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Research Article

Finite p-groups in which the normalizer of each nonnormal subgroup is small

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Abstract

Let G be a finite non-Dedekindian p-group which satisfies $N_G(H) = HZ(G)$ for each nonnormal subgroup H, and we call it an NS-group. In this paper, it is proved that an NS-group is the product of a minimal nonabelian group and the center.

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1. Introduction

The groups all of whose subgroups are normal have been determined and are well known as Dedekindian groups. Let G be a finite non-Dedekindian group. For any nonnormal subgroup H of G, there exists a series

$$1 \leq H_G \leq H \leq N_G(H) \leq G \qquad (*).$$

Noticing the left-hand side of the series $1 \leq H_G < H$, Cutolo *et al*. studied p-groups G in which $|H : H_G| \leq p$ for every subgroup H, called core-p p-groups [3, 4]. For another, Zhao *et al*[9, 10] studied finite groups G with $H_G = 1$ for any nonnormal subgroup H. And Yang, An and Lv [6] gave the characterization of p-groups G in which $|H_G| \leq p^i$ for any nonnormal subgroup H.

Considering the right-hand side of the series $H \leq N_G(H) \leq G$ above, Berkovich proposed the following problem in his book of finite p-groups.

Problem 1.1. ([1] Problem 116) Classify the p-groups G such that $|N_G(H) : H| = p$ for all nonnormal subgroups $H < G$.

This problem was solved by Li and Zhang (see [5]). Moreover, Zhang and Gao [7] studied the generalized problem:

Classify the p-groups G such that $|N_G(H) : H| = p^i$ for all nonnormal subgroups $H < G$, where i is a fixed integer.

Furthermore, Zhang and Guo [8] investigated the p-groups G such that $|N_G(H) : H| \leq p^i$ for all nonnormal subgroups $H < G$, where $p > 2$ and i is a fixed integer.

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In fact, the above series could be refined. For example, we could add $H \cap Z(G)$ between 1 and H_G. Considering the series

$$1 \leq H \cap Z(G) \leq H_G,$$

the authors investigated the p-groups G such that $|H \cap Z(G)| \leq p^i$ for every nonnormal subgroup H of G, see [11]. On the other side, the p-groups G which satisfy $H \cap Z(G) = H_G$ for any nonnormal subgroup H are also studied in [12].

In this paper, we consider the series

$$H \leq HZ(G) \leq N_G(H).$$

We study the p-group G which satisfies $N_G(H) = HZ(G)$ for each nonnormal subgroup H, call it an *NS-group* and prove that an NS-group is a Dedekindian group or the product of a minimal nonabelian group and the center.

All groups considered in the following are finite p-groups. Let G be a p-group. The nilpotent class, the minimal number of generators, the exponent and the Frattini subgroup of G are denoted by $c(G)$, $d(G)$, $exp(G)$ and $\Phi(G)$, respectively. And $\Omega(G) = \langle a \in G | a^p = 1\rangle, \mho_i(G) = \langle a^{p^i} | a \in G\rangle$. Let C_{p^m} and C_p^n denote a cyclic p-group of order p^m and an elementary abelian p-group of order p^n, respectively. The notation is standard, refer to [2].

2. The structure of NS-groups

In this section, we try to give the classification of finite p-groups G with $N_G(H) = HZ(G)$ for each nonnormal subgroup H.

Lemma 2.1. *Let G be a finite p-group. If G is an NS-group, then $c(G) \leq 2$.*

Proof. It is easy to see that $\bar{G} = G/Z(G)$ is Dedekindian since $K < N_G(K)$ for any subgroup K. If $c(G) > 2$, then we may assume that $\bar{G} = G/Z(G) = \langle \bar{a}, \bar{b} | \bar{a}^4 = 1, \bar{b}^2 = \bar{a}^2, [\bar{a}, \bar{b}] = \bar{b}^2\rangle \times \bar{A} \cong Q_8 \times C_2^n$ and $G = \langle a, b, A, Z(G)\rangle$. Since for any $c \in A$, $[b^2, c] = [b, c]^2[b, c, b] = [b, c]^2 = [b, c^2] = 1$ and $[b^2, a] = [a^2 z, a] = 1$, where $z \in Z(G)$, we see that $b^2 \in Z(G)$ and $G/Z(G)$ is abelian, a contradiction. So $c(G) \leq 2$. □

Lemma 2.2. *Let G be a finite p-group. If G is an NS-group, then each quotient group of G is also an NS-group.*

Proof. For any normal subgroup $N \trianglelefteq G$, we consider $\overline{G} = G/N$. Let $N \leq H$ and $\overline{H} \ntrianglelefteq \overline{G}$. Then $H \ntrianglelefteq G$ and so $HZ(G) = N_G(H)$. Therefore, $\overline{HZ(G)} = \overline{N_G(H)} = N_{\overline{G}}(\overline{H})$. It follows from $\overline{HZ(G)} = \overline{H}\,\overline{Z(G)} \leq \overline{H}Z(\overline{G})$ that $N_{\overline{G}}(\overline{H}) \leq \overline{H}Z(\overline{G})$. So $N_{\overline{G}}(\overline{H}) = \overline{H}Z(\overline{G})$. □

Lemma 2.3. *Let G be a non-Dedekindian p-group. If G is an NS-group, then $exp(G') = p$.*

Proof. When $p > 2$, by Lemma 2.1, we see that $c(G) = 2$ and then G is regular. Thus we may assume that $G = \langle a_1, a_2, ..., a_n\rangle$, where $\langle a_i\rangle \cap \langle a_j\rangle = 1$ and $1 \leq i, j \leq n$. And $G' = \langle [a_i, a_j] | 1 \leq i, j \leq n\rangle \leq Z(G)$.

If $exp(G') = p^k, k > 1$, then we may assume that $[a_j, a_i] = c$, $o(c) = p^k, o(a_j) > p$ and $\langle a_i\rangle \cap \langle c\rangle = 1$. Therefore, $H = \lg a_i, \mho_1(G')\rangle \ntrianglelefteq G$. Then we see $a_j^p \in N_G(H)$ from $[a_j^p, g] = [a_j, g]^p \in \mho_1(G')$. And we claim that $a_j^p \notin HZ(G)$. If not, then $a_j^p = a_i^{-s} z$, where $z \in Z(G)$, and s is an integer. Hence $a_j^p a_i^s \in Z(G)$. Thus $1 = [a_j^p a_i^s, a_i] = [a_j^p, a_i][a_i^s, a_i] = c^p$, a contradiction. So $a_j^p \notin HZ(G)$ and $N_G(H) \neq HZ(G)$, which contradicts that $N_G(H) = HZ(G)$ for each nonnormal subgroup H. So $exp(G') = p$.

When $p = 2$, let G be a minimal counterexample. If $exp(G') \geq 2^3$, then we consider $\bar{G} = G/(\mho_2(G'))$. Then $exp(\bar{G}') = 2^2$. By using Lemma 2.2 and $|\bar{G}| < |G|$, we see $exp(\bar{G}') = 2$, a contradiction. So $exp(G') = 2^2$.

Since $c(G) = 2$, there exist elements $g_1, g_2 \notin \Phi(G)$ such that $[g_1, g_2] = c, c^4 = 1, c \in Z(G)$. Then let $M = \langle g_1, g_2\rangle$ and $d(M) = 2, M' \cong C_4$. Now we take $a, b \in M$ such that $\{aM', bM'\}$ is the basis of $\bar{M} = M/M'$ and $o(a)o(b)$ is minimal. And we may assume that $o(a) = 2^n, o(b) = 2^m, n \geq m, [a, b] = d \in Z(G), o(d) = 4$. We claim that $\langle a\rangle \cap \langle b\rangle = 1$.

If not, then $\langle a\rangle \cap \langle b\rangle \leq M' \leq Z(G)$ is of order 2 or 4.

Case 1. $|\langle a\rangle \cap \langle b\rangle| = 4$. Then we may assume that $d = a^{2^{n-2}} = b^{2^{m-2}}$.

By $[a, b^2] = [a, b]^2 = d^2 \neq 1$ and $d = b^{2^{m-2}} \in Z(G)$, we see that $m \geq 4$. Noticing the element $a^{2^{n-m}} b^{-1}$, we see that

$$(a^{2^{n-m}} b^{-1})^{2^{m-2}} = a^{2^{n-2}} b^{-2^{m-2}} [a^{2^{n-m}}, b]^{\binom{2^{m-2}}{2}} = [a^{2^{n-m}}, b]^{\binom{2^{m-2}}{2}}$$

and then $(a^{2^{n-m}} b^{-1})^{2^{m-1}} = [a^{2^{n-m}}, b]^{2\binom{2^{m-2}}{2}} = 1$. Thus $o((a^{2^{n-m}} b^{-1})) \leq 2^{m-1}$. Let $b_1 = a^{2^{n-m}} b^{-1}$. $\{aM', b_1M'\}$ is the basis of $\bar{M} = M/M'$ and $o(a)o(b_1) < o(a)o(b)$, which contradicts with the minimality of $o(a)o(b)$.

Case 2. $|\langle a\rangle \cap \langle b\rangle| = 2$. Then we may assume that $d^2 = a^{2^{n-1}} = b^{2^{m-1}}$.

Note that $(a^{2^{n-m}} b^{-1})^{2^{m-1}} = a^{2^{n-1}} b^{-2^{m-1}} [a^{2^{n-m}}, b]^{\binom{2^{m-1}}{2}} = [a^{2^{n-m}}, b]^{\binom{2^{m-1}}{2}}$. Since $[a, b^2] = [a, b]^2 = d^2 \neq 1$ and $d^2 = b^{2^{m-1}} \in Z(G)$, we see that $m - 1 \geq 2$.

If $m - 1 = 2$, then $m = 3$ and $n \geq 3$.

If $n = 3$, then we consider the subgroup $H = \langle a, \mho_1(G')\rangle \ntrianglelefteq G$. Hence we see $b^2 \in N_G(H)$ from $[b^2, g] = [b, g]^2 \in \mho_1(G')$. And we claim that $b^2 \notin HZ(G)$. If not, then $b^2 = a^{-s}z$, where $z \in Z(G)$, and s is an integer. Hence $b^2a^s \in Z(G)$. Thus $1 = [b^2a^s, a] = [b^2, a][a^s, a] = c^2$, a contradiction. So $b^2 \notin HZ(G)$ and $N_G(H) \neq HZ(G)$, which contradicts that $N_G(H) = HZ(G)$ for each nonnormal subgroup H.

If $n > 3$, then $(a^{2^{n-m}} b^{-1})^{2^{m-1}} = [a^{2^{n-m}}, b]^{\binom{2^{m-1}}{2}} = [a, b]^{2^{n-m}\binom{2^{m-1}}{2}} = 1$. Let $b_1 = a^{2^{n-m}} b^{-1}$. Then $\{aM', b_1M'\}$ is the basis of $\bar{M} = M/M'$ and $o(a)o(b_1) < o(a)o(b)$, which contradicts with the minimality of $o(a)o(b)$.

If $m - 1 \geq 3$, then $o(a^{2^{n-m}} b^{-1}) \leq 2^{m-1}$. Let $b_1 = a^{2^{n-m}} b^{-1}$. Then $\{aM', b_1M'\}$ is the basis of $\bar{M} = M/M'$ and $o(a)o(b_1) < o(a)o(b)$, which contradicts with the minimality of $o(a)o(b)$.

So $\langle a\rangle \cap \langle b\rangle = 1$ and we may assume that $\langle a\rangle \cap \langle d\rangle = 1$. We consider the subgroup $H = \langle a, \mho_1(G')\rangle \ntrianglelefteq G$. Then we see $b^2 \in N_G(H)$ from $[b^2, g] = [b, g]^2 \in \mho_1(G')$. And we claim that $b^2 \notin HZ(G)$. If not, then $b^2 = a^{-s}z$, where $z \in Z(G)$, and s is an integer. Hence $b^2a^s \in Z(G)$. Thus $1 = [b^2a^s, a] = [b^2, a][a^s, a] = d^2$, a contradiction. So $b^2 \notin HZ(G)$ and $N_G(H) \neq HZ(G)$, which contradicts that $N_G(H) = HZ(G)$ for each nonnormal subgroup H, so $exp(G') = 2$. The proof is complete.

□

Then by Lemmas 2.1 and 2.3, we get the following lemma.

Lemma 2.4. *Let G be a non-Dedekindian p-group. If G is an NS-group, then $\Phi(G) \leq Z(G)$.*

Proof. By Lemma 2.1, we see $G' \leq Z(G)$. It follows that $[a^p, g] = [a, g]^p = 1$ for any elements $a, g \in G$ from Lemma 2.3. Then $\mho_1(G) = \langle g^p | g \in G\rangle \leq Z(G)$. Thus $\Phi(G) = G'\mho_1(G) \leq Z(G)$. □

Lemma 2.5. *Let G be a non-Dedekindian p-group. If G is an NS-group, then $G/Z(G) \cong C_p \times C_p$.*

Proof. Since G is a non-Dedekindian p-group, there exist elements $a, b \in G$ such that $\langle a\rangle \ntrianglelefteq G$ and $[a, b] = z \neq 1$.

If $G/Z(G) \ncong C_p \times C_p$, then $G/Z(G) \cong C_p^t, t \geq 3$ by Lemma 2.4. Hence we assume that $G/Z(G) = \overline{G} = \langle \bar{a}, \bar{b}, \overline{c_1} ... \overline{c_s}\rangle, s = t - 2 \geq 1$ and $G = \langle a, b, c_1, ..., c_s, Z(G)\rangle$.

If $[a, c_1] = 1$, then $c_1 \in N_G(\langle a \rangle)$. And by $C_p \times C_p \cong \langle \overline{a}, \overline{c_1} \rangle \leq G/Z(G)$, we see $c_1 \notin \langle a \rangle Z(G)$, which contradicts that $N_G(H) = HZ(G)$ for each nonnormal subgroup H.

So $[a, c_1] \neq 1$. Furthermore, we claim that $\langle [a, c_1] \rangle \neq \langle z \rangle$. If not, then $[a, c_1] = z^n, (n, p) = 1$. Hence $[a, b^{-n}c_1] = [a, b^{-n}][a, c_1] = z^{-n}z^n = 1$. Thus $b^{-n}c_1 \in N_G(\langle a \rangle)$. And by $C_p \times C_p \times C_p \cong \langle \overline{a}, \overline{b}, \overline{c_1} \rangle \leq G/Z(G)$, we see $b^{-n}c_1 \notin \langle a \rangle Z(G)$, which contradicts that $N_G(H) = HZ(G)$ for each nonnormal subgroup H.

By considering the subgroup $H = \langle a, z \rangle$, we see that $[a, c_1] \in H$. If $[a, c_1] \notin H$, then $H \ntrianglelefteq G$. We see that $b \in N_G(H)$ by $[a, b] = z$. And $b \notin HZ(G)$, a contradiction.

Then we may assume that $[a, c_1] = a^i z^j$ where i, j are integers and $\langle a^i \rangle = \Omega(\langle a \rangle)$. Thus $[a, b^{-j}c_1] = [a, b^{-j}][a, c_1] = z^{-j}a^i z^j = a^i$, which implies that $b^{-j}c_1 \in N_G(\langle a \rangle)$. Noting that $\langle a \rangle \ntrianglelefteq G$ and $b^{-j}c_1 \notin \langle a \rangle Z(G)$, which contradicts that $N_G(H) = HZ(G)$ for each nonnormal subgroup H.

So $G/Z(G) \cong C_p \times C_p$. The proof is complete. □

Theorem 2.6. *Let G be a non-Dedekindian p-group. Then G is an NS-group if and only if G is the product of a minimal nonabelian group and the center.*

Proof. If G satisfies that $N_G(H) = HZ(G)$ for each nonnormal subgroup H, then, by Lemma 2.5, we see $G = \langle a, b, Z(G) \rangle$. It follows from Lemmas 2.1 and 2.3 that G is the product of a minimal nonabelian group and the center.

On the other hand, it is easy to see that $G/Z(G) \cong C_p \times C_p$. For each $H \ntrianglelefteq G$, there exists an element $a \in H$ such that $a \notin Z(G)$. Then $HZ(G) \lessdot G$. And it is easy to see $HZ(G) \leq N_G(H) \neq G$. So $N_G(H) = HZ(G)$. □

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