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Klein Stratification of Orbit Spaces: Examples

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ABSTRACT

We consider the Klein stratification of orbit spaces M/G, defined by the action of local diffeomorphisms. We show that the Klein strata on M/G, where the singular orbits are isolated points and the stabilizer group acts transitively on the unit sphere, are precisely the union of the points where M/G has the same dimension.

Keywords: *Orbit Spaces, stratifications, diffeology.*

AMS Subject Classification (2020): *Primary: 58A35 ; Secondary: 57S15.*

Introduction

The theory of stratified spaces, when examined through the framework of diffeology theory, provides fascinating insights. Notably, a natural stratification, defined by the action of diffeomorphisms, already exists in any diffeological space. This stratification is known as the 'Klein stratification' and was initially introduced in [6, 1.42]. The Klein stratification adheres to the fundamental frontier condition and allows for several variants based on the same underlying principle, as explored in [7]. Consequently, every diffeological space possesses a structural stratification, making it amenable to the study of its intrinsic properties. Hence as diffeological spaces, orbit space are stratified by the action of local diffeomorphisms too.

On the other hand, any Lie group acting properly on a smooth manifold M induces a stratification of M based on the orbit types [10].

This raises an intriguing question: Which is the correspondence between the singular orbits defined by the group action and the Klein strata on orbit spaces? At present, we have a partial response to this question.

We explicitly describe the structure of Klein strata on the orbit space. This analysis specifically addresses situations where the singular orbits consist of isolated points and where the stabilizer group acts transitively on the unit sphere. It's worth noting that this transitivity condition is the main of the two cases outlined in [1, Cor. 6.3.], when singular orbits are isolated points.

Theorem. Let G be a compact Lie group acting on a manifold M such that the singular orbits are the isolated points and the stabilizer group acts transitively on the unit sphere. Two points x and x' in M/G are on the same Klein stratum if and only if M/G has the same dimension at these two points.

One of the advantages of treating the quotient space not as a differential space but as a diffeological quotient space is that it lifts the degeneracy of singular points. Concerning the differential quotient, the dimension at singular orbits represents a new invariant. This is a diffeological invariant not present in Sikorski's structure. It's possible that it is this invariant which appears in the work currently in preparation by Yael Karshon and Shintaro Kuroki[8].

In the last section, we will delve into specific examples to illustrate these concepts, which include the orbit space of a compact Lie group acting locally linearly on a manifold, the orbit space of $\mathrm{SO}(3)$ acting on TS^2, and the orbit space of $\mathrm{SO}(3)$ acting on $\mathrm{S}^2 \times \mathrm{S}^2$.

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Diffeological Stratified Spaces

General properties of stratifed spaces in diffeology and the general framewok for the theory of stratifications is introduced in [2]. Firstly we will give the definition of the natural extension to diffeology of the usual notion of topological stratified space.

Definition 1. Consider a diffeological space X, we call standard stratification of X any locally finite partition in strata $\mathcal{S}$ of X, such that:

(1) Each stratum is a manifold for the induced diffeology.
(2) Each stratum is locally closed for the D-topology.
(3) The strata satisfy the frontier condition: for all $\mathrm{S}, \mathrm{S}' \in \mathcal{S}$,

$$\mathrm{S} \cap \overline{\mathrm{S}}' \neq \emptyset \Rightarrow \mathrm{S} \subset \overline{\mathrm{S}}'.$$

In other words the closure of a stratum is a union of strata. We remind that the D-topology of a diffeological space is the finest topology that makes the plots continuous [3, I.2.3] and [6, 2.8]. A subset $A \subset X$ is open for the D-topology (or D-open) if $\mathrm{P}^{-1}(\mathrm{A})$ is open for any plot P in X.

Remark 1. In classical differential geometry it is important to require that the strata are manifolds since it is the only smooth structure known by the theory, and it is essential that the space of strata to be T_0 when it comes to the uniform structure of strata. These requirements become irrelevant in diffeology since diffeology has the capacity to discriminate between various types of strata, even when the space of non-manifold strata lacks a T_0 property. Consequently, we can assess the conditions in this definition independently [2].

Klein Stratification

We will particularly concentrate on the class of stratified spaces that are defined by the action of the diffeomorphism groups of the space itself.

Definition 2. Let X be a diffeological space, the Klein strata of X are defined as the the orbits of the group $\mathrm{Diff}(\mathrm{X})$ of diffeomorphisms of X. The space of Klein strata will be denoted by $\mathcal{S}_{\mathrm{K}}$.

We can however weaken this definition by considering local diffeomorphisms. In this work, we use the definition of Klein stratification by local diffeomorphisms.

Klein Stratification of Orbit Spaces

In this section, we compare the Klein stratification of the orbit spaces with the stratification by orbit types, whose strata are defined as the germs of points whose stabilizers are conjugate [10].

We will first introduce the Slice Theorem, a crucial tool for analyzing the structure of orbits and the underlying homogeneous spaces. Let G be a Lie group, H a closed subgroup, and E an euclidean space. The equivariant vector bundle

$$G \times_H V$$

over G/H is obtained as the quotient of $G \times V$ by the anti-diagonal H-action $h \cdot (g, v) = (gh^{-1}, h \cdot v)$. The G-action on $G \times_H V$ is $g \cdot [g', v] = [gg', v]$. We know the follwing theorem [9].

Slice Theorem (R. Palais). *Let G be a Lie group acting properly on a manifold M. Fix $x \in M$. Let H be the stabiliser of x, and let $V = T_xM/T_x(G \cdot x)$ be the normal space to the orbit $G \cdot x$ at x, equipped with the linear H-action that is induced by the linear isotropy action of H on T_xM. Then there exist a G-invariant open neighbourhood U of x and a G-equivariant diffeomorphism $F\colon U \to G \times_H V$ that takes x to $[1, 0]$.*

Proposition 1. *The map $(G \times_H E)/G \to E/H$ defined by $G[e, \xi] \mapsto H(\xi)$ is a diffeomorphism.*

Proof. First, note that every G-orbit in $G \times_H E$ passes through a point of the form $[e, \xi]$. It suffices to consider points of the form $[e, \xi]$, and we will denote their set as Σ.

$$\Sigma = \{[e, \xi], \xi \in \mathrm{E}\}$$

The group H preserves Σ:

$$\{g \in \mathrm{G}, g\Sigma = \Sigma\} = \{g \in \mathrm{G}, g[e, \xi] = [e, \xi']\}$$

because we have $g[e, \xi] = [e, \xi']$ if and only if $g \in \mathrm{H}$. We will show that $(G \times_H E)/G \to E/H$ defined by $G[e, \xi] \mapsto H(\xi)$ is injective. Let ξ and ξ' in E/H such that $\xi = \xi'$, which means there exists $h \in \mathrm{H}$ such that $\xi = h\xi'$. We have $[e, \xi'] = [e, h\xi] = [h, \xi] = h[e, \xi]$. So the map $(G \times_H E)/G \to E/H$ is injective.

Let $G \times H$ act on $G \times V$ where G acts by left multiplication on the first factor and where H acts by the anti-diagonal action $h : (g, v) \mapsto \left(gh^{-1}, h \cdot v\right)$. Let us denote by $\pi \colon G \times_H E \to (G \times_H E)/G$ the projection from $G \times_H E$ onto its quotient, by $\pi_H : G \times V \to G \times_H V$ the quotient by the H-action, and by $\mathrm{pr}_2 : G \times V \to V$ the projection to the second factor.

Taking the quotient by G and then by H, and taking the quotient by H and then by G give the following commuting diagram:

$$\begin{array}{ccccc}
 & & G \times E & & \\
 & \swarrow^{\pi_H} & \downarrow & \searrow^{pr_2} & \\
G \times_H E & & & & E \\
\downarrow \pi & & & & \downarrow \\
(G \times_H E)/G & \longrightarrow & G \times E/G \times H & \longrightarrow & \mathrm{E}/\mathrm{H}
\end{array}$$

Now, thanks to the uniqueness of quotients [6, 1.52], $(G \times_H E)/G \to E/H$ is a diffeomorphism between $(G \times_H E)/G$ equipped with the quotient diffeology and E/H, equipped with the quotient diffeology. □

Case of Isolated Singularities

Our study focuses on a specific case characterized by singular orbits consisting of isolated points, where the stabilizer acts transitively on the unit sphere. In this context, we emphasize the discriminative role of dimension at singular orbits.

Theorem 1. *Let* G *be a compact Lie group acting on an* n*-manifold* M *such that the singular orbits are the isolated points and the stabilizer group acts transitively on the unit sphere. Two points* x *and* x' *in* M/G *are on the same Klein stratum if and only if* M/G *has the same dimension at these two points.*

Proof. Let g be a local diffeomorphism, defined on some D-open $\mathcal{O}$, such that $g(x) = x'$.

$$\begin{array}{ccc}
(G \times_H E)/G & \xrightarrow{g} & (G \times_{H'} E')/G \\
f \downarrow & & \downarrow f' \\
\mathrm{E}/\mathrm{H} & \longrightarrow & \mathrm{E}'/\mathrm{H}' \\
\downarrow & & \downarrow \\
[0, \infty) & \longrightarrow & [0, \infty)
\end{array}$$

We have demonstrated in Proposition 1. that both maps, f and f', are diffeomorphisms. Since H acts transitively on the unit sphere, it follows that E/H is diffeomorphic to $\mathbf{R}^n/\mathrm{O(n)}$.

Furthermore, there exists a diffeomorphism between $\mathbf{R}^n/\mathrm{O(n)}$, equipped with the quotient diffeology, and $[0, \infty)$, equipped with the pushforward of the standard diffeology of $\mathbf{R}^n$ through the norm-square map $\|.\|^2$ [6, Ex.50]. This establishes a diffeomorphism of the half-line $[0, \infty)$.

We should note that the dimension plays a crucial role in characterizing this diffeomorphism of the half-line [6, Ex.64]. As the dimension is a local invariant, we can conclude that there exists a diffeomorphism between points x and x' in M/G if and only if they have the same dimension in M/G.

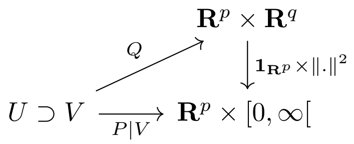
Therefore, we can assert that x and x' in M/G belong to the same Klein stratum if and only if M/G have the same dimension at these two points. □

We have the following result about the dimension of E/H for the general case, where the stabilizer group does not act transitively on the unit sphere.

Proposition 2. $\dim_{(x,y)}(\mathbb{R}^p \times \mathbb{R}^q/\mathrm{O}(q)) = \begin{cases} p+1 & y \neq 0 \\ p+q & y = 0 \end{cases}.$

Proof. Let us prove first that $\mathbf{1}_{\mathbf{R}^p} \times \|.\|^2 \colon \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^p \times [0,\infty[$ defined by $(x,y) \mapsto (x, \|y\|^2)$ is a generating family for the space $\mathbf{R}^p \times [0,\infty[$ equipped with the pushforward diffeolgy of the smooth diffeology of $\mathbf{R}^p \times \mathbf{R}^q$ by the map $\mathbf{1}_{\mathbf{R}^p} \times \|.\|^2$.

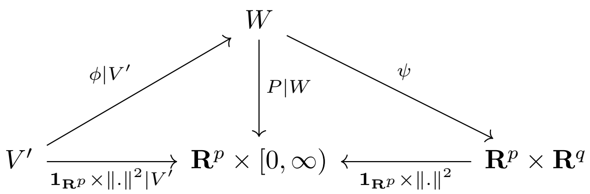
Let $\mathrm{P} \colon \mathrm{U} \to \mathbf{R}^p \times \mathbf{R}^q$ be a plot of $\mathbf{R}^p \times \mathbf{R}^q$. Let $\mathrm{r} \in \mathrm{U}$. There exists an open neighborhood V of r such that, either $\mathrm{P} \mid \mathrm{V}$ is a constant parametrization, or there exists a plot $\mathrm{Q} \colon \mathrm{V} \to \mathbf{R}^p \times [0,\infty[$ such that $\mathrm{P} \mid \mathrm{V} = \mathbf{1}_{\mathbf{R}^p} \times \|.\|^2 \circ Q$. This is exactly the criterion of generation.



Now, we will show that the plot $\mathbf{1}_{\mathbf{R}^p} \times \|.\|^2$ cannot be lifted locally at the point 0 along an m-plot, with $\mathrm{m} < \mathrm{p}+\mathrm{q}$.

Let us assume that the plot $\mathbf{1}_{\mathbf{R}^p} \times \|.\|^2$ can be lifted at the point 0 along an m-plot $\mathrm{P} : \mathrm{U} \to \mathbf{R}^p \times [0,\infty)$, with $\mathrm{m} < \mathrm{p}+\mathrm{q}$. Let $\phi : \mathrm{V} \to \mathrm{U}$ be a smooth parametrization such that $\mathrm{P} \circ \phi = (\mathbf{1}_{\mathbf{R}^p} \times \|.\|^2) \mid \mathrm{V}$. We can assume without loss of generality that $\mathrm{P}\,(0_{\mathrm{m}}) = 0$ and $\phi\,(0_{\mathrm{p+q}}) = 0_{\mathrm{m}}$.

Now, since P is a plot of $\mathbf{R}^p \times [0,\infty)$, it can be lifted locally at the point 0_{m} along $\mathbf{1}_{\mathbf{R}^p} \times \|.\|^2$. Let $\psi : \mathrm{W} \to \mathbf{R}^p \times \mathbf{R}^q$ be a smooth parametrization such that $0_{\mathrm{m}} \in \mathrm{W}$ and $\mathbf{1}_{\mathbf{R}^p} \times \|.\|^2 \circ \psi = \mathrm{P} \mid \mathrm{W}$. Let us introduce $\mathrm{V}' = \phi^{-1}(\mathrm{W})$. We have then the following commutative diagram.



Now, denoting by $\mathrm{F} = \psi \circ \phi \mid \mathrm{V}'$, we get $\mathbf{1}_{\mathbf{R}^p} \times \|.\|^2 \mid \mathrm{V}' = \mathbf{1}_{\mathbf{R}^p} \times \|.\|^2 \circ \mathrm{F}$, with $\mathrm{F} \in \mathcal{C}^\infty(\mathrm{V}', \mathbf{R}^n)$, $0_{\mathrm{p+q}} \in \mathrm{V}'$ and $\mathrm{F}\,(0_{\mathrm{p+q}}) = 0_{\mathrm{p+q}}$, that is,

$$(x_1, \ldots, x_p, \|x_{p+1}, \ldots, x_{p+q}\|^2) = (F(x_1), \ldots, F(x_p), \|\mathrm{F}(x_{p+1}), \ldots \mathrm{F}(x_{p+q})\|^2).$$

The derivative of this identity gives

$$\delta x_i = \mathrm{DF}(x_i) \text{ for } i \in \{1, \ldots, p\} \text{ and}$$

$$x \cdot \delta x = \mathrm{F}(x) \cdot \mathrm{D}(\mathrm{F})(x)(\delta x), \text{ for all } x = (x_{p+1}, \cdots, x_{p+q})$$

whoses coordinates are the restrictions of points in V' to last q coordiantes, and for all $\delta x \in \mathbf{R}^q$.

The second derivative, computed at the point 0_q, where F vanishes, gives then

$$\mathbf{1}_q = \mathrm{M}^t\mathrm{M}, \text{ with } \mathrm{M} = \mathrm{D}(\mathrm{F} \mid 0_{\mathrm{p}} \times \mathbf{R}^q))\,(0_q)\,,$$

where M^t is the transposed matrix of M. But $\mathrm{D}(\mathrm{F} \mid 0_{\mathrm{p}} \times \mathbf{R}^q))\,(0_q) = \mathrm{D}(\psi)\,(0_m) \circ \mathrm{D}(\phi)\,(0_q)$. Let us denote $\mathrm{A} = \mathrm{D}(\psi)\,(0_m)$ and $\mathrm{B} = \mathrm{D}(\phi)\,(0_q)\,, \mathrm{A} \in \mathrm{L}\,(\mathbf{R}^{m-p}, \mathbf{R}^q)$ and $\mathrm{B} \in \mathrm{L}\,(\mathbf{R}^q, \mathbf{R}^{m-p})$. Thus $\mathrm{M} = \mathrm{AB}$ and the previous identity $\mathbf{1}_q = \mathrm{M}^t\mathrm{M}$ becomes $\mathbf{1}_q = \mathrm{B}^t\,\mathrm{A}^t\mathrm{AB}$. But the rank of B is less or equal to q which is, by hypothesis, strictly less than q, which would imply that the rank of $\mathbf{1}_q$ is strictly less than q. And this is not true: the rank of $\mathbf{1}_q$ is q. Therefore, the plot $\mathbf{1}_{\mathbf{R}^p} \times \|.\|^2$ cannot be lifted locally at the point 0 by a m-plot of $\mathbf{R}^p \times [0,\infty)$ with $m < p+q$. □

Examples

We will explore several examples to demonstrate how the Klein stratification can offer valuable insights into the structure of the resulting orbit space.

Example 1

In this example, we will concentrate on analyzing the orbit space in the context where G is a compact Lie group and acts locally linearly on a manifold M. That is, M has an atlas for which the group acts linearly inside the domain of the chart. In a way, this situation looks like the situation on orbifolds. We will introduce a proposition that clarifies this particular case.

Proposition 3. *Let G be a compact Lie group acting locally linearly on a manifold M. If the stabilzers of two points x and x' are conjugate, then their orbits are on the same Klein stratum in M/G, and the projection map $\pi\colon \mathrm{M} \to \mathrm{M}/\mathrm{G}$ from $(\mathrm{M}, \mathcal{S}_{\mathrm{OT}})$ to $(\mathrm{M}/\mathrm{G}, \mathcal{S}_{\mathrm{K}})$ is a stratified map.*

Remark 2. Note that the principal orbits form the 'principal stratum' on M/G.

Proof. Let x and $x' \in \mathrm{M}$. Let $\mathrm{f} : \mathrm{U} \to \mathrm{M}$ be a chart of M with $\mathrm{f}(r) = x$ and let $\mathrm{f} : \mathrm{U}' \to \mathrm{M}$ be a chart of M with $\mathrm{f}(r') = x'$. Denote by H the stabilizer group of x and H' the stabilizer group of x'. As their stabilizer groups conjugate, there is $\mathrm{g} \in \mathrm{G}$ such that $\mathrm{H}' = \mathrm{g}\mathrm{H}\mathrm{g}^{-1}$. Define locally around r

$$\psi : s \mapsto r' + \mathrm{g}(s - r).$$

The map ψ is clearly a diffeomorphism. Let $\gamma \in \mathrm{H}$, then $\psi(\gamma s) = r' + \mathrm{g}(\gamma(s) - r) = r' + \mathrm{g}\gamma(s - r)$ since $\gamma(r) = r$. On the other hand, $\gamma'\psi(s) = \mathrm{g}\gamma\mathrm{g}^{-1}\,[r' + \mathrm{g}(s - r)] = r' + \mathrm{g}\gamma(s - r)$, where $\gamma' \in \mathrm{H}'$ such that $\gamma' = g\gamma g^{-1}$, since $\mathrm{g}\gamma\mathrm{g}^{-1}r' = r'$. Hence, $\psi(\gamma s) = \gamma'\psi(s)$ the map ψ descends to the orbit space M/G into a local diffeomorphism.

Now we will show that the projection map π from $(\mathrm{M}, \mathcal{S}_{\mathrm{OT}})$ to $(\mathrm{M}/\mathrm{G}, \mathcal{S}_{\mathrm{K}})$ is a stratified map [2] that is there is a map φ from $\mathcal{S}_{\mathrm{OT}}$ to $\mathcal{S}_{\mathrm{K}}$ such that:

$$s_{\mathrm{K}} \circ \pi = \varphi \circ s_{\mathrm{OT}}$$

where s_{K} denote the projection from the space to the space of Klein strata and s_{OT} the projection from the space to the space of orbit type strata.

Let x and $x' \in \mathrm{M}$ such that $s_{\mathrm{OT}}(x) = s_{\mathrm{OT}}(x')$ which means their stabilizers conjugate, then we have established the existence of a local diffeomorphism between $\pi(x)$ and $\pi(x')$, affirming that they are in the same Klein strata. Hence, we can define the map φ such that $\varphi(s_{\mathrm{OT}}(x)) = s_{\mathrm{K}}(\pi(x))$, ensuring the commutivity of the following diagram:

$$\begin{array}{ccc} \mathrm{M} & \xrightarrow{\ \pi\ } & \mathrm{M}/\mathrm{G} \\ {\scriptstyle s_{\mathrm{OT}}}\downarrow & & \downarrow{\scriptstyle s_{\mathrm{K}}} \\ \mathcal{S}_{\mathrm{OT}} & \xrightarrow{\ \varphi\ } & \mathcal{S}_{\mathrm{K}} \end{array}$$

□

Remark 3. As a natural continuation of the preceding proposition, which established in this particular case that the stratification by orbit types is a diffeological stratification which is geometric and all the strata are submanifolds for the induced diffeology as a consequence of [1, Thm. 3.3]. Furthermore, due to the compact action, these strata are locally closed. This stratification is denoted by the labels $[B]$-$[F]$-$[G]$-$[M]$-$[T_0]$, which are terminologies introduced in [2] to describe and categorize specific aspects of this stratification.

Example 2

Consider the tangent space TS^2 of the 2-sphere, that is, $\mathrm{TS}^2 = \{(u, v) \in \mathrm{S}^2 \times \mathbf{R}^3 \mid u \cdot v = 0\}$. Let us denote by $\pi : \mathrm{TS}^2 \to \mathrm{TS}^2/\mathrm{SO}(3)$ the projection from TS^2 onto its quotient. Since, $\|w'\| = \|w\|$ if and only if $w' = \mathrm{A}w$, with $\mathrm{A} \in \mathrm{SO}(3)$, there exists a bijection $f : \mathrm{TS}^2/\mathrm{SO}(3) \to [0, \infty)$ such that $f \circ \pi = \|.\|$, where $< v, v >= \|v\|^2$. Now, thanks to the uniqueness of quotients (op.cit.) f is a diffeomorphism between TS^2 equipped with the quotient diffeology and $[0, \infty)$, equipped with the pushforward of the standard diffeology of TS^2 by the map $\|.\|$.

$$\begin{array}{ccc} \mathrm{TS}^2 & & \\ {\scriptstyle \pi}\downarrow & \searrow^{\|.\|^2} & \\ \mathrm{TS}^2/\mathrm{SO}(3) & \xrightarrow[\ f\ ]{} & [0, \infty) \end{array}$$

The dimension of $\mathrm{TS}^2/\mathrm{SO}(3)$ at its points is:

$$\dim_0(\mathrm{TS}^2/\mathrm{SO}(3)) = 2 \text{ and } \dim_t(\mathrm{TS}^2/\mathrm{SO}(3)) = 1 \text{ if } t \neq 0.$$

The map $(A, v) \mapsto (Ae_1, Av)$, from $\mathrm{SO}(3) \times e_1^{\perp}$ to $\mathrm{S}^2 \times \mathbb{R}^3$, takes its values in TS^2 and represents the associated fiber bundle $\mathrm{SO}(3) \times_{\mathrm{SO}(2,e_1)} e_1^{\perp}$ [6].

Let us prove that $\psi\colon \mathrm{TS}^2 \to \mathrm{S}^2 \times \mathrm{S}^2$ defined by $(u, w) \to (u, v)$, where $v = \alpha(u + w)$ such that $\|\alpha(u + w)\| = 1$ is an induction. First of all, the map ψ is injective. The inverse is given by

$$\psi^{-1}(u, v) = (u, \frac{1}{\alpha}v - u).$$

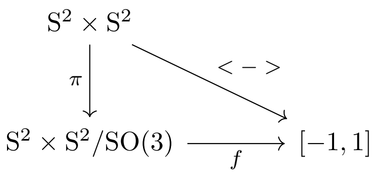
Now, let P be a plot of $\mathrm{S}^2 \times \mathrm{S}^2$ with values in $\psi(\mathrm{TS}^2)$ which means $P(r) = (Q(r), \alpha(Q(r) + F(r)))$, where Q is plot of S^2 and F is a plot of $\mathbf{R}^3$. Then $\psi^{-1} \circ P(r) = (Q(r), \frac{1}{\alpha}\mathrm{F}(r) - Q(r))$. It's clearly a plot of $\mathrm{S}^2 \times \mathbb{R}^3$, thus ψ is an induction. Consequently, the map ψ is SO(3)-equivariant. On the other hand, $\mathrm{S}^2 \times \mathrm{S}^2/\mathrm{SO}(3)$ is obtained by gluing two copies of TS^2 along the boundary via an equivariant diffeomorphism [4].

Example 3

Let $\mathrm{SO}(3)$ be the group of direct rotations of the space $\mathbf{R}^3$, that is, the group of real 3×3 matrices A such that $\bar{\mathrm{A}}\mathrm{A} = \mathbf{1}_{\mathbf{R}^3}$ and $\det(\mathrm{A}) = +1$, where the bar denotes the transposition.

We consider the quotient of $\mathrm{S}^2 \times \mathrm{S}^2$ under the action of $\mathrm{SO}(3)$. The quotient $\mathrm{S}^2 \times \mathrm{S}^2/\mathrm{SO}(3)$ is equivalent to the set $[-1, 1]$ equipped with the pushforward of the smooth diffeology of $\mathrm{S}^2 \times \mathrm{S}^2$ by the map $<,>\colon (x, y) \mapsto < x, y >$.

Let us denote by $\pi : \mathrm{S}^2 \times \mathrm{S}^2 \to \mathrm{S}^2 \times \mathrm{S}^2/\mathrm{SO}(3)$ the projection from $\mathrm{S}^2 \times \mathrm{S}^2$ onto its quotient. Since, $< x, y > = < x', y' >$ if and only if $x' = \mathrm{A}x$ and $y' = \mathrm{A}y$, with $\mathrm{A} \in \mathrm{SO}(3)$, there exists a bijection $f\colon \mathrm{S}^2 \times \mathrm{S}^2/\mathrm{SO}(3) \to [-1, 1]$. such that $f \circ \pi_n = <,>$. Now, thanks to the uniqueness of quotients (op. cit.), f is a diffeomorphism between $\mathrm{S}^2 \times \mathrm{S}^2/\mathrm{SO}(3)$ equipped with the quotient diffeology and $[-1, 1]$, equipped with the pushforward of the smooth diffeology of $\mathrm{S}^2 \times \mathrm{S}^2$ by the map $<,>$.



The points -1 and 1 represent two singular orbits $\{(u, -u)\}$ and $\{(u, u)\}$.
The dimension of $\mathrm{S}^2 \times \mathrm{S}^2/\mathrm{SO}(3)$ at its points is:

$$\dim_{-1}(\mathrm{S}^2 \times \mathrm{S}^2/\mathrm{SO}(3)) = \dim_{+1}(\mathrm{S}^2 \times \mathrm{S}^2/\mathrm{SO}(3)) = 2 \text{ and } \dim_t(\mathrm{S}^2 \times \mathrm{S}^2/\mathrm{SO}(3)) = 1 \text{ if } t \neq \pm 1.$$

A remark we can do, for the quotient diffeology, these spaces are not manifolds with boundary and corners because the interval $[0, \infty[$ has dimension ∞ at 0 and 1 otherwise. On the contrary, for Sirkoski's differential structure, the orbit space is a manifold with corners, as all the quotients of $\mathbb{R}^n/\mathrm{SO(n)}$ are equivalent.

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Author's contributions

Author read and approved the final manuscript.

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