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A New Exponentiated Extended Family of Distributions with Applications

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Abstract

In this article, we propose a new family of distributions called "A New Exponentiated Extended G family of distributions" in short EEX-G family, which generalize the extended-G of Corideiro et al. (2013). We gave the infinite mixture representation of the EEX-G cdf and pdf in terms of base line cdf and pdf. We study most of its mathematical properties. We give the explicit expression on ith order statistics as a linear combination of baseline cdf and pdf and model parameters are estimated by maximum likelihood method. The flexibility of the family is explained by four special models with their plots of density and hazard rate function. Finally the performance of the family is checked by fitting one of the special model on two real data sets.

1. INTRODUCTION

In the last few years, several ways of generating new probability distribution from classic ones were developed and discussed by using a function of baseline cdf. Some well-known generators are the Marshall-Olkin generated (MO-G) by Marshall and Olkin (1997), the beta-G by Eugene et al. (2002), Kumaraswamy-G (Kw-G for short) by Cordeiro and Castro (2011), the McDonald-G (Mc-G) by Alexander et al. (2012), the gamma-G by Zografos and Balakrishnan (2009), the transformer (T-X) by Alzaatreh et al. (2013), the extended-G by Cordeiro et al. (2013), the Weibull-G by Bourguignon et al. (2014), the exponentiated half-logistic by Cordeiro et al. (2014), the Kw-odd log-logistic-G by Alizadeh et al. (2015), type I half-logistic-G by Cordeiro et al. (2015), Garhy-G by Elgarhy et al. (2016), KwWeibull-G by Hassan and Elgarhy (2016a), exponentiated Weibull-G by Hassan and Elgarhy (2016b), type II half logistic–G by Hassan et al. (2017).

Ristic and Balakrishnan, (2011) proposed an alternative gamma generator for any continuous distribution $G(x)$ which is defined as

$$F(x)=1-\frac{1}{\Gamma(\delta)}\int_0^{-\log G(x,\zeta)} t^{\delta-1}e^{-t}dt, \tag{1}$$

where $\Gamma(\delta)=\int_0^\infty t^{\delta-1}e^{-t}dt$ is the gamma function.

This paper aims to introduce an Exponentiated version of the Extended-G (EX-G) distributions which is called as Exponentiated Extended-G (EEX-G) family of distributions. The paper is organized as. In section 2, we defined the new EEX-G family. In section 3, we give infinite mixture representation of the EEX-G cdf and pdf in terms of base line cdf and pdf. In section 4, we give some mathematical properties including *rth* moment, *rth* incomplete moment, moment generating function and mean deviations. In section 5, the

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density of *ith* order statistics is given and model parameters are estimated by ML method. In section 6, four special models are given along with their plots of density and hazard rate function. In section 7, application is carried out on two real data sets.

2. THE EEX-G FAMILY

In this section, (EEX-G) family is introduced. Kumaraswamy (1980) presented the Kw distribution with the following pdf

$$r(\mathrm{t}) = \alpha\beta t^{\alpha-1}\left(1-t^{\alpha}\right)^{\beta-1}, \qquad 0<t<1\,;\alpha,\beta>0. \tag{2}$$

Now, we obtain the new family by replacing the generator $r(\mathrm{t})$ defined in (1) by the pdf generator defined in (2) and replacing $W(G(x))$ in (1) with $\left[1-G(x)\right]^{b}$ as follows:

$$F(x)=\int_{\left(\bar{G}(x)\right)^{b}}^{1}\alpha\beta t^{\alpha-1}\left(1-t^{\alpha}\right)^{\beta-1}dt=1-\int_{0}^{\left(\bar{G}(x)\right)^{b}}\alpha\beta t^{\alpha-1}\left(1-t^{\alpha}\right)^{\beta-1}dt.$$

Then, we obtain the cumulative density function (cdf) of the EEX-G family as

$$F(x)=\left(1-\left(1-G(x)\right)^{b\,\alpha}\right)^{\beta}, X>0. \tag{3}$$

where $b,\alpha,\beta>0$ are three shape parameters. The cdf (3) provides a wider family of continuous distributions. The pdf corresponding to (3) is given by

$$f(x)=b\alpha\beta g(x)\left(1-G(x)\right)^{b\alpha-1}\left[1-\left(1-G(x)\right)^{b\alpha}\right]^{\beta-1},\ \ x>0\,;\mathrm{b},\alpha,\beta>0. \tag{4}$$

Hereafter, a random variable X with pdf (4) is denoted by *X~EEX-G*. For $b=1,$ the EEX-G family reduces to EX-G family of distributions which is obtained by Cordeiro et al. (2013).

The survival, hazard and cumulative hazard rate functions corresponding to (3) are, respectively, given by

$$\overline{F}\left(x\right)=1-\left[1-\left(1-G(x)\right)^{b\alpha}\right]^{\beta},\quad x>0\,;\mathrm{b},\alpha,\beta>0,$$

$$h(x)=\frac{b\alpha\beta g(x)\left(1-G(x)\right)^{b\alpha-1}\left[1-\left(1-G(x)\right)^{b\alpha}\right]^{\beta-1}}{1-\left[1-\left(1-G(x)\right)^{b\alpha}\right]^{\beta}},$$

and

$$H(x)=-\ln\left(1-F(x)\right)=-\ln\left(1-\left[1-\left(1-G(x)\right)^{b\alpha}\right]^{\beta}\right).$$

The quantile function of the EEX-G family is $Q(u)=F^{-1}(u)$ of X is derived as

$$u=\left[1-\left(1-G(Q(u))\right)^{b\alpha}\right]^{\beta}.$$

After some algebra, the quantile function will be

$$Q(u) = G^{-1}\left(1-\left(1-u^{\frac{1}{\beta}}\right)^{\frac{1}{b\alpha}}\right), \tag{5}$$

where u follows uniform distribution on the interval $(0,1)$ and $G^{-1}(.)$ is the inverse function of $G(.)$. In particular, *Q(0.5)* is the median of the family and defined by

$$Median = G^{-1}\left(1-\left(1-(0.5)^{\frac{1}{\beta}}\right)^{\frac{1}{b\alpha}}\right).$$

3. MATHEMATICAL PROPERTIES

In this section some mathematical properties of the EEX-G family are given,

a. Infinite Mixture Representation

In this subsection, useful expansions for the pdf and cdf of the EEX-G family of distributions is provided. First, we obtain an expansion for the cdf defined in (3). The generalized binomial expansion is given by

$$(1-z)^{\beta-1} = \sum_{i=0}^{\infty}(-1)^i \binom{\beta-1}{i} z^i, \quad \text{for } |z|<1,$$

By applying the generalize binomial theorem to (4), the expression $(1-G(x))^{b\,\alpha}$ becomes

$$\left(1-G(x)\right)^{b\,\alpha} = \sum_{j=0}^{\infty}\binom{b\,\alpha}{j}(-1)^i\,G^j(x) \tag{6}$$

Rewriting

$$\sum_{j=0}^{\infty}\binom{b\,\alpha}{j}(-1)^i\,G^j(x) = \sum_{j=0}^{\infty} a_j\,G^j(x)$$

Where $a_j = \binom{b\,\alpha}{j}(-1)^i$

Now (6) becomes

$$(1-G(x))^{b\,\alpha} = \sum_{j=0}^{\infty} a_j\,G^j(x) \tag{7}$$

Substituting (7) in (3), we have

$$F(x) = \left(1-\sum_{j=0}^{\infty} a_j\,G^j(x)\right)^{\beta}$$

Rewriting

$$F(x) = \left(\sum_{j=0}^{\infty} b_j\,G^j(x)\right)^{\beta}$$

Where $b_0 = 1-a_0$ and for $j \geq 1$, $b_j = a_j$. Using power series expansion (Gradshteyn and Ryzhik, 2000) page 17, we have

$$F(x) = \sum_{j=0}^{\infty} c_{j:\beta}\,G^j(x)$$

where $c_{0:\beta} = b_0^{\beta}$, $c_m = \frac{1}{m\, b_0}\sum_{k=1}^{m}(k\beta - m + k)b_k c_{m-k}$ for $m \geq 1$

Rewriting the above expression, we have

$$F(x) = \sum_{j=0}^{\infty} c_{j:\beta}\, G^{j}(x) = \sum_{j=0}^{\infty} c_{j:\beta}\, H_j(x) \tag{8}$$

Similarly, the density in Eq. (4) can be expressed as

$$f(x) = \sum_{j=0}^{\infty} c_{j:\beta}\,(j-1)g(x)G^{j-1}(x) = \sum_{j=0}^{\infty} c_{j:\beta}\, h_{j-1(x)} \tag{9}$$

Where $H_j(x)$ and $h_{j-1(x)}$ are the exp-G cdf and pdf of the baseline distributions.

Now, we obtain an expansion for $[F(x)]^h$ where h is an integer and β is a real non-integer. Again, the binomial expansion is worked out for $[F(x)]^h$ as follows:

$$F^h(x) = \left(1 - \left(1 - G(x)\right)^{b\,\alpha}\right)^{\beta h}$$

Then, the binomial expansion is applied to $\left(1 - G(x)\right)^{b\alpha k}$ and we have

$$F^h(x) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\binom{\beta h}{j}\binom{b\alpha j}{i}(-1)^{i+j}G^{i}(x).$$

Finally, we can write

$$F^h(x) = \sum_{i=0}^{\infty} w_i G^{i}(x) \ , \tag{10}$$

where $w_i = \sum_{j=0}^{\infty}\binom{\beta h}{j}\binom{b\alpha j}{i}(-1)^{i+j}$.

b. Moments

The *rth* moment of the EEX-G family can be obtained by using following expression

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x)dx.$$

Using the infinite mixture representation of the pdf in (x) we have

$$\mu_r = \sum_{j=0}^{\infty} c_{j:\beta}\, \int_{-\infty}^{\infty} x^r h_{j-1(x)}dx.$$

$$\mu_r = \sum_{j=0}^{\infty} c_{j:\beta}\ \gamma_r. \tag{11}$$

Where $\gamma_r = \int_{-\infty}^{\infty} x^r h_{j-1(x)}dx.$

The *rth* moments can be obtained by using quantile function

$$\mu_r = \sum_{j=0}^{\infty} c_{j:\beta}\, \int_0^1 u^{j-1}(Q_G(u))^r du$$

c. Moment Generating Function

The moment generating function of EE family can be obtained as

$$M_0(t) = \sum_{j=0}^{\infty} c_{j:\beta} \int_{-\infty}^{\infty} e^{tx} h_{j-1(x)} dx$$

$$M_0(t) = \sum_{j=0}^{\infty} c_{j:\beta} \; M_x(t) \tag{12}$$

where $M_x(t) = \int_{-\infty}^{\infty} e^{tx} h_{j-1(x)} dx$.

The moment generating function of EEX-G family can be obtained by using quantile function.

$$M_0(t) = \sum_{j=0}^{\infty} c_{j:\beta} \int_0^1 e^{t(Q_G(u))} u^{j-1} \, du$$

d. Incomplete Moments

The *rth* incomplete moment of the EEX-G family can be obtained by using following expression

$$T_r(x) = \int_{-\infty}^{x} x^r f(x) dx$$

Using the infinite mixture representation of the pdf in (x) we have

$$T_r(x) = \sum_{j=0}^{\infty} c_{j:\beta} \int_{-\infty}^{x} x^r h_{j-1(x)} dx.$$

$$T_r(x) = \sum_{j=0}^{\infty} c_{j:\beta} \; \varphi_r. \tag{13}$$

Where $\varphi_r = \int_{-\infty}^{x} x^r h_{j-1(x)} dx$.

The rth incomplete can be obtained by using quantile function

$$T_r(x) = \sum_{j=0}^{\infty} c_{j:\beta} \int_0^q u^{j-1} (Q_G(u))^r du$$

e. The Mean Deviation

The mean deviations of the EEX-G family can be obtained by following expressions

$$\delta_1(X) = 2\mu F(\mu) - 2T(\mu) \text{ and } \delta_2(X) = \mu - 2T(\mathrm{M}),$$

Where $\mu = E(x)$ is mean can easily be obtained from Eq. (x), $F(\mu)$ can be obtained from Eq.(3) and $M = median$ given in Eq. (x) and $T(x)$ is the first incomplete moment given in Eq.(x).

f. Order Statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let $X_{(1)} < X_{(2)} < ... < X_{(n)}$ be an ordered random sample from a population of size *n* following the EEX-G family. The density of *ith* order statistic is

$$f(x_{i:n}) = \frac{n!}{(i-1)!\,(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j f(x) F^{j+i-1}(x)$$

Using pdf and cdf of EEX-G family in (.) and (.), we have

$$f(x_{i:n})=\frac{n!}{(i-1)!\,(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^{j}\left[\alpha\beta b\; g(x)\big(1-G(x)\big)^{b\,\alpha-1}\left(1-\big(1-G(x)\big)^{b\,\alpha}\right)^{\beta-1}\right]$$
$$\times\left[\left(1-\big(1-G(x)\big)^{b\,\alpha}\right)^{\beta}\right]^{j+i-1}$$

Using generalized binomial expansion and after some algebra, we have

$$f(x_{i:n})=\sum_{l=0}^{\infty}V_l h_l(x)$$

Where $V_l=\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{j}\binom{\beta(i+j)-1}{m}\binom{b\alpha(m+1)-1}{l}\frac{(-1)^{j+m+l}}{l+1}$

and $h_j(x)=(j+1)g(x)G^{l}(x)$.

Moments of the *ith* order statistics can be defined as

$$E(x_{i:n})=\int_{-\infty}^{\infty}{x_{i:n}}^{r}f(x_{i:n})\,dx_{i:n} \tag{14}$$

Substituting (14) in (13), we have

$$E(x_{i:n})=\sum_{l=0}^{\infty}V_l\int_{-\infty}^{\infty}{x_{i:n}}^{r}h_l(x_{i:n})\,dx_{i:n}$$

g. The Probability Weighted Moments

The probability weighted moment of EEX-G family can be obtained as

$$\tau_{r,s}=E[X^{r}\,\mathrm{F}(x)^{s}]=\int_{-\infty}^{\infty}x^{r}\,\mathrm{f}(x)(\mathrm{F}(x))^{s}\,dx.$$

Substituting (8) and (9) into (10), replacing h with s, we have

$$\tau_{r,s}=\int_{-\infty}^{\infty}X^{r}\sum_{i=0}^{\infty}w_iG^{i}(x)\sum_{j=0}^{\infty}c_{j:\beta}\,h_{j-1(x)}\,dx$$

$$\tau_{r,s}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}j\,w_ic_{j:\beta}\int_{-\infty}^{\infty}X^{r}g(x)G^{i+j-1}(x)dx \tag{15}$$

Then, we have

$$\tau_{r,s}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}j\,w_ic_{j:\beta}\;\gamma_{r,i+j}$$

Where $\gamma_{r,i+j}=\int_{-\infty}^{\infty}X^{r}g(x)G^{i+j-1}(x)dx$

Additionally, different form is yielded by using quantile function as follows:

$$\tau_{r,s}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}j\,w_ic_{j:\beta}\int_{0}^{1}u^{i+j}(Q_G(u))^{r}du$$

4. SPECIAL DISTRIBUTIONS

In this section, we will give some special models of EEX-G family.

a. The EEX-Uniform (EEX-U) Distribution

If Uniform distribution is the base line distribution having pdf and cdf $g(x,\theta)=\dfrac{1}{\theta}$, $0<x<\theta$ and $G(x,\theta)=\dfrac{x}{\theta}$, then the cdf and pdf of EEX-U distribution are, respectively, given below

$$F(x)=\left[1-\left(1-\frac{x}{\theta}\right)^{b\alpha}\right]^{\beta},$$

and

$$f\left(x\right)=\frac{b\alpha\beta}{\theta}\left(1-\frac{x}{\theta}\right)^{b\alpha-1}\left[1-\left(1-\frac{x}{\theta}\right)^{b\alpha}\right]^{\beta-1},\ 0<x<\theta.$$

In Figure 1 (a) and Figure 2 (a) the plots of density and hazard rate function are given. The density of EEX-U is U shape, J-shape and symmetrical and hrf is increasing and bathtub.

b. The EEX –Burr XII (EEX-BXII) Distribution

If Burr XII distribution is the base line distribution having pdf and cdf $\mathrm{g}(x,c,\mu,\sigma)=c\sigma\mu^{-c}x^{c-1}[1+(\frac{x}{\mu})^{c}]^{-\sigma-1},\ \ c,\mu,\sigma>0,$ and $G(x,c,\mu,\sigma)=1-[1+(\frac{x}{\mu})^{c}]^{-\sigma}.$, then the cdf and pdf of EEX-B distribution are, respectively, given below

$$F(x)=\left[1-\left(\left(1+\left(\frac{x}{\mu}\right)^{c}\right)^{-\sigma}\right)^{b\alpha}\right]^{\beta},\ \ x>0,$$

and

$$f(x)=b\alpha\beta c\sigma\mu^{-c}x^{c-1}\left[1+\left(\frac{x}{\mu}\right)^{c}\right]^{-\sigma-1}\left(\left(1+\left(\frac{x}{\mu}\right)^{c}\right)^{-\sigma}\right)^{b\alpha-1}\left[1-\left(\left(1+\left(\frac{x}{\mu}\right)^{c}\right)^{-\sigma}\right)^{b\alpha}\right]^{\beta-1}.$$

In Figure 1 (b) and Figure 2 (b) the plots of density and hazard rate function are given. The density of EEX-B is reversed J, symmetrical and right skewed and hrf is decreasing and upside-down bathtub.

c. The EEX-Weibull (EEX-W) Distribution

If Burr XII distribution is the base line distribution having pdf and cdf $g(x,\lambda,\gamma)=\lambda\gamma x^{\gamma-1}e^{-\lambda x^{\gamma}}$ and $(x,\lambda,\gamma)=1-e^{-\lambda x^{\gamma}}$, then the cdf and pdf of EEX-W distribution are, respectively, given below

$$F(x)=\left[1-e^{-\lambda b\alpha x^{\gamma}}\right]^{\beta},\quad x>0,$$

and

$$f(x)=b\alpha\beta\lambda\gamma x^{\gamma-1}e^{-\lambda b\alpha x^{\gamma}}\left[1-e^{-\lambda b\alpha x^{\gamma}}\right]^{\beta-1},\ \mathrm{b},\alpha,\beta,\lambda,\gamma>0$$

In Figure 1 (c) and Figure 2 (c) the plots of density and hazard rate function are given. The density of EEX-

W is reversed J, symmetrical and right skewed and hrf increasing and decreasing.

d. The EEX-Quasi Lindley (EEX-QL) Distribution

If Burr XII distribution is the base line distribution having pdf and cdf $g(x,\theta)=\frac{\theta}{p+1}(p+\theta x)e^{-\theta x}$ and $G(x,\theta)=1-e^{-\theta x}[1+\frac{\theta x}{p+1}]$, , then the cdf and pdf of EEX-QL distribution are, respectively, given below

$$F(x)=\left[1-\left(e^{-\theta x}\left[1+\frac{\theta x}{p+1}\right]\right)^{b\alpha}\right]^{\beta},\ x>0,$$

and

$$f(x)=\frac{b\alpha\beta\theta}{p+1}(p+\theta x)e^{-\theta b\alpha x}\left(1+\frac{\theta x}{p+1}\right)^{b\alpha-1}\left[1-\left(e^{-\theta x}\left[1+\frac{\theta x}{p+1}\right]\right)^{b\alpha}\right]^{\beta-1}.$$

In Figure 1 (d) and Figure 2 (d) the plots of density and hazard rate function are given. The density of EEX-QL is reversed J and right skewed and hrf is increasing and decreasing.

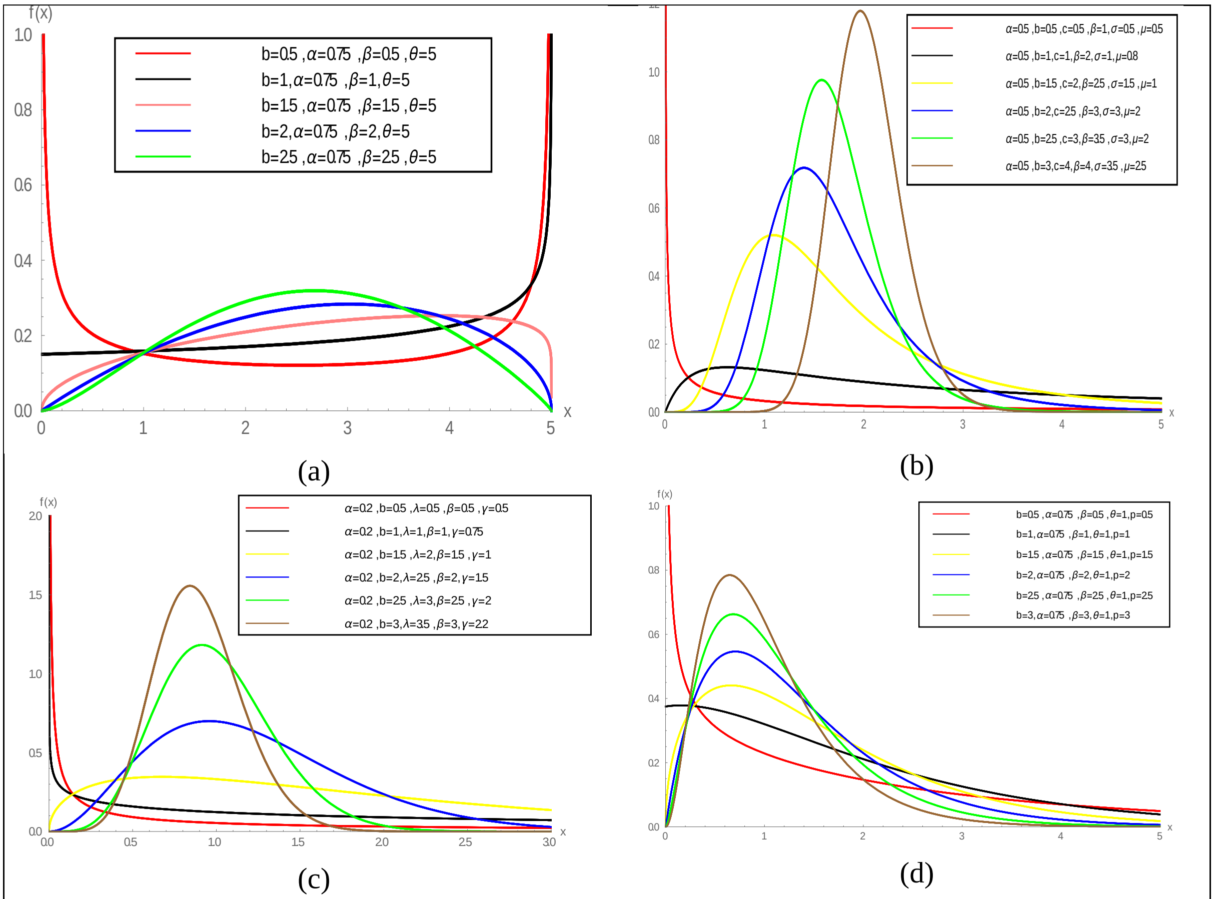

Figure 1. *Plots of (a) EEX-U (b) EEX-BXII (c) EEX-W (d) EEX-QL densities.*

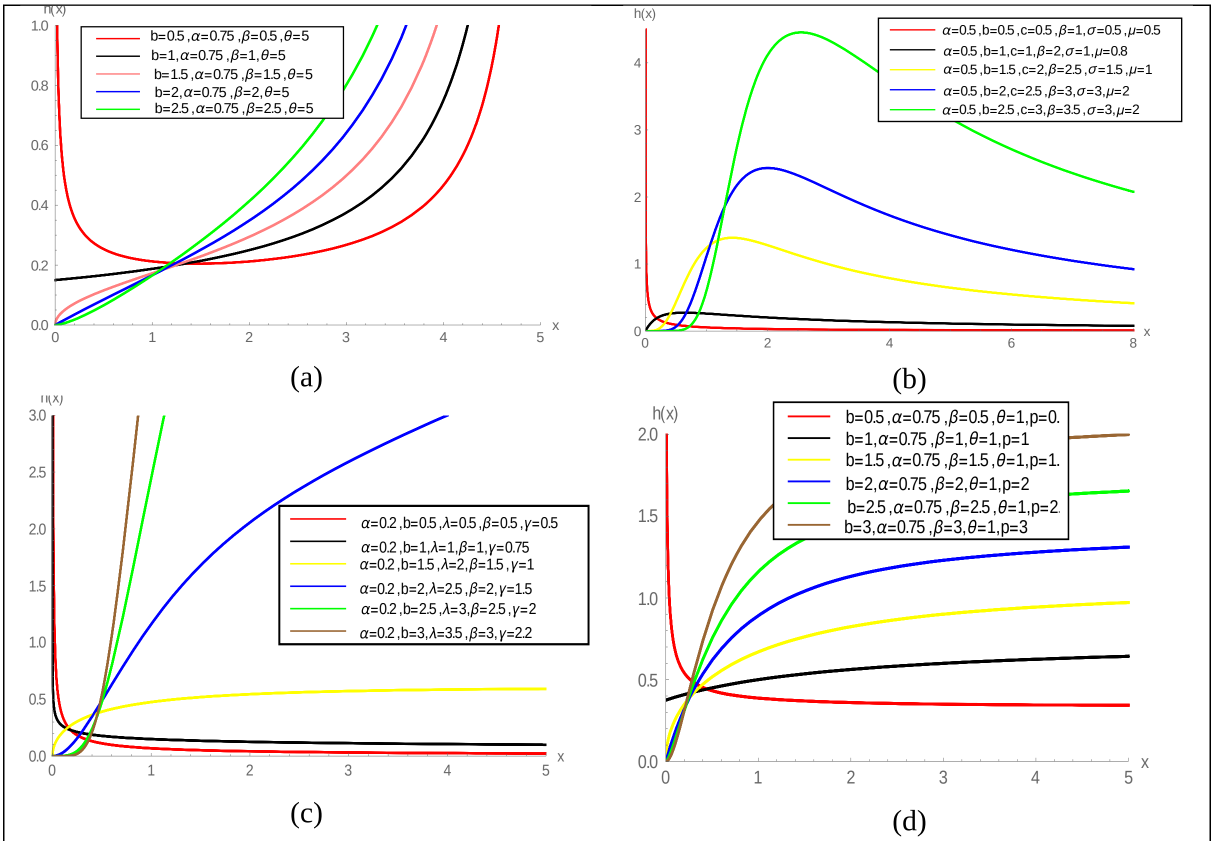

Figure 2. *Plots of (a) EEX-U (b) EEX-BXII (c) EEX-W (d) EEX-QL hazard rate.*

From Figures 1 and 2 we conclude that the density shapes of EEX-G model are left-skewed, right-skewed, decreasing, symmetric and J-shaped and the hazard rate shape are increasing, decreasing, left-skewed, right-skewed, symmetric, and bathtub shaped. This attractive flexibility makes the hrf of the EEX-G useful and suitable for non-monotone empirical hazard behaviors which are more likely to be encountered or observed in real life situations.

5. MAXIMUM LIKELIHOOD METHOD

In this section, the method of maximum likelihood method is used to estimate the EEX-G parameters $\Theta = (b, \alpha, \beta, \xi\ )^T$. The maximum likelihood estimates (MLEs) enjoy desirable properties that can be used when constructing confidence intervals and regions and deliver simple approximations that work well in finite samples. The resulting approximation for the MLEs in distribution theory is easily handled either analytically or numerically. If $x_1, , \ldots, x_n$be a sample of size n from the EEX-G family, then the log-likelihood function for the vector of parameters is

$$\ln L(\Theta) = n\ln b + n\ln\alpha + n\ln\beta + \sum_{i=1}^{n}\ln g(x_i,\zeta) + (b\alpha-1)\sum_{i=1}^{n}\ln\left[1-G(x_i,\zeta)\right] + (\beta-1)\sum_{i=1}^{n}\ln\left[1-\left[1-G(x_i,\zeta)\right]^{b\alpha}\right],$$

The components of score vector $U(\Theta) = (U_b, U_\alpha, U_\beta, U_\zeta)$ are

$$U_b = \frac{n}{b} + \alpha\sum_{i=1}^{n}\ln\left[1-G(x_i,\zeta)\right] - \alpha(\beta-1)\sum_{i=1}^{n}\frac{\left[1-G(x_i,\zeta)\right]^{b\alpha}\ln\left[1-G(x_i,\zeta)\right]}{\left[1-\left[1-G(x_i,\zeta)\right]^{b\alpha}\right]},$$

$$U_{\alpha}=\frac{n}{\alpha}+b\sum_{i=1}^{n}\ln\left[1-G(x_i,\zeta)\right]-b\left(\beta-1\right)\sum_{i=1}^{n}\frac{\left[1-G(x_i,\zeta)\right]^{b\alpha}\ln\left[1-G(x_i,\zeta)\right]}{\left[1-\left[1-G(x_i,\zeta)\right]^{b\alpha}\right]}$$

$$U_{\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\ln\left[1-\left[1-G(x_i,\zeta)\right]^{b\alpha}\right],$$

$$U_{\xi_k}=\frac{n}{\alpha}+\sum_{i=1}^{n}\frac{g_k'(x_i,\zeta)}{g(x_i,\zeta)}-\left(b\alpha-1\right)\sum_{i=1}^{n}\frac{G_k'(x_i,\zeta)}{1-G(x_i,\zeta)}+b\alpha\left(\beta-1\right)\sum_{i=1}^{n}\frac{\left[1-G(x_i,\zeta)\right]^{b\alpha-1}G_k'(x_i,\zeta)}{\left[1-\left[1-G(x_i,\zeta)\right]^{b\alpha}\right]}.$$

Setting U_b, U_α, U_β and U_ζ equal to zero and solving the equations simultaneously yields the maximum likelihood estimates. Note that these equations cannot be solved analytically and statistical software can be used to solve those numerically using iterative methods.

6. APPLICATION

In this section, we provide two applications to real data to illustrate the flexibility of the EEX-G family and EEX-W distribution. The first data set consists of a random sample of 128 bladder cancer patients. The data set reported in Hashmi and Memon (2016).

The second data set represents by Silva et al. (2010) the maximum annual flood discharges in units of 1000 cubic feet per second, of the North Saskachevan River at Edmonton, over a period of 48 years.

The MLEs are computed using Quasi-Newton Code for Bound Constrained Optimization (L-BFGS-B) and the log-likelihood function evaluated. The goodness-of-fit measures; Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Anderson-Darling (A*), Cramér–von Mises (W*) are computed. The lower the values of these criteria, the better the fit. We compare the EEX-W distribution with those of the Weibull (W), Gamma (G), beta exponential (BE), transmuted exponentiated Weibull (TEW), exponentiated transmuted generalized Rayleigh (ETGR) (Afify et al., 2015), gamma Weibull (GW), exponentiated Weibull (EW) and transmuted additive Weibull (TAW). The MLEs and some statistics of the models for the first data set are presented in Tables 1 and 2, respectively.

Table 1. *The MLEs for the first data set.*

Distribution	Estimates				
	$\hat{\alpha}$	$\hat{b}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\lambda}$
EEX-W	0.6304	1.28136	0.65441	2.79601	0.561649
TEW	0.7449	-	1.13333	0.0478274	0.000011
BE	1.4485	0.179191	-	-	0.645544
ETGR	7.3762	-	0.0494	0.0473	0.118
W	-	-	1.04783	-	10.651
G	-	-	1.17251	-	7.98766

Table 2. *Some statistics for the models fitted to the first data set.*

Model	Goodness of fit criteria				
	AIC	BIC	L	A	W
EEX-W	831.36	845.620	-410.680	0.27198	0.04050
TEW	832.92	849.925	-411.958	0.56339	0.08826
BE	832.69	847.244	-412.344	0.56139	0.09984
ETGR	866.35	877.758	-429.175	2.36077	0:39794
W	832.17	846.878	-414.087	0.96345	0.15430
G	832.74	846.44	-413.368	0.77625	0.13606

Tables 3 and 4 present the MLEs and some statistics of the models for the second data set.

Table 3. *The MLEs for the second data set.*

Distribution	Estimates				
	$\hat{\alpha}$	$\hat{b}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\lambda}$
EEX-W	1.76775	12.2004	0.224467	35778.6	0.217492
TAW	0.000236	1.94727	5.86338*10^-13	0.535489	0.703578
EW	5.86338*10^-13	0.00074	-	-	1.77242
W	-	-	1.77242	-	1350.96
GW	4.11135	-	0.721162	-	3.52303

Table 4. *Some statistics for the models fitted to the second data set.*

Model	Goodness of fit criteria				
	AIC	BIC	L	A	W
EEX-W	440.568	449.924	-215.284	0.18857	0.025970
TAW	458.199	467.555	-224.100	1.46318	0.213104
EW	457.413	463.027	-225.706	1.80212	0.279709
W	455.413	459.155	-225.706	1.80212	0.279709
GW	446.724	452.338	-220.362	0.94575	0.141279

The values in Tables 2 and 4 indicate that the EEX-W model has the lowest values for AIC, BIC, A and W among all fitted models (for the two real data sets). So, the EEX-W models could be chosen as the best models. Then, the estimated pdfs and cdfs are displayed in Figures 3 and 4. It is clear from Figures 3 and 4 that the new EEX-W distribution provides the best fits to both data sets.

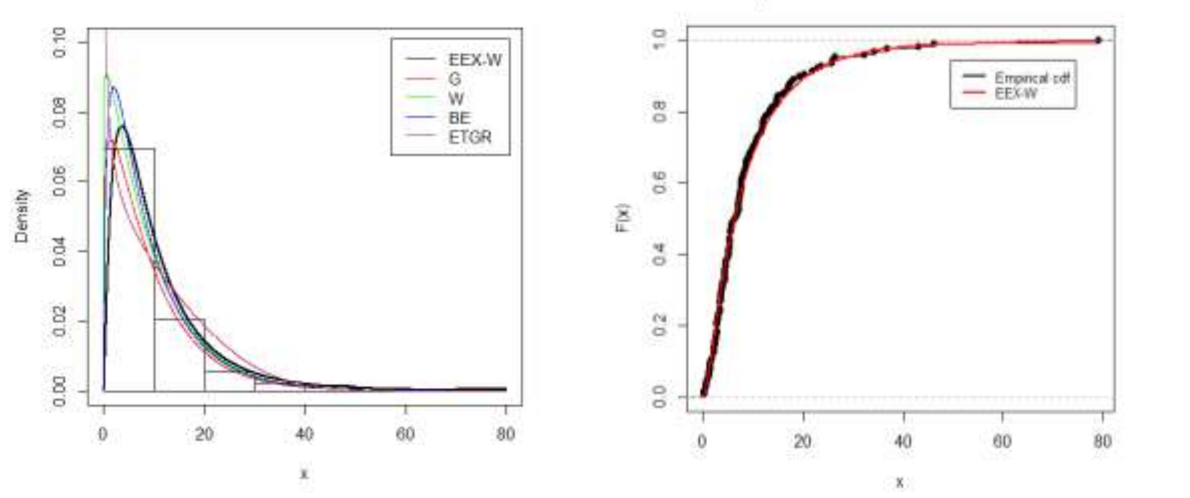

Figure 3. *Plots of the estimated pdfs and cdf for the EEX-W and their sub-models for the data set-1*

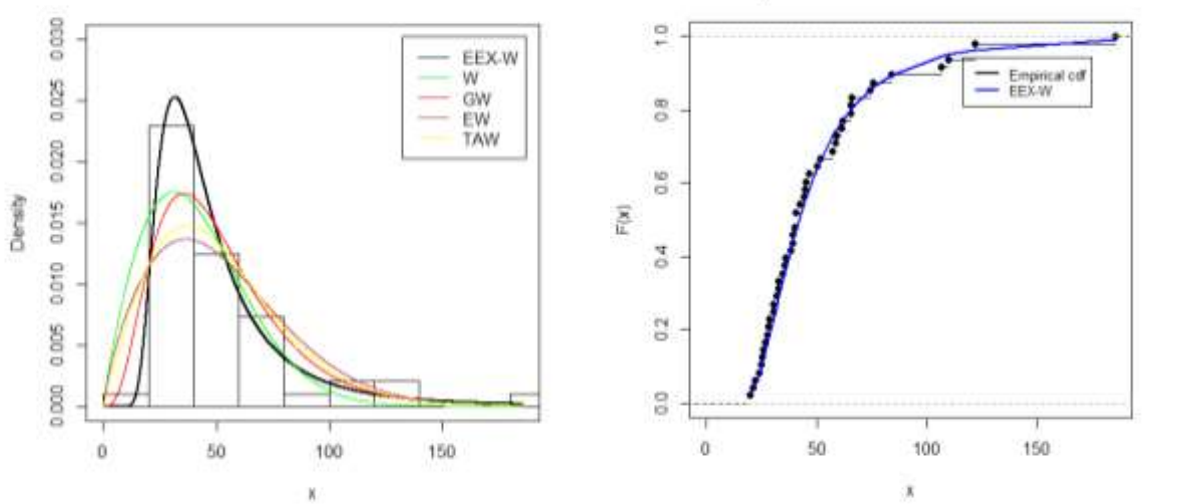

Figure 4. *Plots of the estimated (a) pdfs and (b) cdfs for the EEX-W and their sub-models for the data set-2.*

7. CONCLUSIONS

In this paper, we discuss a special family of absolutely continuous distribution namely the exponentiated extended-G (EEX-G) family of distributions. This family of distributions have been obtained by adding one extra shape parameter to extend the extended-G class. Properties of the EEX-G were discussed, such as, expressions for the density function, moments, mean deviation, quantile function and order statistics. The maximum likelihood method is employed for estimating the model parameters. Four special models namely, exponentiated extended Uniform, exponentiated extended Burr XII and exponentiated extended quasi Lindley, are provided. Further, the derived properties of the generated family are valid to these selected models. We also provide two real life applications for a specific member of the EEX-G family. Results of the applications nicely exhibit the fact that the EEX-G family performs better in some situations in comparison to its parent model. We wish a broadly statistical application in some area for this new generalization.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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APPENDIX

The elements of the observed Fisher information matrix $I(\Theta)$, are given by

$$I_{a\beta}=b\alpha\sum_{i=1}^{n}\frac{\left(\ln G(x_i,\zeta)\right)\left[1-G(x_i,\zeta)\right]^{b\alpha-1}G(x_i,\zeta)}{\left[1-\left[1-G(x_i,\zeta)\right]^{b\alpha}\right]},$$

$$I_{b\beta}=-\alpha\sum_{i=1}^{n}\frac{\left[1-G(x_i,\zeta)\right]^{b\alpha}\ln\left(1-G(x_i,\zeta)\right)}{\left[1-\left[1-G(x_i,\zeta)\right]^{b\alpha}\right]},$$

$$I_{\alpha\beta}=-b\sum_{i=1}^{n}\frac{\left[1-G(x_i,\zeta)\right]^{b\alpha}\ln\left(1-G(x_i,\zeta)\right)}{\left[1-\left[1-G(x_i,\zeta)\right]^{b\alpha}\right]},$$

$$U_{\beta\beta}=\frac{-n}{\beta^2},$$

$$I_{\alpha\alpha}=-\frac{n}{\alpha^2}-b^2\left(\beta-1\right)\sum_{i=1}^{n}\left\{\frac{\left(\ln\left(1-G(x_i,\zeta)\right)\right)^2\left[1-G(x_i,\zeta)\right]^{b\alpha}}{\left[1-\left[1-G(x_i,\zeta)\right]^{b\alpha}\right]}+\left[\frac{\ln\left(1-G(x_i,\zeta)\right)\left[1-G(x_i,\zeta)\right]^{b\alpha}}{\left[1-\left[1-G(x_i,\zeta)\right]^{b\alpha}\right]}\right]^2\right\},$$

$$I_{bb}=-\frac{n}{b^2}-\alpha^2\left(\beta-1\right)\sum_{i=1}^{n}\left\{\frac{\left(\ln\left(1-G(x_i,\zeta)\right)\right)^2\left[1-G(x_i,\zeta)\right]^{b\alpha}}{\left[1-\left[1-\mathrm{H}(x_i,\zeta)\right]^{b\alpha}\right]}+\left[\frac{\ln\left(1-G(x_i,\zeta)\right)\left[1-G(x_i,\zeta)\right]^{b\alpha}}{\left[1-\left[1-G(x_i,\zeta)\right]^{b\alpha}\right]}\right]^2\right\},$$

$$I_{b\alpha}=\sum_{i=1}^{n}\ln\left(1-G(x_i,\zeta)^a\right)-\left(\beta-1\right)\sum_{i=1}^{n}\left\{\frac{\ln\left(1-G(x_i,\zeta)\right)\left[1-G(x_i,\zeta)\right]^{b\alpha}}{\left[1-\left[1-\mathrm{H}(x_i,\zeta)\right]^{b\alpha}\right]}\right\}$$
$$-\alpha b\left(\beta-1\right)\sum_{i=1}^{n}\left\{\frac{\left(\ln\left(1-G(x_i,\zeta)\right)\right)^2\left[1-G(x_i,\zeta)\right]^{b\alpha}}{\left[1-\left[1-G(x_i,\zeta)\right]^{b\alpha}\right]}+\left[\frac{\ln\left(1-G(x_i,\zeta)\right)\left[1-G(x_i,\zeta)\right]^{b\alpha}}{\left[1-\left[1-G(x_i,\zeta)\right]^{b\alpha}\right]}\right]^2\right\},$$

$$I_{\alpha\xi_k}=-b\sum_{i=1}^{n}\frac{G'_k(x_i,\zeta)}{1-G(x_i,\zeta)}+b\left(\beta-1\right)\sum_{i=1}^{n}\frac{\left[1-G(x_i,\zeta)\right]^{b\alpha-1}G'_k(x_i,\zeta)}{1-\left[1-\mathrm{H}(x_i,\zeta)\right]^{b\alpha}}$$
$$+b^2\alpha\left(\beta-1\right)\sum_{i=1}^{n}\left\{\frac{\left[1-G(x_i,\zeta)\right]^{b\alpha-1}\ln\left(1-G(x_i,\zeta)\right)G'_k(x_i,\zeta)}{\left[1-\left[1-G(x_i,\zeta)\right]^{b\alpha}\right]}+\frac{\left[1-G(x_i,\zeta)\right]^{2b\alpha-1}\ln\left(1-G(x_i,\zeta)\right)G'_k(x_i,\zeta)}{\left[1-\left[1-G(x_i,\zeta)\right]^{b\alpha}\right]^2}\right\},$$

$$I_{\beta\xi_k} = b\alpha \sum_{i=1}^{n} \frac{\left[1-G(x_i,\zeta)\right]^{b\alpha-1} G'_k(x_i,\zeta)}{1-\left[1-G(x_i,\zeta)\right]^{b\alpha}},$$

and

$$\begin{aligned}
U_{\xi_k\xi_l} &= \sum_{i=1}^{n} \frac{g''_{k,l}(x_i,\zeta)g(x_i,\zeta) - g'_k(x_i,\zeta)g'_l(x_i,\zeta)}{g(x_i,\zeta)^2} \\
&\quad -\left(b\alpha-1\right)\sum_{i=1}^{n}\left\{\frac{G''_{k,l}(x_i,\zeta)}{1-G(x_i,\zeta)} + \frac{G'_k(x_i,\zeta)G'_l(x_i,\zeta)}{\left[1-G(x_i,\zeta)\right]^2}\right\} \\
&\quad + b\alpha\left(\beta-1\right)\sum_{i=1}^{n}\left\{\frac{-\left(b\alpha-1\right)\left[1-G(x_i,\zeta)\right]^{b\alpha-2} G'_k(x_i,\zeta)G'_l(x_i,\zeta)}{\left[1-\left[1-G(x_i,\zeta)\right]^{b\alpha}\right]}\right. \\
&\qquad + \frac{\left[1-G(x_i,\zeta)\right]^{b\alpha-1} G''_{k,l}(x_i,\zeta)}{\left[1-\left[1-G(x_i,\zeta)\right]^{b\alpha}\right]} \\
&\qquad \left. - \frac{b\alpha\left[1-G(x_i,\zeta)\right]^{2b\alpha-2} G'_k(x_i,\zeta)G'_l(x_i,\zeta)}{\left[1-\left[1-G(x_i,\zeta)\right]^{b\alpha}\right]^2}\right\}.
\end{aligned}$$

where

$$G'_k(x_i,\zeta) = \frac{\partial}{\partial\zeta_k}G(x_i,\zeta), \quad G'_l(x_i,\zeta) = \frac{\partial}{\partial\zeta_l}G(x_i,\zeta), \quad G''_{k,l}(x_i,\zeta) = \frac{\partial^2}{\partial\zeta_k\partial\zeta_l}G(x_i,\zeta),$$

$$g''_{k,l}(x_i,\zeta) = \frac{\partial^2}{\partial\zeta_k\partial\zeta_l}G(x_i,\zeta), \quad g'_k(x_i,\zeta) = \frac{\partial}{\partial\zeta_k}g(x_i,\zeta), \quad g'_l(x_i,\zeta) = \frac{\partial}{\partial\zeta_l}g(x_i,\zeta).$$

Data set-1: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34,5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Data set-2: 19.885, 20.940, 21.820, 23.700, 24.888, 25.460, 25.760, 26.720, 27.500, 28.100, 28.600, 30.200, 30.380, 31.500, 32.600, 32.680, 34.400, 35.347, 35.700, 38.100, 39.020, 39.200, 40.000, 40.400, 40.400, 42.250, 44.020, 44.730, 44.900, 46.300, 50.330, 51.442, 57.220, 58.700, 58.800, 61.200, 61.740, 65.440, 65.597, 66.000, 74.100, 75.800, 84.100, 106.600, 109.700, 121.970, 121.970, 185.560.