Bitlis Eren Üniversitesi Fen Bilimleri Dergisi
BİTLİS EREN UNIVERSITY JOURNAL OF SCIENCE
ISSN: 2147-3129/e-ISSN: 2147-3188
VOLUME: 13 NO: 1 PAGE: 161-169 YEAR: 2024
DOI:10.17798/bitlisfen.1364241

# On Conic Equations Under Bernstein Operators

Tuncay TUNC[1*], Ghofran ALHAZZORI[2]

[1] *Department of Mathematics, Faculty of Science, Mersin University, Mersin, Türkiye*
[2] *Department of Mathematics, Institute of Science, Mersin University, Mersin, Türkiye*
*(ORCID: 0000-0002-3061-7197) (ORCID: 0000-0003-0342-1575)*

**Keywords:** Bivariate Bernstein polynomials, Conic equations, Shape-preserving approximation, Korovkin type theorem.

**Abstract**
One of the most important problems in approximation theory in mathematical analysis is the determination of sequences of polynomials that converge to functions and have the same geometric properties. This type of approximation is called the shape-preserving approximation. These types of problems are usually handled depending on the convexity of the functions, the degree of smoothness depending on the order of differentiability, or whether it satisfies a functional equation. The problem addressed in this paper belongs to the third class. A quadratic bivariate algebraic equation denotes geometrically some well-known shapes such as circle, ellipse, hyperbola and parabola. Such equations are known as conic equations. In this study, it is investigated whether conic equations transform into a conic equation under bivariate Bernstein polynomials, and if so, which conic equation it transforms into.

## 1. Introduction

One of the important problems in approximation theory is to identify sequences of polynomials that converge to functions and have the same geometric properties. This type of approximation is called the shape-preserving approximation. The issue of shape-preserving approximation with algebraic polynomials has a long history and probably begins in 1925 with a result of Pal [1] stating that any convex function defined in the interval $[a,b]$ can be properly approximated by convex polynomials in that interval. The first structural answer to Pal's conclusion seems to have been given by Popoviciu [2] in 1937 with the help of Bernstein polynomials: For $f:[0,1]\to\mathbb{R}$ and $n\in\mathbb{N}$, the nth Bernstein polynomial is defined by

$$B_n(f;x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}$$

The sequence of Bernstein polynomials is the best-known that has the property of shape-preserving approximation. There have been many studies on whether the sequence of Bernstein polynomials preserves which geometric properties, especially convexity types. For $k\in\mathbb{N}$, Popoviciu showed that the Bernstein polynomials $B_n(f)$ are also $k$-convex for each $n\in\mathbb{N}$, if $f$ is a $k$-convex function [2]. Over time, many mathematicians have made great efforts to contribute to this topic. The studies of Lupaş [3], Leviatan [4,5], Kocic and Milovanovic [6] and Hu-Yu [7] can be mentioned as good examples of studies on the shape-preserving approximation of univariate real functions with polynomials.

Despite the large number of articles in the literature, there are not many books dealing with the shape-preserving approximation in the bivariate or multivariate case. First, Gal [8] dealt with it. Using the method of forming Bernstein polynomials dependent on univariate functions, Hildebrant and Schoenberg [9] defined bivariate Bernstein polynomials with double indices

$$B_{n,m}(f;x,y)=\sum_{k=0}^{n}\sum_{j=0}^{m} f\left(\frac{k}{n},\frac{j}{m}\right)\binom{n}{k}\binom{m}{j}x^k \times(1-x)^{n-k}y^j(1-y)^{m-j}$$

*Corresponding author: *ttunc@mersin.edu.tr*

*Received: 21.09.2023, Accepted: 22.01.2024*

dependent on bivariate functions defined on the $[0,1]\times[01]$ and examined their approximation properties. Kingsley [10] showed that this set of operators converges uniformly to the partial derivatives of the function f. $(r,s)$ −convex functions defined by Popoviciu [11] are protected under $B_{n,m}$ operators [8]. Tunç and Uzun [12] obtained results that B-convex functions are not conserved under these operators, but B-concave functions are preserved in some special cases.

In this study, we have examined whether the geometric shapes specified by the conic equations were preserved under Bernstein operators mentioned above.

## 2. Material and Method

The moment formulas given below are correct for the bivariate Bernstein operators with double indices that are defined by

$$B_{n,m}(f;x,y)=\sum_{k=0}^{n}\sum_{j=0}^{m} f\left(\frac{k}{n},\frac{j}{m}\right)\binom{n}{k}\binom{m}{j}x^k \times(1-x)^{n-k}\,y^j(1-y)^{m-j}.$$

Let us define the functions $e_{i,j}$, $i,j\in\mathbb{N}_0$, by $e_{i,j}(x,y)=x^iy^j$ on $\mathbb{R}^2$.

**Lemma 2.1.** ([9])

i. $B_{n,m}(e_{i,j};x,y)=e_{i,j}(x,y)$, for all $i,j\in\{0,1\}$

ii. $B_{n,m}(e_{0,2};x,y)=e_{0,2}(x,y)+\frac{y(1-y)}{m}$,

iii. $B_{n,m}(e_{2,0};x,y)=e_{2,0}(x,y)+\frac{x(1-x)}{n}$

**Teorem 2.2.** ([9]) If $f$ is a continuous function on $[0,1]\times[0,1]$ then the sequence $(B_{n,m}(f))$ converges uniformly to $f$ function on $[0,1]\times[0,1]$.

In the article referenced for the proof of this theorem, it is said that it can be done in a similar way by referring to the work of Bernstein [13]. However, a simpler method can be proved. [14] and Lemma 2.1 can be used for this.

## 3. Results and Discussion

### 3.1. Circles Under Bivariate Bernstein Operators

In this section, firstly, conic equations will be discussed under the bivariate Bernstein Operators with double index. For $A,B,C,D,E,F\in\mathbb{R}$, the equation

$$f(x,y)=Ax^2+Bxy+Cy^2+Dx+Ey+F=0 \quad (1)$$

looks like following under the $B_{n,m}$ operator:

$$B_{n,m}(f;x,y)=A'x^2+B'xy+C'y^2+D'x+E'y+F'=0$$

where

$$A'=\frac{n-1}{n}A;\; B'=B;\; C'=\frac{m-1}{m}C;\; D'=D+\frac{A}{n};\; E'=E+\frac{C}{m};\quad F'=F \quad (2)$$

In Theorem 2.2, the double-index bivariate Bernstein operators are defined according to their values in the domain of the function, and it is said that the sequence formed by these operators converges properly in the domain of the function. However, the equations in Lemma 2.1 are valid at every point of the plane. So the following theorem is true.

**Theorem 3.1.** For the function $f$ defined by (1), the sequence $(B_{n,m}(f))$ is uniformly convergent to the function $f$ on every compact subset of the plane.

**Proof.** It is easily obtained from Lemma 2.1.

Let

$$\Delta_0=D^2+E^2-4F;$$
$$\Delta_1=D+E+2F.$$

It is well known that, if $A=C=1$, $B=0$ and $\Delta_0>0$ then the equation (1) indicates a circle in the cartesian plane.

**Theorem 3.2.** Let the equation $f(x,y)=0$ given by (1), where $A=C=1$, specify a circle and let $n\in\mathbb{N}_2:=\{2,3,4,\ldots\}$. In this case, the necessary and sufficient condition for the equation $B_{n,n}(f;x,y)=0$ to specify a circle is

$$n^2\Delta_0+2n\Delta_1+2>0.$$

**Proof.** Since $A=C=1$ and $B=0$, if $m=n$ is taken in equations (2), the appearance of equation (1) under the $B_{n,n}$ operator will be as follows

$$B_{n,n}(f;x,y)=\left(\frac{n-1}{n}\right)x^2+\left(\frac{n-1}{n}\right)y^2+\frac{Dn+1}{n}x+\frac{En+1}{n}y+F=0.$$

For $n > 1$, we get

$$\frac{n-1}{n}\left[x^2+y^2+\frac{Dn+1}{n-1}x+\frac{En+1}{n-1}y+\frac{n}{n-1}F\right] = 0.$$

If we write

$$D'' = \frac{Dn+1}{n-1}, \qquad E'' = \frac{En+1}{n-1}, \qquad F'' = \frac{nF}{n-1},$$

by simple calculations

$$\begin{aligned} D''^2 + E''^2 - 4F'' &= \left(\frac{Dn+1}{n-1}\right)^2 + \left(\frac{En+1}{n-1}\right)^2 - 4\left(\frac{nF}{n-1}\right) \\ &= \frac{(Dn+1)^2}{(n-1)^2} + \frac{(En+1)^2}{(n-1)^2} - \frac{4nF(n-1)}{(n-1)^2} \\ &= \frac{1}{(n-1)^2}[(Dn+1)^2 + (En+1)^2 - 4nF(n-1)] \\ &= \frac{1}{(n-1)^2}[D^2n^2 + E^2n^2 - 4Fn^2 + 2Dn + 2En + 4Fn + 2] \\ &= \frac{1}{(n-1)^2}[n^2\Delta_0 + 2n\Delta_1 + 2] \end{aligned}$$

are obtained. So that we get desired result.

**Remarks 3.3.**

1. In Theorem 3.2, if $n = 1$, the equation $B_{n,n}\,(f;x,y) = 0$ indicates a line. The equation of this line is $(D+1)x + (E+1)y + F = 0$.
2. In Theorem 3.2, if $n > 1$, the equation $B_{n,n}\,(f;x,y) = 0$ refers to the circle with radius $\frac{\sqrt{n^2\Delta_0+2n\Delta_1+2}}{2n-2}$ centered at $\left(\frac{Dn+1}{2n-2},\frac{En+1}{2n-2}\right)$.
3. Since $\Delta_0 > 0$, the condition $n^2\Delta_0 + 2n\Delta_1 + 2 > 0$ in Theorem 3.2 will be satisfied for every sufficiently large natural number $n$. Therefore, if the equation $f(x,y) = 0$ indicates a circle, the equation $B_{n,n}\,(f;x,y) = 0$ indicates a circle except for a finite number of $n$.

Since $\Delta_0 > 0$, the following result can be easily obtained from Theorem 3.2.

**Corollary 3.4.** Under the conditions of Theorem 3.2, if $\Delta_1 > -1/n$, then $B_{n,n}\,(f;x,y) = 0$ indicates a circle.

**Example 3.5.** If the circle specified by the equation $f(x,y) = x^2 + y^2 + 4x - 6y - 12 = 0$ is taken into account, since $\Delta_0 = 100$ and $\Delta_1 = -26$, from Corollary 3.4, it is not clear that the equations $B_{n,n}\,(f;x,y) = 0$ specify a circle. However, since

$$n^2\Delta_0 + 2n\Delta_1 + 2 = 100n^2\Delta_0 - 52n\Delta_1 + 2 > 0,$$

the equation $B_{n,n}\,(f;x,y) = 0$ specify a circle by Theorem 3.2. (see Figure 1)

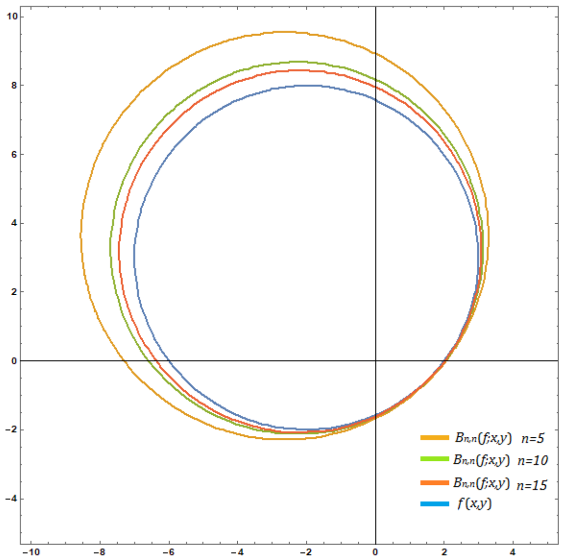


**Figure 1.** Images of the circle given in Example 3.5 under the operator $B_{n,n}$ for $n = 5,10,15$.

**Theorem 3.6.** Let the equation $f(x,y) = 0$ given by (1), where $A = C = 1$, specify a circle and let $n, m \in \mathbb{N}_2$. If $n \neq m$, then the equation $B_{n,m}\,(f;x,y) = 0$ specifies an ellipse.

**Proof.** Since $A = C = 1$ and $B = 0$, from the equations (2), the appearance of equation (1) under the $B_{n,m}$ operators will be as follows

$$B_{n,m}(f;x,y) = \left(\frac{n-1}{n}\right)x^2 + \left(\frac{m-1}{m}\right)y^2 + \frac{Dn+1}{n}x + \frac{Em+1}{m}y + F = 0.$$

For $n, m > 1$, we get

$$B'^2 - 4A'C' = -4A'C' = -4\left(\frac{n-1}{n}\right)\left(\frac{m-1}{m}\right) < 0.$$

Since $n \neq m$, the equation $B_{n,m}(f;x,y) = 0$ specifies an ellipse.

**Example 3.7.** If the circle specified by the equation $f(x,y) = x^2 + y^2 + 4x - 6y - 12 = 0$ is taken into account, the equations

$$B_{n,m}(f;x,y) = \left(\frac{n-1}{n}\right)x^2 + \left(\frac{m-1}{m}\right)y^2 + \frac{4n+1}{n}x + \frac{1-6m}{m}y - 12 = 0$$

specify ellipses under the condition $n \neq m$ (see Figure 2).

**Theorem 3.8.** Let the equation $f(x,y) = 0$ given by (1), where $A = C = 1$, specify a circle and let $n, m \in \mathbb{N}$. If $n \neq m$ and $min\{n,m\} = 1$ then the equation $B_{n,m}(f;x,y) = 0$ specifies a parabola.

**Proof.** Since it does not violate generality, let $n = 1$ and $m > 1$. Since $A = C = 1$ and $B = 0$, from the equations (2), the appearance of equation (1) under the $B_{n,m}$ operators will be as follows

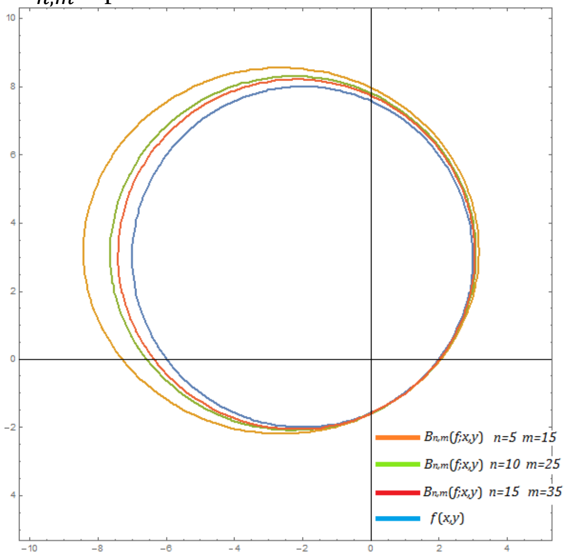


**Figure 2.** Images of the circle given in Example 3.7 under the operators $B_{n,m}$ for $(n,m) = (5,15), (10,25), (15,35)$.

$$B_{n,m}(f;x,y) = \left(\frac{m-1}{m}\right)y^2 + (D+1)x + \frac{Em+1}{m}y + F = 0.$$

Since $B'^2 - 4A'C' = 0$, the equation $B_{n,m}(f;x,y) = 0$ specifies a parabola.

**Example 3.9.** If the circle specified by the equation $f(x,y) = x^2 + y^2 + 4x - 6y - 12 = 0$ is taken into account, the equations

$$B_{1,m}(f;x,y) = \left(\frac{m-1}{m}\right)y^2 + 5x + \frac{1-6m}{m}y - 12 = 0$$

specify parabolas under the condition $m > 1$ (see Figure 3).

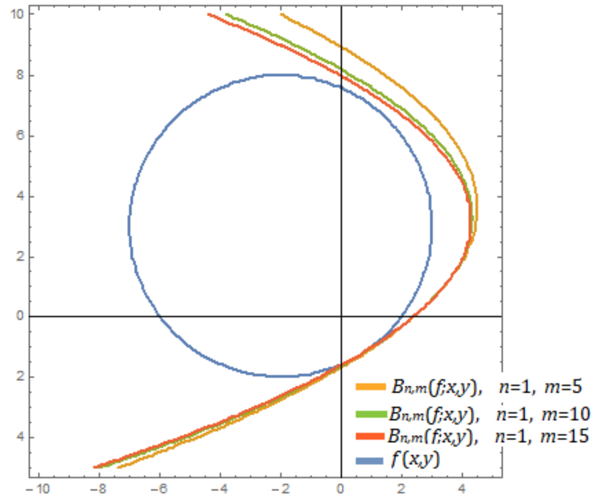


**Figure 3.** Images of the circle given in Example 3.9 under the operators $B_{n,m}$ for $(n,m) = (1,5), (1,10), (1,15)$.

### 3.2. Ellipses Under Bivariate Bernstein Operators

**Theorem 3.10.** Let the equation $f(x,y) = 0$ given by (1) specify an ellipse. Then the equation $B_{n,m}(f;x,y) = 0$ specifies

- **i.** an ellipse for all $n, m \in \mathbb{N}_2$ with the condition $\frac{B^2}{4AC} < \left(\frac{n-1}{n}\right)\left(\frac{m-1}{m}\right)$,
- **ii.** a hyperbola for all $n, m \in \mathbb{N}_2$ with the condition $\frac{B^2}{4AC} > \left(\frac{n-1}{n}\right)\left(\frac{m-1}{m}\right)$,
- **iii.** a parabola for all $n, m \in \mathbb{N}_2$ with the condition $\frac{B^2}{4AC} = \left(\frac{n-1}{n}\right)\left(\frac{m-1}{m}\right)$.

**Proof.** Since the equation $f(x,y) = 0$ given by (1) indicates an ellipse, the inequality $B^2 - 4AC < 0$ and thus $AC > 0$ is satisfied. From the equations (2), the appearance of equation (1) under the $B_{n,m}$ operators will be as follows

$$B_{n,m}(f;x,y) = \left(\frac{n-1}{n}\right)Ax^2 + Bxy + \left(\frac{m-1}{m}\right)Cy^2 + \frac{Dn+A}{n}x + \frac{Em+C}{m}y + F = 0.$$

Since

$$B'^2 - 4A'C' = B^2 - 4AC\left(\frac{n-1}{n}\right)\left(\frac{m-1}{m}\right)$$

for all $n, m \in \mathbb{N}_2$, we have desired results.

**Example 3.11.** If the ellipse specified by the equation $f(x,y) = 15x^2 + 15xy + 6y^2 + 4x - 6y - 12 = 0$ is taken into account, we get $B^2/4AC = 5/8$. Hence, according to Theorem 3.10, the equation

$$B_{n,m}(f;x,y) = 15\left(\frac{n-1}{n}\right)x^2 + 15xy + 6\left(\frac{m-1}{m}\right)y^2 + \left(\frac{15}{n} + 4\right)x + \left(\frac{6}{m} - 6\right)y - 12 = 0$$

specifies an ellipse for $n = 6,\ m = 8$, a parabola for $n = 4,\ m = 6$ and a hyperbola for $n = 2,\ m = 4$ (see Figure 4)

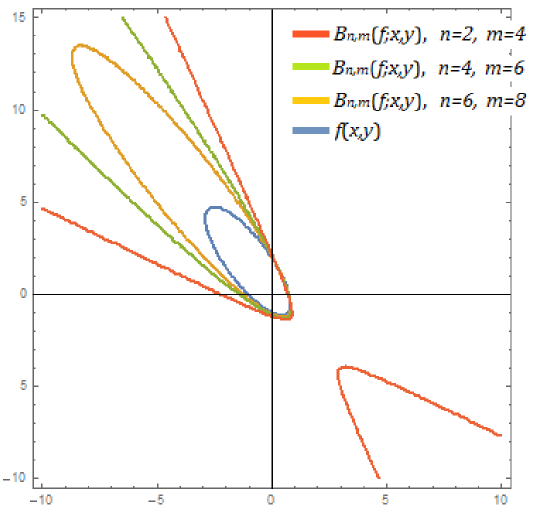


**Figure 4.** Images of the ellipse given in Example 3.11 under the operators $B_{n,m}$ for $(n,m) = (2,4),\ (4,6), (6,8)$.

**Theorem 3.12.** Let the equation $f(x,y) = 0$ given by (1) specify an ellipse. Then, if $n \neq m$ and $\min\{n,m\} = 1$, the equation $B_{n,m}\,(f;x,y) = 0$ specifies

- **i.** a hyperbola for $B \neq 0$,
- **ii.** a parabola for $B = 0$.

**Proof.** Since it does not violate generality, let $n = 1$ and $m > 1$. From the equations (2), the appearance of equation (1) under the $B_{n,m}$ operators will be as follows

$$B_{n,m}(f;x,y) = Bxy + \left(\frac{m-1}{m}\right)Cy^2 + (D+A)x + \frac{Em + C}{m}y + F = 0.$$

Since $B'^2 - 4A'C' = B^2$, the equation $B_{n,m}\,(f;x,y) = 0$ specifies a parabola for $B = 0$ while a hyperbola for $B \neq 0$.

**Example 3.13.**
**(a)** If the ellipse specified by the equation

$$f(x,y) = 15x^2 + 15xy + 6y^2 + 4x - 6y - 12 = 0$$

is taken into account, the equation

$$B_{1,m}(f;x,y) = 15xy + 6\left(\frac{m-1}{m}\right)y^2 + 19x + \left(\frac{6}{m} - 6\right)y - 12 = 0$$

specifies a hyperbola for each $m > 1$ (see Figure 5).

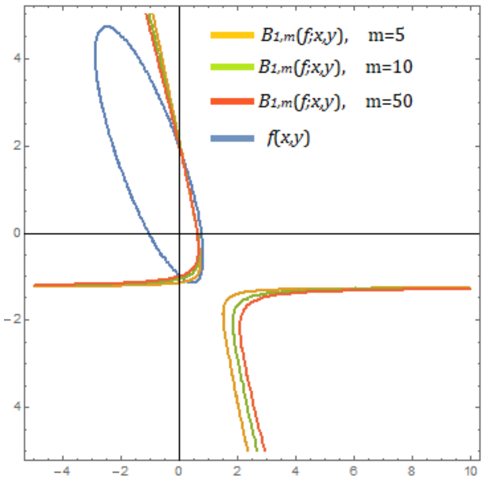


**Figure 5.** Images of the ellipse given in Example 3.13(a) under the operators $B_{1,m}$ for $m = 5, 10, 50$

**(b)** If the ellipse specified by the equation

$$f(x,y) = 15x^2 + 6y^2 + 4x - 6y - 12 = 0$$

is taken into account, the equation

$$B_{1,m}(f;x,y) = 6\left(\frac{m-1}{m}\right)y^2 + 19x + \left(\frac{6}{m} - 6\right)y - 12 = 0$$

specifies a parabol for each $m > 1$ (see Figure 6).

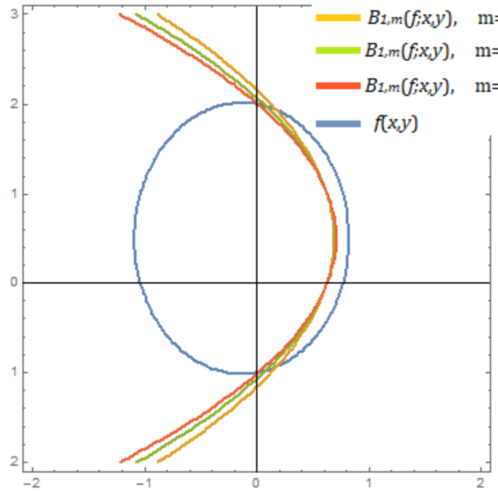


**Figure 6.** Images of the ellipse given in Example 3.13(b) under the operators $B_{1,m}$ for $m = 5, 10, 50$

### 3.3. Hyperbolas Under Bivariate Bernstein Operators

**Theorem 3.14.** If the equation $f(x,y) = 0$ given by (1) specifies a hyperbola, then the equations $B_{n,m}(f;x,y) = 0$ specify hyperbola for all $n, m \in \mathbb{N}_2$.

**Proof.** Since the equation $f(x,y) = 0$ given by (1) indicates a hyperbola, the inequality $B^2 - 4AC > 0$ is satisfied. From the equations (2), the appearance of equation (1) under the $B_{n,m}$ operators will be as follows

$$B_{n,m}(f;x,y) = \left(\frac{n-1}{n}\right)Ax^2 + Bxy + \left(\frac{m-1}{m}\right)Cy^2 + \frac{Dn+A}{n}x + \frac{Em+C}{m}y + F = 0.$$

Since $0 < \left(\frac{n-1}{n}\right)\left(\frac{m-1}{m}\right) < 1$, then we have

$$B'^2 - 4A'C' = B^2 - 4AC\left(\frac{n-1}{n}\right)\left(\frac{m-1}{m}\right) > 0$$

for all $n, m \in \mathbb{N}_2$.

**Remarks 3.15.**

1. Let the equation $f(x,y) = 0$ given by (1) specify a hyperbola and let $AC < 0$. In this case, the possibility of $B = 0$ arises. If this situation occurs, the equation $B_{n,m}(f;x,y) = 0$ for every $n, m \in \mathbb{N}$ with $min\{n,m\} = 1$ and $n \neq m$ also indicates a parabola. Because under these conditions will be $B'^2 - 4A'C' = 0$.
2. If $B \neq 0$, Theorem 3.14 is valid for every $n, m \in \mathbb{N}$ with $max\{n,m\} \neq 1$.

**Example 3.15.**

**(a)** If the hyperbola specified by the equation $f(x,y) = 15x^2 + 15xy + 2y^2 + 4x - 6y - 12 = 0$ is taken into account, the equations

$$B_{n,m}(f;x,y) = 15\left(\frac{n-1}{n}\right)x^2 + 15xy + 2\left(\frac{m-1}{m}\right)y^2 + \left(\frac{15}{n} + 4\right)x + \left(\frac{2}{m} - 6\right)y - 12 = 0$$

specify hyperbola for all $n, m \in \mathbb{N}_2$. Note that $AC = 15.2 = 30 > 0$ (see Figure 7).

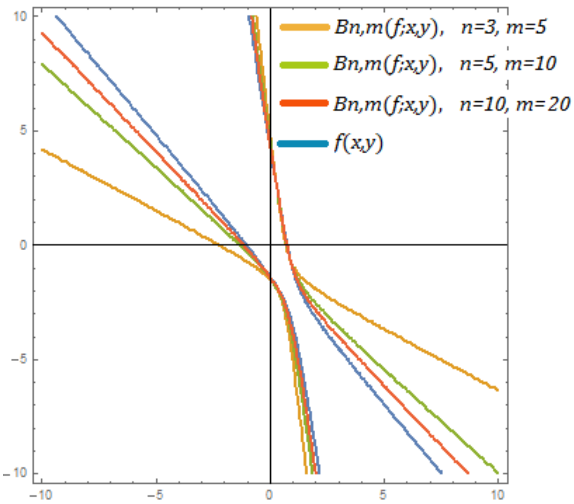


**Figure 7.** Images of the hyperbola given in Example 3.15(a) under the operators $B_{n,m}$ for $(n,m) = (3,5), (5,10), (10,20)$.

**(b)** If the hyperbola specified by the equation

$$f(x,y) = 15x^2 + 15xy + 4x - 6y - 12 = 0$$

is taken into account, the equations

$$B_{n,m}(f;x,y) = 15\left(\frac{n-1}{n}\right)x^2 + 15xy + \left(\frac{15}{n} + 4\right)x - 6y - 12 = 0$$

specify hyperbola for all $n, m \in \mathbb{N}_2$. Note that $AC = 0$ (see Figure 8).

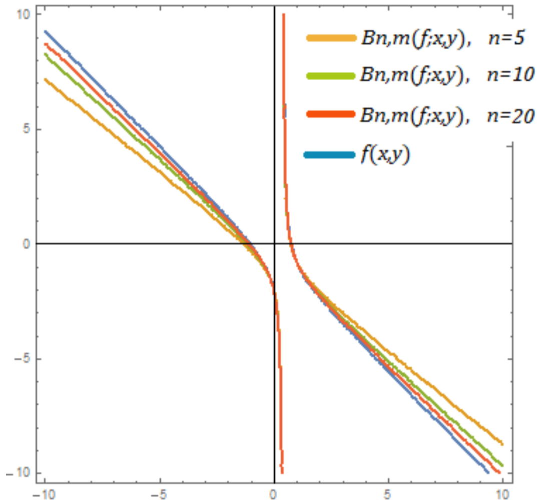


**Figure 8.** Images of the hyperbola given in Example 3.15(b) under the operators $B_{n,m}$ for $(n,m) = (5,m),(10,m),(20,m)$.

**(c)** If the hyperbola specified by the equation

$$f(x,y) = 15x^2 + 15xy + 4x - 6y - 12 = 0$$

is taken into account, the equations

$$B_{n,m}(f;x,y) = 15\left(\frac{n-1}{n}\right)x^2 - 2\left(\frac{m-1}{m}\right)y^2 + \left(\frac{15}{n}+4\right)x + \left(-\frac{2}{m}-6\right)y - 12 = 0$$

specify hyperbola for all $n,m \in \mathbb{N}_2$. Note that $AC = -30 < 0$ (see Figure 9). On the other hand, the equations

$$B_{1,m}(f;x,y) = -2\left(\frac{m-1}{m}\right)y^2 + 19x + \left(-\frac{2}{m} - 6\right)y - 12 = 0,$$

$$B_{n,1}(f;x,y) = 15\left(\frac{n-1}{n}\right)x^2 + \left(\frac{15}{n}+4\right)x - 8y - 12 = 0$$

specify parabolas for all $n,m \in \mathbb{N}_2$ (see Figure 10).

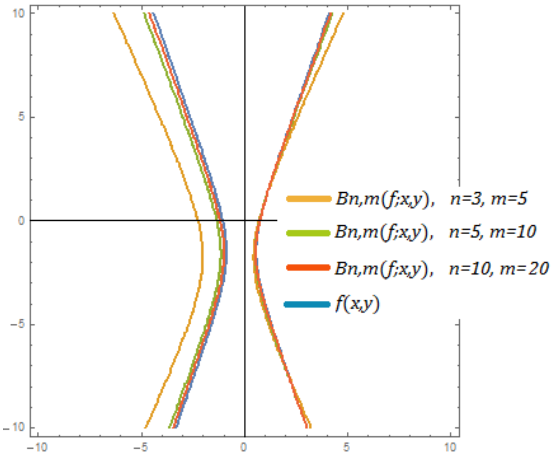


**Figure 9.** Images of the hyperbola given in Example 3.15(c) under the operators $B_{n,m}$ for $(n,m) = (3,5),(5,10),(10,20)$.

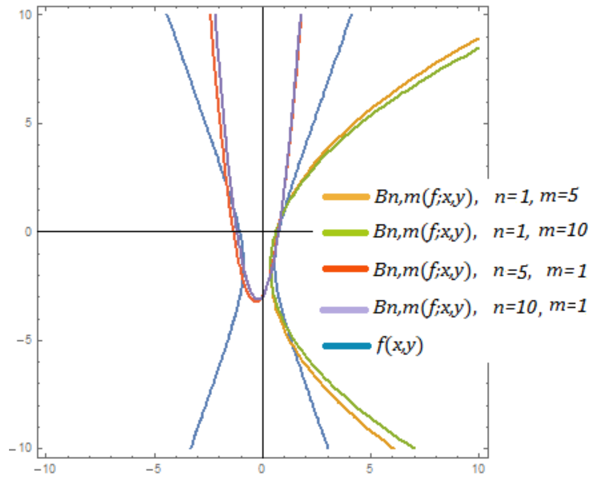


**Figure 10.** Images of the hyperbola given in Example 3.15(c) under the operators $B_{n,m}$ for $(n,m) = (1,5),(1,10),(5,1),(10,1)$.

### 3.4. Parabolas Under Bivariate Bernstein Operators

**Theorem 3.16.** Let the equation $f(x,y) = 0$ given by (1) specify a parabola. Then, for all $n,m \in \mathbb{N}_2$, the equation $B_{n,m}(f;x,y) = 0$ specifies

**i.** a hyperbola if $AC \neq 0$,
**ii.** a parabola if $AC = 0$.

**Proof.** Since the equation $f(x,y) = 0$ given by (1) indicates a parabola, the equality $B^2 - 4AC = 0$ is satisfied. Therefore, it must be $AC \geq 0$ and $A$ and $C$ cannot be zero at the same time. From the equations (2), the appearance of equation (1) under the $B_{n,m}$ operators will be as follows

$$B_{n,m}(f;x,y) = \left(\frac{n-1}{n}\right)Ax^2 + Bxy + \left(\frac{m-1}{m}\right)Cy^2 + \frac{Dn+A}{n}x + \frac{Em+C}{m}y + F = 0.$$

Hence

$$B'^2 - 4A'C' = B^2 - 4AC\left(\frac{n-1}{n}\right)\left(\frac{m-1}{m}\right)$$

for all $n, m \in \mathbb{N}_2$. If $AC > 0$, then since $0 < \left(\frac{n-1}{n}\right)\left(\frac{m-1}{m}\right) < 1$, we have

$$B'^2 - 4A'C' = B^2 - 4AC\left(\frac{n-1}{n}\right)\left(\frac{m-1}{m}\right) > B^2 - 4AC = 0$$

hence the equation $B_{n,m}(f;x,y) = 0$ specifies a hyperbola. If $AC = 0$, since $B = 0$, the equality $B'^2 - 4A'C' = 0$ is obtained, therefore the equation $B_{n,m}(f;x,y) = 0$ specifies a parabola.

**Remark 3.17**. Let the equation $f(x,y) = 0$ given by (1) specify a parabola and let $AC$ >0. In this case, the equation $B_{n,m}(f;x,y) = 0$ for every $n, m \in \mathbb{N}$ with $min\{n,m\} = 1$ and $n \neq m$ also indicates a hyperbola. Because under these conditions $B'^2 - 4A'C'$ will be $B^2 > 0$.

**Example 3.18.**
**(a)** If the parabola specified by the equation $f(x,y) = 3x^2 - 6xy + 3y^2 + 2x - 7 = 0$ is taken into account, it is clear that $AC = 3.3 = 9 > 0$ and the equations

$$B_{n,m}(f;x,y) = 3\left(\frac{n-1}{n}\right)x^2 - 6xy + 3\left(\frac{m-1}{m}\right)y^2 + \left(\frac{3}{n}+2\right)x + \frac{3}{m}y - 7 = 0$$

specify hyperbola for all $n, m \in \mathbb{N}_2$ (see Figure 11).

**(b)** If the parabola specified by the equation $f(x,y) = 3y^2 + 2x - 7 = 0$ is taken into account, it is clear that $AC = 0$ and the equations

$$B_{n,m}(f;x,y) = 3\left(\frac{m-1}{m}\right)y^2 + 2x + \frac{3}{m}y - 7 = 0$$

specify hyperbola for all $n, m \in \mathbb{N}_2$(see Figure 12).

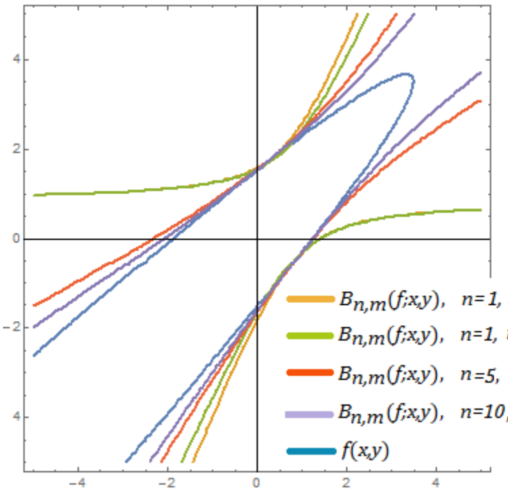


**Figure 11.** Images of the parabola given in Example 3.17(a) under the operators $B_{n,m}$ for $(n,m) = (1,5), (1,10), (5,10), (10,15)$.

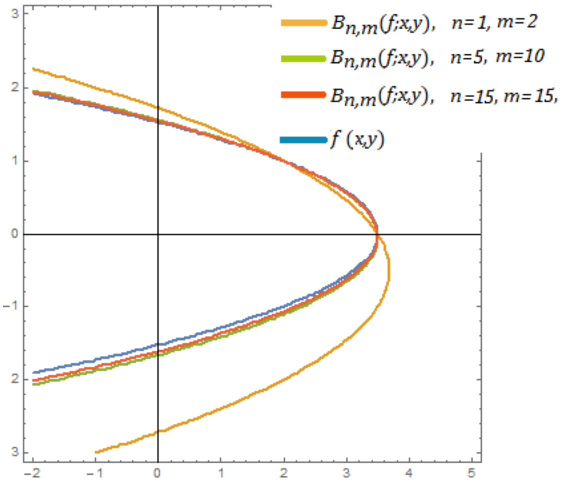


**Figure 12.** Images of the parabola given in Example 3.17(b) under the operators $B_{n,m}$ for $(n,m) = (1,2), (5,10), (15,15)$.

## 4. Conclusion and Suggestions

The equation $B_{n,m}(f;x,y) = 0$ is a quadratic two-variable equation, where $f(x,y) = 0$ is a conic equation and $B_{n,m}$ is a double-indexed two-variable Bernstein operator. If the equation $f(x,y) = 0$ indicates a circle, then the equations $B_{n,n}(f;x,y) = 0$ specify a circle for sufficiently large numbers $n$. However, in cases where the indices are different, that is, for $n \neq m$, $B_{n,m}(f;x,y) = 0$ equations indicate an ellipse or parabola. If the equation $f(x,y) = 0$ specifies an ellipse, the equations $B_{n,m}(f;x,y) = 0$ specifies an ellipse, hyperbola or parabola in certain cases. If the equation $f(x,y) = 0$ specifies a hyperbola, the equations $B_{n,m}(f;x,y)=0$ specifies hyperbola. Finally, if the equation $f(x,y)=0$ specifies a parabola, the equations $B_{n,m}(f;x,y)=0$ specifies a hyperbola or parabola, depending on whether the $AC$ product is zero or not.

**Conflict of Interest Statement**

There is no conflict of interest between the authors.

**Statement of Research and Publication Ethics**

The study is complied with research and publication ethics

**References**


[1] J. Pal, "Approksimation of konvekse Funktioner ved konvekse Polynomier," *Mat. Tidsskrift, B,* pp. 60–65, 1925.

[2] T. Popoviciu, "About the Best Polynomial Aprroximation of Continious Functions," *Mathematical Monography*, Sect. Mat. Univ. Cluj, III, 1937.

[3] L. Lupaş, "A property of the S. N. Bernstein Operator." *Mathematica (Cluj)*, vol. 9, no. 32, pp. 299-301, 1967.

[4] D. Leviatan, "Shape preserving approximation by polynomials and splines, in: Approximation Theory and Function Series," *Bolyai Soc. Math. Stud.* vol. 5, no. 1, pp. 63–84, 1996.

[5] D. Leviatan, "Shape preserving approximation by polynomials," *J. Comput. Appl. Math.* vol. 121, no. 1–2, pp. 73–94, 2000.

[6] L. M. Kocic and G. M. Milovanovic, "Shape preserving approximation by polynomials and splines," *Comput. Math. Appl.*, vol. 33, vo. 11, pp. 59–97, 1997.

[7] Y. K. Hu and X. M. Yu, "Copositive polynomial approximation revisited," *Applied Mathematics Reviews* (G.A. Anastassiou ed.), vol. I, pp. 157–174, 2000.

[8] S. G. Gal, *Shape Preserving Approximation By Real and Complex Polynomials*. 1th ed.; Springer Science and Business Media. 2008.

[9] T. H. Hildebrandt and I. J. Schoenberg, "On Linear Functional Operations and the Moment Problem for a Finite Interval in One or Several Dimensions." *Annals of Mathematics*, vol. 34, no. 2, pp. 317-328, 1933.

[10] E. H. Kingsley, "Bernstein polynomials for functions of two variables of class C(k)." *Proceedings of the American Mathematical Society*, vol. 12, no. 1, pp. 64-71, 1951.

[11] T. Popoviciu, "Sur Quelques Proprietes Des Fonctions D'une Ou De Deux Variables Reelles". *Mathematica (Cluj).* vol. 8, no. 1, pp. 1-85, 1934.

[12] M. Uzun, and T. Tunc, "B-Convexity and B-Concavity Preserving Property Of Two-Dimensional Bernstein Operators." 7th Int. IFS and Contemporary Mathematics Conf., May, 25-29, 2021, Türkiye, 1-7.

[13] S.N. Bernstein, "Demonstration Du Theoreme De Weierstrass Fondee Sur Le Calcul De Probabilities," *Comm. Soc. Math. Kharkov*, vol. 13, no.2, pp. 1-2, 1912.

[14] D. D. Stancu, "A method for obtaining polynomials of Bernstein type of two variables." *The American Mathematical Monthly*, vol. 70, no. 3, pp. 260-264, 1963.