac{u}{2}\right)A\otimes 1+\frac{u}{2}1\otimes B\right)\right.\\ &\left.-f'\left(\frac{u}{2}A\otimes 1+\left(1-\frac{u}{2}\right)1\otimes B\right)\right]du.\end{aligned}$$

Proof. We start to the Montgomery identity for real valued absolutely continuous functions on $[a,b]$ that can be easily proved integrating by parts in the right side of the equality,

$$(b-a)f(x)-\int_a^b f(t)\,dt=\int_a^x (t-a)f'(t)\,dt+\int_x^b (t-b)f'(t)\,dt \qquad (2.11)$$

for $a\le x\le b$.

If we use the change of variable $t=(1-u)a+ux$, then we have $dt=(x-a)\,du$ and

$$\int_a^x (t-a)f'(t)\,dt=(x-a)^2\int_0^1 uf'((1-u)a+ux)\,du.$$

If we use the change of variable $t=(1-u)x+ub$, then we have $dt=(b-x)\,du$ and

$$\int_x^b (t-b)f'(t)\,dt=-(b-x)^2\int_0^1 (1-u)f'((1-u)x+ub)\,du.$$

By (2.11) we get

$$(b-a) f(x) - (b-a) \int_0^1 f((1-u) a + ub) du \tag{2.12}$$
$$= (x-a)^2 \int_0^1 uf'((1-u) a + ux) du$$
$$- (b-x)^2 \int_0^1 (1-u) f'((1-u) x + ub) du.$$

If we take $x = (1-\lambda) a + \lambda b$, $\lambda \in [0,1]$ in (2.12), then we get

$$(b-a) f((1-\lambda) a + \lambda b) - (b-a) \int_0^1 f((1-u) a + ub) du \tag{2.13}$$
$$= (b-a)^2 \lambda^2 \int_0^1 uf'((1-u) a + u[(1-\lambda) a + \lambda b]) du$$
$$- (b-a)^2 (1-\lambda)^2 \int_0^1 (1-u) f'((1-u)[(1-\lambda) a + \lambda b] + ub) du$$
$$= (b-a)^2 \lambda^2 \int_0^1 uf'((1-u\lambda) a + u\lambda b) du$$
$$- (b-a)^2 (1-\lambda)^2 \int_0^1 (1-u) f'((1-u)(1-\lambda) a + (\lambda + (1-\lambda) u) b) du.$$

Therefore, for all $a, b \in I$ and $\lambda \in [0,1]$,

$$f((1-\lambda) a + \lambda b) - \int_0^1 f((1-u) a + ub) du \tag{2.14}$$
$$= (b-a) \lambda^2 \int_0^1 uf'((1-u\lambda) a + u\lambda b) du$$
$$- (b-a)(1-\lambda)^2 \int_0^1 (1-u) f'((1-u)(1-\lambda) a + (\lambda + (1-\lambda) u) b) du$$
$$= \lambda^2 (b-a) \int_0^1 uf'((1-u\lambda) a + u\lambda b) du$$
$$- (1-\lambda)^2 (b-a) \int_0^1 uf'(u(1-\lambda) a + (1-(1-\lambda) u) b) du,$$

where for the last equality we change the variable $1-u$ with u in the second previous integral.

Assume that A and B have the spectral resolutions

$$A = \int_I t dE(t) \text{ and } B = \int_I s dF(s).$$

If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$ in (2.14), then we get

$$\begin{aligned}
&\int_I \int_I f\left((1-\lambda) t+\lambda s\right) dE_t \otimes dF_s \qquad (2.15)\\
&-\int_I \int_I \left(\int_0^1 f\left((1-u) t+u s\right) du\right) dE_t \otimes dF_s\\
&=\lambda^2 \int_I \int_I \left((s-t) \int_0^1 u f'\left((1-u\lambda) t+u\lambda s\right) du\right) dE_t \otimes dF_s\\
&-(1-\lambda)^2\\
&\times \int_I \int_I \left((s-t) \int_0^1 u f'\left(u(1-\lambda) t+(1-(1-\lambda) u) s\right) du\right) dE_t \otimes dF_s,
\end{aligned}$$

for all $\lambda \in [0,1]$.

By utilizing the Fubini's theorem and Lemma 2.1 for appropriate choices of the functions involved, we have successively

$$\int_I \int_I f\left((1-\lambda) t+\lambda s\right) dE_t \otimes dF_s = f\left((1-\lambda) A\otimes 1+\lambda 1\otimes B\right),$$

$$\begin{aligned}
&\int_I \int_I \left(\int_0^1 f\left((1-u) t+u s\right) du\right) dE_t \otimes dF_s\\
&=\int_0^1 \left(\int_I \int_I f\left((1-u) t+u s\right) dE_t \otimes dF_s\right) du\\
&=\int_0^1 f\left((1-u) A\otimes 1+u 1\otimes B\right) du,
\end{aligned}$$

$$\begin{aligned}
&\int_I \int_I \left((s-t) \int_0^1 u f'\left((1-u\lambda) t+u\lambda s\right) du\right) dE_t \otimes dF_s\\
&=\int_0^1 u \left(\int_I \int_I (s-t) f'\left((1-u\lambda) t+u\lambda s\right) dE_t \otimes dF_s\right) du\\
&=(1\otimes B-A\otimes 1) \int_0^1 u f'\left((1-u\lambda) A\otimes 1+u\lambda 1\otimes B\right) du
\end{aligned}$$

and

$$\begin{aligned}
&\int_I \int_I \left((s-t) \int_0^1 u f'\left(u(1-\lambda) t+(1-(1-\lambda) u) s\right) du\right) dE_t \otimes dF_s\\
&=\int_0^1 u \left(\int_I \int_I (s-t) f'\left((1-\lambda) t+(1-(1-\lambda) u) s\right) dE_t \otimes dF_s\right) du\\
&=(1\otimes B-A\otimes 1) \int_0^1 u \left(f'\left((1-\lambda) A\otimes 1+(1-(1-\lambda) u) 1\otimes B\right)\right) du.
\end{aligned}$$

By employing (2.15), we then get the desired result (2.9). □

Theorem 2.3. *Assume that f is continuously differentiable on I with $\|f'\|_{I,\infty} := \sup_{t\in I} |f'(t)| < \infty$ and A, B are selfadjoint operators with $Sp(A)$, $Sp(B) \subset I$,* *then*

$$\left\| f\left(\left(1-\lambda \right) A\otimes 1+\lambda 1\otimes B\right) -\int_{0}^{1}f\left(\left(1-u\right) A\otimes 1+u1\otimes B\right) du\right\| \tag{2.16}$$
$$\leq \left\| f'\right\|_{I,\infty }\left[\frac{1}{4}+\left(\lambda -\frac{1}{2}\right) ^{2}\right] \left\| 1\otimes B-A\otimes 1\right\|$$

for $\lambda \in \left[0,1\right]$.

In particular, we have the midpoint inequality

$$\left\| f\left(\frac{A\otimes 1+1\otimes B}{2}\right) -\int_{0}^{1}f\left(\left(1-u\right) A\otimes 1+u1\otimes B\right) du\right\| \tag{2.17}$$
$$\leq \frac{1}{4}\left\| f'\right\|_{I,\infty }\left\| 1\otimes B-A\otimes 1\right\| .$$

Proof. If we take the operator norm and use the triangle inequality, we get

$$\left\| f\left(\left(1-\lambda \right) A\otimes 1+\lambda 1\otimes B\right) -\int_{0}^{1}f\left(\left(1-u\right) A\otimes 1+u1\otimes B\right) du\right\| \tag{2.18}$$
$$\leq \lambda ^{2}\left\| 1\otimes B-A\otimes 1\right\|$$
$$\times \left\| \int_{0}^{1}uf'\left(\left(1-u\lambda \right) A\otimes 1+u\lambda 1\otimes B\right) du\right\|$$
$$+\left(1-\lambda \right) ^{2}\left\| 1\otimes B-A\otimes 1\right\|$$
$$\times \left\| \int_{0}^{1}uf'\left(u\left(1-\lambda \right) A\otimes 1+\left(1-\left(1-\lambda \right) u\right) 1\otimes B\right) du\right\| ,$$

for all $\lambda \in \left[0,1\right]$.

By the properties of the integral and norm, we have

$$\left\| \int_{0}^{1}uf'\left(\left(1-u\lambda \right) A\otimes 1+u\lambda 1\otimes B\right) du\right\| \tag{2.19}$$
$$\leq \int_{0}^{1}u\left\| f'\left(\left(1-u\lambda \right) A\otimes 1+u\lambda 1\otimes B\right) \right\| du$$

and

$$\left\| \int_{0}^{1}uf'\left(u\left(1-\lambda \right) A\otimes 1+\left(1-\left(1-\lambda \right) u\right) 1\otimes B\right) du\right\| \tag{2.20}$$
$$\leq \int_{0}^{1}u\left\| f'\left(u\left(1-\lambda \right) A\otimes 1+\left(1-\left(1-\lambda \right) u\right) 1\otimes B\right) \right\| du.$$

Observe that, by Lemma 2.1

$$\left\vert f'\left(\left(1-u\lambda \right) A\otimes 1+u\lambda 1\otimes B\right) \right\vert =\int_{I}\int_{I}\left\vert f'\left(\left(1-u\lambda \right) t+u\lambda s\right) \right\vert dE_{t}\otimes dF_{s}$$

for $u,\lambda \in \left[0,1\right]$.

Since

$$\left\vert f'\left(\left(1-u\lambda \right) t+u\lambda s\right) \right\vert \leq \left\| f'\right\|_{I,\infty }$$

for $u,\lambda \in \left[0,1\right]$ and $t,s\in I$.

If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, then we get

$$\begin{aligned}
&\left|f'\left(\left(1-u\lambda\right)A\otimes 1+u\lambda 1\otimes B\right)\right| \quad (2.21)\\
&=\int_I\int_I \left|f'\left(\left(1-u\lambda\right)t+u\lambda s\right)\right| dE_t\otimes dF_s \leq \left\|f'\right\|_{I,\infty}\int_I\int_I dE_t\otimes dF_s\\
&=\left\|f'\right\|_{I,\infty}
\end{aligned}$$

for $u,\lambda\in[0,1]$. This implies that

$$\left\|f'\left(\left(1-u\lambda\right)A\otimes 1+u\lambda 1\otimes B\right)\right\|\leq\left\|f'\right\|_{I,\infty}$$

for $u,\lambda\in[0,1]$ which gives

$$\int_0^1 u\left\|f'\left(\left(1-u\lambda\right)A\otimes 1+u\lambda 1\otimes B\right)\right\|du\leq\left\|f'\right\|_{I,\infty}\int_0^1 udu=\frac{1}{2}\left\|f'\right\|_{I,\infty}$$

Similarly, we have

$$\int_0^1 u\left\|f'\left(u\left(1-\lambda\right)A\otimes 1+\left(1-\left(1-\lambda\right)u\right)1\otimes B\right)\right\|du\leq\frac{1}{2}\left\|f'\right\|_{I,\infty}.$$

By (2.18)-(2.20) we derive

$$\begin{aligned}
&\left\|f\left(\left(1-\lambda\right)A\otimes 1+\lambda 1\otimes B\right)-\int_0^1 f\left(\left(1-u\right)A\otimes 1+u1\otimes B\right)du\right\|\\
&\leq\frac{1}{2}\left\|f'\right\|_{I,\infty}\left\|1\otimes B-A\otimes 1\right\|\left[\lambda^2+\left(1-\lambda\right)^2\right]\\
&=\left\|1\otimes B-A\otimes 1\right\|\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\left\|f'\right\|_{I,\infty},
\end{aligned}$$

which proves (2.16). □

3. Related Results

We start by the following result:

Theorem 3.1. *Assume that f is continuously differentiable on I with $|f'|$ is convex on I, A and B are selfadjoint operators with $Sp(A)$, $Sp(B)\subset I$, then*

$$\begin{aligned}
&\left\|f\left(\left(1-\lambda\right)A\otimes 1+\lambda 1\otimes B\right)-\int_0^1 f\left(\left(1-u\right)A\otimes 1+u1\otimes B\right)du\right\| \quad (3.1)\\
&\leq\left\|1\otimes B-A\otimes 1\right\|\left[p\left(1-\lambda\right)\left\|f'\left(A\right)\right\|+p\left(\lambda\right)\left\|f'\left(B\right)\right\|\right],
\end{aligned}$$

for $\lambda\in[0,1]$, where

$$p\left(\lambda\right)=\frac{1}{3}\left[\lambda^3-\left(1-\lambda\right)^3\right]+\frac{1}{2}\left(1-\lambda\right)^2,\ \lambda\in[0,1].$$

In particular, for $\lambda=\frac{1}{2}$, we get the midpoint inequality:

$$\begin{aligned}
&\left\|f\left(\frac{A\otimes 1+1\otimes B}{2}\right)-\int_0^1 f\left(\left(1-u\right)A\otimes 1+u1\otimes B\right)du\right\| \quad (3.2)\\
&\leq\frac{1}{8}\left\|1\otimes B-A\otimes 1\right\|\left[\left\|f'\left(A\right)\right\|+\left\|f'\left(B\right)\right\|\right].
\end{aligned}$$

Proof. Since $|f'|$ is convex on I, then we get

$$|f'((1-u\lambda)t+u\lambda s)| \leq (1-u\lambda)|f'(t)|+u\lambda|f'(s)|$$

for all for $u,\lambda\in[0,1]$ and $t,s\in I$.

If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, then we get

$$\begin{aligned}
&|f'((1-u\lambda)A\otimes 1+u\lambda 1\otimes B)| \quad (3.3)\\
&=\int_I\int_I |f'((1-u\lambda)t+u\lambda s)|\,dE_t\otimes dF_s\\
&\leq \int_I\int_I [(1-u\lambda)|f'(t)|+u\lambda|f'(s)|]\,dE_t\otimes dF_s\\
&=(1-u\lambda)|f'(A)|\otimes 1+u\lambda 1\otimes|f'(B)|
\end{aligned}$$

for all for $u,\lambda\in[0,1]$.

If we take the norm in (3.3), then we get

$$\begin{aligned}
&\|f'((1-u\lambda)A\otimes 1+u\lambda 1\otimes B)\| \quad (3.4)\\
&\leq \|(1-u\lambda)|f'(A)|\otimes 1+u\lambda 1\otimes|f'(B)|\|\\
&\leq (1-u\lambda)\||f'(A)|\otimes 1\|+u\lambda\|1\otimes|f'(B)|\|\\
&=(1-u\lambda)\|f'(A)\|+u\lambda\|f'(B)\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^1 u\|f'((1-u\lambda)A\otimes 1+u\lambda 1\otimes B)\|\,du\\
&\leq \|f'(A)\|\int_0^1 u(1-u\lambda)\,du+\|f'(B)\|\lambda\int_0^1 u^2du\\
&=\left(\frac{1}{2}-\frac{\lambda}{3}\right)\|f'(A)\|+\|f'(B)\|\frac{\lambda}{3}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^1 u\|f'(u(1-\lambda)A\otimes 1+(1-(1-\lambda)u)1\otimes B)\|\,du\\
&\leq \int_0^1 u[u(1-\lambda)\|f'(A)\|+(1-(1-\lambda)u)\|f'(B)\|]\\
&=\frac{1}{3}(1-\lambda)\|f'(A)\|+\left(\frac{1}{2}-\frac{(1-\lambda)}{3}\right)\|f'(B)\|.
\end{aligned}$$

From (2.18) we get

$$\begin{aligned}
&\left\|f((1-\lambda)A\otimes 1+\lambda 1\otimes B)-\int_0^1 f((1-u)A\otimes 1+u1\otimes B)\,du\right\|\\
&\leq \lambda^2\|1\otimes B-A\otimes 1\|\left[\left(\frac{1}{2}-\frac{\lambda}{3}\right)\|f'(A)\|+\|f'(B)\|\frac{\lambda}{3}\right]\\
&+(1-\lambda)^2\|1\otimes B-A\otimes 1\|\\
&\times\left[\frac{1}{3}(1-\lambda)\|f'(A)\|+\left(\frac{1}{2}-\frac{(1-\lambda)}{3}\right)\|f'(B)\|\right]
\end{aligned}$$

$$= \|1 \otimes B - A \otimes 1\| \\ \times \left\{ \lambda^2 \left[\left(\frac{1}{2} - \frac{\lambda}{3} \right) \|f'(A)\| + \|f'(B)\| \frac{\lambda}{3} \right] \right. \\ \left. + (1-\lambda)^2 \left[\frac{1}{3}(1-\lambda) \|f'(A)\| + \left(\frac{1}{2} - \frac{(1-\lambda)}{3} \right) \|f'(B)\| \right] \right\} \\ = \|1 \otimes B - A \otimes 1\| \\ \times \left\{ \left[\frac{1}{3}(1-\lambda)^3 + \lambda^2 \left(\frac{1}{2} - \frac{\lambda}{3} \right) \right] \|f'(A)\| \right. \\ \left. + \left[\frac{1}{3}\lambda^3 + (1-\lambda)^2 \left(\frac{1}{2} - \frac{1-\lambda}{3} \right) \right] \|f'(B)\| \right\},$$

which gives the desired result (3.1). □

We recall that the function $g : I \to \mathbb{R}$ is *quasi-convex*, if

$$g((1-\lambda)t + \lambda s) \le \max\{g(t), g(s)\} = \frac{1}{2}(g(t) + g(s) + |g(t) - g(s)|)$$

for all $t, s \in I$ and $\lambda \in [0,1]$.

Theorem 3.2. *Assume that f is continuously differentiable on I with $|f'|$ is quasi-convex on I, A and B are selfadjoint operators with $Sp(A), Sp(B) \subset I$, then*

$$\left\| f((1-\lambda) A \otimes 1 + \lambda 1 \otimes B) - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B) du \right\| \tag{3.5}$$
$$\le \frac{1}{2} \left[\frac{1}{4} + \left(\lambda - \frac{1}{2} \right)^2 \right] \|1 \otimes B - A \otimes 1\| \\ \times (\||f'(A)| \otimes 1 + 1 \otimes |f'(B)|\| + \||f'(A)| \otimes 1 - 1 \otimes |f'(B)|\|)$$

for $\lambda \in [0,1]$.

In particular, we have the midpoint inequality:

$$\left\| f\left(\frac{A \otimes 1 + 1 \otimes B}{2} \right) - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B) du \right\| \tag{3.6}$$
$$\le \frac{1}{8} \|1 \otimes B - A \otimes 1\| \\ \times (\||f'(A)| \otimes 1 + 1 \otimes |f'(B)|\| + \||f'(A)| \otimes 1 - 1 \otimes |f'(B)|\|).$$

Proof. Since $|f'|$ is quasi-convex on I, then we get

$$|f'((1-u\lambda)t + u\lambda s)| \le \frac{1}{2}(|f'(t)| + |f'(s)| + ||f'(t)| - |f'(s)||)$$

for all for $u, \lambda \in [0,1]$ and $t, s \in I$.

If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, then we get

$$|f'((1-u\lambda) A \otimes 1 + u\lambda 1 \otimes B)| \tag{3.7}$$
$$= \int_I \int_I |f'((1-u\lambda)t + u\lambda s)| dE_t \otimes dF_s \\ \le \frac{1}{2} \int_I \int_I (|f'(t)| + |f'(s)| + ||f'(t)| - |f'(s)||) dE_t \otimes dF_s \\ = \frac{1}{2}(|f'(A)| \otimes 1 + 1 \otimes |f'(B)| + ||f'(A)| \otimes 1 - 1 \otimes |f'(B)||)$$

for all for $u, \lambda \in [0,1]$.

If we take the norm, then we get

$$\begin{aligned}
&\|f'((1-u\lambda)A\otimes 1+u\lambda 1\otimes B)\| \\
&\leq \frac{1}{2}\|(|f'(A)|\otimes 1+1\otimes |f'(B)|+||f'(A)|\otimes 1-1\otimes |f'(B)||)\| \\
&\leq \frac{1}{2}(\||f'(A)|\otimes 1+1\otimes |f'(B)|\|+\||f'(A)|\otimes 1-1\otimes |f'(B)|\|)
\end{aligned}$$

for all for $u, \lambda \in [0,1]$.

Therefore

$$\begin{aligned}
&\int_0^1 u\|f'((1-u\lambda)A\otimes 1+u\lambda 1\otimes B)\|du \\
&\leq \frac{1}{2}(\||f'(A)|\otimes 1+1\otimes |f'(B)|\|+\||f'(A)|\otimes 1-1\otimes |f'(B)|\|)\int_0^1 udu \\
&= \frac{1}{4}(\||f'(A)|\otimes 1+1\otimes |f'(B)|\|+\||f'(A)|\otimes 1-1\otimes |f'(B)|\|)
\end{aligned}$$

and, in a similar way

$$\begin{aligned}
&\int_0^1 u\|f'(u(1-\lambda)A\otimes 1+(1-(1-\lambda)u)1\otimes B)\|du \\
&\leq \frac{1}{4}(\||f'(A)|\otimes 1+1\otimes |f'(B)|\|+\||f'(A)|\otimes 1-1\otimes |f'(B)|\|).
\end{aligned}$$

By utilizing (2.18) we then get

$$\begin{aligned}
&\left\|f((1-\lambda)A\otimes 1+\lambda 1\otimes B)-\int_0^1 f((1-u)A\otimes 1+u1\otimes B)du\right\| \\
&\leq \lambda^2\|1\otimes B-A\otimes 1\| \\
&\times \frac{1}{4}(\||f'(A)|\otimes 1+1\otimes |f'(B)|\|+\||f'(A)|\otimes 1-1\otimes |f'(B)|\|) \\
&+(1-\lambda)^2\|1\otimes B-A\otimes 1\| \\
&\times \frac{1}{4}(\||f'(A)|\otimes 1+1\otimes |f'(B)|\|+\||f'(A)|\otimes 1-1\otimes |f'(B)|\|) \\
&= \frac{1}{4}\left(\lambda^2+(1-\lambda)^2\right)\|1\otimes B-A\otimes 1\| \\
&\times (\||f'(A)|\otimes 1+1\otimes |f'(B)|\|+\||f'(A)|\otimes 1-1\otimes |f'(B)|\|) \\
&= \frac{1}{2}\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|1\otimes B-A\otimes 1\| \\
&\times (\||f'(A)|\otimes 1+1\otimes |f'(B)|\|+\||f'(A)|\otimes 1-1\otimes |f'(B)|\|),
\end{aligned}$$

which proves the desired inequality (3.5). □

4. Examples

It is known that if U and V are commuting, i.e. $UV = VU$, then the exponential function satisfies the property

$$\exp(U)\exp(V) = \exp(V)\exp(U) = \exp(U+V).$$

Also, if U is invertible and $a, b \in \mathbb{R}$ with $a < b$ then

$$\int_a^b \exp(tU)\,dt = U^{-1}\left[\exp(bU) - \exp(aU)\right].$$

Moreover, if U and V are commuting and $V - U$ is invertible, then

$$\begin{aligned}\int_0^1 \exp((1-s)U + sV)\,ds &= \int_0^1 \exp(s(V-U))\exp(U)\,ds\\ &= \left(\int_0^1 \exp(s(V-U))\,ds\right)\exp(U)\\ &= (V-U)^{-1}\left[\exp(V-U) - I\right]\exp(U)\\ &= (V-U)^{-1}\left[\exp(V) - \exp(U)\right].\end{aligned}$$

Since the operators $U = A\otimes 1$ and $V = 1\otimes B$ are commutative and if $1\otimes B - A\otimes 1$ is invertible, then

$$\begin{aligned}&\int_0^1 \exp((1-u)A\otimes 1 + u1\otimes B)\,du\\ &= (1\otimes B - A\otimes 1)^{-1}\left[\exp(1\otimes B) - \exp(A\otimes 1)\right].\end{aligned}$$

If A, B are selfadjoint operators with $Sp(A)$, $Sp(B) \subset [m, M]$ and $1\otimes B - A\otimes 1$ is invertible, then by (2.16)

$$\begin{aligned}&\|\exp((1-\lambda)A\otimes 1 + \lambda 1\otimes B)\\ &- (1\otimes B - A\otimes 1)^{-1}\left[\exp(1\otimes B) - \exp(A\otimes 1)\right]\Big\|\\ &\le \exp(M)\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right]\|1\otimes B - A\otimes 1\|,\end{aligned} \tag{4.1}$$

for $\lambda \in [0,1]$.

In particular,

$$\begin{aligned}&\left\|\exp\left(\frac{A\otimes 1 + 1\otimes B}{2}\right)\right.\\ &- (1\otimes B - A\otimes 1)^{-1}\left[\exp(1\otimes B) - \exp(A\otimes 1)\right]\Big\|\\ &\le \frac{1}{4}\exp(M)\|1\otimes B - A\otimes 1\|.\end{aligned} \tag{4.2}$$

Since for $f(t) = \exp t$, $t \in \mathbb{R}$, $|f'|$ is convex, then by Theorem 3.1 we get

$$\begin{aligned}&\|\exp((1-\lambda)A\otimes 1 + \lambda 1\otimes B)\\ &- (1\otimes B - A\otimes 1)^{-1}\left[\exp(1\otimes B) - \exp(A\otimes 1)\right]\Big\|\\ &\le \frac{1}{2}\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right]\|1\otimes B - A\otimes 1\|\\ &\times\left(\|\exp(A)\otimes 1 + 1\otimes\exp(B)\| + \|\exp(A)\otimes 1 - 1\otimes\exp(B)\|\right)\end{aligned} \tag{4.3}$$

for $\lambda \in [0,1]$.

In particular,

$$\left\|\exp\left(\frac{A\otimes 1+1\otimes B}{2}\right)\right. \tag{4.4}$$
$$\left.-\left(1\otimes B-A\otimes 1\right)^{-1}\left[\exp\left(1\otimes B\right)-\exp\left(A\otimes 1\right)\right]\right\|$$
$$\leq\frac{1}{8}\left\|1\otimes B-A\otimes 1\right\|$$
$$\times\left(\left\|\exp\left(A\right)\otimes 1+1\otimes\exp\left(B\right)\right\|+\left\|\exp\left(A\right)\otimes 1-1\otimes\exp\left(B\right)\right\|\right)$$

provided that $1\otimes B-A\otimes 1$ is invertible.

5. Conclusion

In this paper we established various Ostrowski type tensorial norm inequalities for continuous functions of selfadjoint operators in Hilbert spaces. Some examples for the operator exponential are also given.

References

[1] T. Ando, Concavity of certain maps on positive definite matrices and applications to Hadamard products, Lin. Alg. & Appl. 26 (1979), 203-241.

[2] H. Araki and F. Hansen, Jensen's operator inequality for functions of several variables, Proc. Amer. Math. Soc. 128 (2000), No. 7, 2075-2084.

[3] J. S. Aujila and H. L. Vasudeva, Inequalities involving Hadamard product and operator means, Math. Japon. 42 (1995), 265-272.

[4] N. S. Barnett, P. Cerone and S. S. Dragomir, Some new inequalities for Hermite-Hadamard divergence in information theory. in Stochastic Analysis and Applications. Vol. 3, 7–19, Nova Sci. Publ., Hauppauge, NY, 2003. Preprint RGMIA Res. Rep. Coll. 5 (2002), No. 4, Art. 8, 11 pp. [Online https://rgmia.org/papers/v5n4/NIHHDIT.pdf]

[5] S. S. Dragomir, Bounds for the normalized Jensen functional, Bull. Austral. Math. Soc. 74(3)(2006), 417-478.

[6] S. S. Dragomir, Some tensorial Hermite-Hadamard type inequalities for convex functions of selfadjoint operators in Hilbert spaces, Preprint RGMIA Res. Rep. Coll. 25 (2022), Art. 90, 14 pp. [Online https://rgmia.org/papers/v25/v25a90.pdf]

[7] A. Korányi. On some classes of analytic functions of several variables. Trans. Amer. Math. Soc., 101 (1961), 520–554.

[8] A. Ebadian, I. Nikoufar and M. E. Gordji, Perspectives of matrix convex functions, Proc. Natl. Acad. Sci. USA, 108 (2011), no. 18, 7313–7314.

[9] J. I. Fujii, The Marcus-Khan theorem for Hilbert space operators. Math. Jpn. 41 (1995), 531-535

[10] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space, Element, Zagreb, 2005.

[11] K. Kitamura and Y. Seo, Operator inequalities on Hadamard product associated with Kadison's Schwarz inequalities, Scient. Math.1 (1998), No. 2, 237-241.

[12] I. Nikoufar and M. Shamohammadi, The converse of the Loewner–Heinz inequality via perspective, Lin. & Multilin. Alg., 66 (2018), N0. 2, 243–249.

[13] A. Ostrowski, Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, Comment. Math. Hel, 10 (1938), 226-227.

[14] S. Wada, On some refinement of the Cauchy-Schwarz Inequality, Lin. Alg. & Appl. 420 (2007), 433-440.

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