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Comparison of Test Statistics Based on Different Scale Estimators for the Umbrella Alternative of Scale Parameters

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Abstract

To test null hypothesis against umbrella alternative for scale parameters, a test statistic W_h based on a scale estimator is proposed in [1]. In this study, W_h statistics based on different estimators of scale parameters are compared according to type I error , α , and power. For the nominal $\alpha = 0.05$ and given values of peak of umbrella, number of populations and sample sizes, the results are obtained using the data is generated from two-parameter exponential and normal distributions in simulation design. According to the simulation results, when the data generated from normal distribution, the power values of test statistics based on robust scale estimators are almost the same to the ones based on maximum likelihood estimator of scale parameters as the sample sizes increase. Moreover, the similar results are obtained when the data is generated from the two-parameter exponential distribution.

1. INTRODUCTION

Let $\pi_1, \pi_2, ..., \pi_k$, $i = 1, 2, ..., k$, be independent populations and the cumulative distribution function (cdf) of an observation from population π_i is $F_i(x) = F\left[\left(x - \mu_i\right)/\theta_i\right]$ where θ_i is the scale parameter, μ_i is the location parameter and $F(.)$ is any absolutely continuous cdf. The umbrella ordering $\theta_1 \leq ... \leq \theta_h \geq ... \geq \theta_k$ with at least one strict inequality is assumed to be satisfied for given $1 \leq h \leq k$ where h is the peak of umbrella. Umbrella ordering is important in close-response experiments. For details, the reader may refer to [2-4]. Testing the null hypothesis against the umbrella ordering with at least one strict inequality is studied by many researchers among them are [5-7]. Rank test statistics were proposed for umbrella alternative in [8,9]. In k-sample problems, the test statistic for pattern alternatives that can be transformed to umbrella alternatives is developed in [10]. For testing of equality of location parameters against the umbrella ordering with at least one strict inequality, a likelihood ratio test statistic is proposed in [11]. In [12], the simultaneous confidence intervals for umbrella alternatives for normal means are studied. Testing the equality of normal means against simple tree alternatives is examined in [13].

This study focuses on W_h statistics based on different estimators of the population scale parameters. The frame of the paper is as follows. For testing the null hypothesis $H_0 : \theta_1 = ... = \theta_k$ against the umbrella alternative hypothesis $H_1 : \theta_1 \leq ... \leq \theta_h \geq ... \geq \theta_k$ with at least one strict inequality, the test procedure proposed by [1] is introduced in section 2. Some robust estimators for scale parameter θ_i are given in section 3. The design of carrying out the simulation study and the presentation of the simulation results are given in section 4 and 5, respectively.

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2. SINGH AND LIU'S TEST STATISTIC

Let $X_{i1},...,X_{in}$ be a random sample of a common size n from the i *th* population π_i, $i=1,2,...,k$ and S_i be any suitable estimator based on these random samples for scale parameter θ_i. To test the null hypothesis $H_0:\theta_1=...=\theta_k$ against the umbrella alternative hypothesis $H_1:\theta_1\le...\le\theta_h\ge...\ge\theta_k$ with at least one strict inequality for a given $1\le h\le k$, the test statistic W_h proposed in [1] is

$$W_h=Max\left(\max_{1\le i<j\le h}\left(S_j/S_i\right),\max_{h\le j<i\le k}\left(S_j/S_i\right)\right).$$

The null hypothesis H_0 is rejected at α significance level if $w_h\ge c_{k,h,v,\alpha}$ where w_h is the calculated value from sample for the statistic W_h and $P_0\left(W_h\ge c_{k,h,v,\alpha}\right)=\alpha$ under the null hypothesis. The critical value $c_{k,h,v,\alpha}$ is calculated for $k=3(1)10,\ n=3(1)15(5)40$ and $\alpha=0.01,\ 0.05$ and presented as a table in [1].

3. SOME ROBUST ESTIMATORS FOR SCALE PARAMETER

In the literature, there are many robust estimators for scale parameter θ_i. In this section, $\varepsilon-trimmed$ and $winsorized$ variance estimators will be presented due to having high breakdown point and efficiency.

3.1. Trimmed Mean and Variance

Trimmed estimation of location parameter would be to discard a proportion of the largest and smallest values. More precisely, let $\varepsilon\in[0,1/2)$ and $m=\left[(n-1)\varepsilon\right]$ where [.] stands for the integer part, and define the $\varepsilon-trimmed$ mean as

$${}_t\bar{X}_\varepsilon=\frac{1}{n-2m}\sum_{i=m+1}^{n-m}X_{(i)} \tag{2}$$

where $X_{(i)}$ denotes the $i.$ order statistics [14]. Using $\varepsilon-trimmed$ mean, trimmed variance is defined as, see [15],

$${}_t\hat{\sigma}^2_\varepsilon=\frac{1}{(n-2m-1)}\sum_{i=m+1}^{n-m}\left(X_{(i)}-{}_t\bar{X}_\varepsilon\right)^2 . \tag{3}$$

3.2. Winsorized Mean and Variance

The $\varepsilon-winsorized$ mean $\left(0<\varepsilon<1/2\right)$ of a sample $x_1,x_2,...,x_n$ is defined as follows. First, order the sample, obtaining $x_{(1)}\le x_{(2)}\le...\le x_{(n)}$. Then replace the $\left[n\varepsilon\right]=h$ smallest observations by $x_{(h+1)}$ hence counting this value $(h+1)$ times. Analogously, replace the h largest observations by $x_{(n-h)}$. Then $\varepsilon-winsorized$ mean is calculated as

$${}_w\bar{X}_\varepsilon=\frac{1}{n}\left\{(h+1)X_{(h+1)}+\sum_{i=h+2}^{n-h-1}X_{(i)}+(h+1)X_{(n-h)}\right\}$$

[16] . Winsorized variance suggested by [17] is calculated as

$$ {}_w\hat{\sigma}^2_\varepsilon = \frac{1}{(n-2h-1)}\left\{(h+1)\left(X_{(h+1)} - {}_w\bar{X}_\varepsilon\right)^2 + \sum_{i=h+2}^{n-h-1}\left(X_{(i)} - {}_w\bar{X}_\varepsilon\right) + (h+1)\left(X_{(n-h)} - {}_w\bar{X}_\varepsilon\right)^2\right\} \qquad (4) $$

4. SIMULATION

This simulation study consists of two parts. In first part, 10000 samples, each of which has size n, are generated from standart normal distribution and following exponential distribution

$$ f(x_i / \mu_i, \theta_i) = \frac{1}{\theta_i} e^{-(x_i-\mu_i)/\theta_i} \quad , \theta_i > 0 \ , -\infty < \mu_i < +\infty $$

with parameters $(\mu_i = 0, \theta_i = 1)$ under hyphotesis H0. Wh statistics based on different scale estimators are calculated for these samples. These Wh values are ordered and [10000α]th value is taken as Monte Carlo estimation for criticial value $c_{k,h,v,\alpha}$.

In second part, normal distribution and two-parameter exponential distribution with parameters $(\mu_i = 0, \theta_i)$ are used to generate 10000 samples, each of which has size n, under umbrella alternative hyphotesis H1. Similarly, Wh statistics based on different scale estimators are calculated for these samples. Ratio of Wh values, which are greater than the critical value in the first part, is taken as Monte Carlo estimation of $1-\beta$.

W_h statistic based on S_i, $\varepsilon-trimmed$ and $winsorized$ variance estimators for scale parameter θ_i is used for testing null hypothesis H_0 when the data is generated from the two-parameter exponential distribution. It is known that the minimum variance-unbiased estimator of the scale parameter of the two-parameter exponential distribution is $S_i = \sum_{j=1}^{n}\left(X_{ij} - Y_i\right)/(n-1)$ where $Y_i = \min_{1\le j\le n} X_{ij}$ [1]. $\varepsilon-trimmed$ and $winsorized$ variance estimators are presented in Eq. (3) and Eq. (4). In this study, W_h test statistics based on S_i, $\varepsilon-trimmed$ and $winsorized$ estimators of scale parameter are shown as T_1, T_2 and T_3, respectively.

When the data is generated from the normal distribution, $\varepsilon-trimmed$ and $winsorized$ estimators and the maximum likelihood estimator $S_i^2 = \sum_{j=1}^{n}\left(X_{ij} - \bar{X}_i\right)^2/(n-1)$ instead of S_i, are used for scale parameter. W_h test statistics based on S_i^2, $\varepsilon-trimmed$ and $winsorized$ estimators of scale parameter are shown as T_4, T_5 ve T_6, respectively.

In the simulation studies, the iteration number is set as 10000. In each iteration, n observations are generated as described above under H_0 and H_1. The values of test statistics $T_1, T_2, ..., T_6$ are calculated from this observations. The critical value of each statistic for a given α is calculated as the [10000α]th value as these values are arranged in ascending order. The [.] operator represents the ceil operation. The Monte Carlo estimation of power value, $1-\beta$, is obtained with 10000 iterations by using this critical values for umbrella alternative hypothesis.

Figure 1 and 2 are constructed for $k=5, h=2,3,4$ and $n=5,10,50,100$ values when $\alpha=0.05$. Umbrella hypotheses in these figures are taken as follows for $h=2,3,4$, respectively,

$\theta_1 = 1+t \le \theta_2 = 1+m \ge \theta_3 = 1+t \ge \theta_4 = 1 \ge \theta_5 = 1$

$\theta_1 = 1 \le \theta_2 = 1+t \le \theta_3 = 1+m \ge \theta_4 = 1+t \ge \theta_5 = 1$

$$\theta_1 = 1 \le \theta_2 = 1 \le \theta_3 = 1 + t \le \theta_4 = 1 + m \ge \theta_5 = 1 + t.$$

Similarly, Figure 3 and 4 are constructed for $k = 7, h = 3,4,5$ and $n = 5,10,50,100$ values when $\alpha = 0.05$ and umbrella hyphotheses in these figures are defined as follows for $h = 3,4,5$, respectively,

$$\theta_1 = 1 \le \theta_2 = 1 + t \le \theta_3 = 1 + m \ge \theta_4 = 1 + t \ge \theta_5 = 1 \ge \theta_6 = 1 \ge \theta_7 = 1$$
$$\theta_1 = 1 \le \theta_2 = 1 \le \theta_3 = 1 + t \le \theta_4 = 1 + m \ge \theta_5 = 1 + t \ge \theta_6 = 1 \ge \theta_7 = 1$$
$$\theta_1 = 1 \le \theta_2 = 1 \le \theta_3 = 1 \le \theta_4 = 1 + t \ge \theta_5 = 1 + m \ge \theta_6 = 1 + t \ge \theta_7 = 1$$

For each possible k, h and n values, unique umbrella hyphotheses are constructed by initiating t and m values from 0 and increasing these values by 0.025 and 0.05 respectively in each step. For these hyphotheses, power values of these statistics are shown in vertical axis.

In the Figures 1 and 3, the Monte Carlo estimations of α and $1-\beta$ for T_1, T_2 and T_3 statistics are presented for given values of k, h and n when the data is generated from two-parameter exponential distribution. The results show that the Monte Carlo estimation of α is almost equal to nominal α. Moreover, T_1 statistic has more power than T_2 and T_3 statistics for given all k, h and n values. As the sample size increases, Monte Carlo estimations of the power values related to T_1, T_2 and T_3 statistics also increase and these estimated values are resulted to be quite close to each other. In addition, these estimated power values are observed to be almost similar for different h values when k and n are constants.

In the Figures 2 and 4, the Monte Carlo estimations of α and $1-\beta$ for T_4, T_5 and T_6 statistics are presented for the given values of k, h and n when the data is generated from normal distribution. Similarly, it is observed that the Monte Carlo estimation of α is considerably close to nominal α. For all k, h and n values, it is seen that T_4 statistic has more power than T_5 and T_6 statistics which is an expected result for normal distribution case. As the sample size increases, it is observed that the Monte Carlo estimations of the power values related to T_4, T_5 and T_6 statistics increase and these estimated power values are closer to each other than they are for the two-parameter exponential distribution case in Figures 1 and 3. Additionally, when k and n are constants, these estimated power values are observed to be almost similar for different h values.

According to Figures 1 to 4, the differences among estimated power values of the test statistics T_4, T_5, T_6 are observed to be smaller than they are for the test statistics T_1, T_2, T_3. In addition, the differences among these estimated power values of the test statistics T_4, T_5, T_6 are resulted to decrease faster than they are for the test statistics T_1, T_2, T_3.

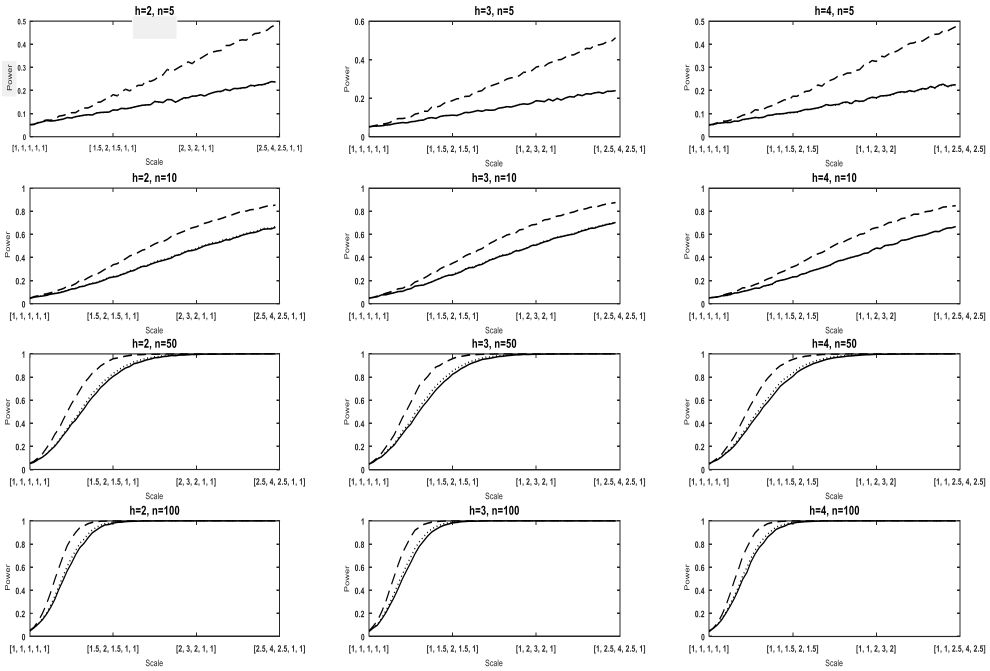

Figure 1. *The Monte Carlo estimation of α and 1-β for k=5, n=5, 10, 50, 100 and h=2, 3, 4 when data is generated from two-parameter exponential distribution. ('- - - ' , '. . .' and ' —— ' represent the power functions of* T_1, T_2 *and* T_3*, respectively)*

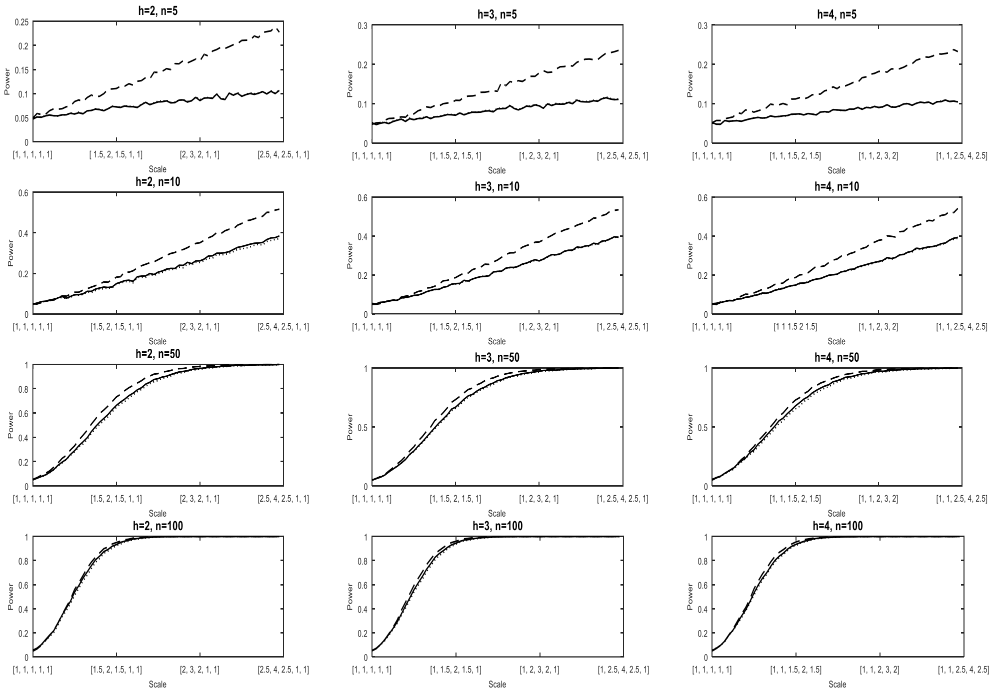

Figure 2. *The Monte Carlo estimation of α and 1-β for k=5, n=5, 10, 50, 100 and h=2, 3, 4 when data is generated from normal distribution. ('- - -' , '. . .' and ' —— ' represent the power functions of* T_4, T_5 *and* T_6*, respectively)*

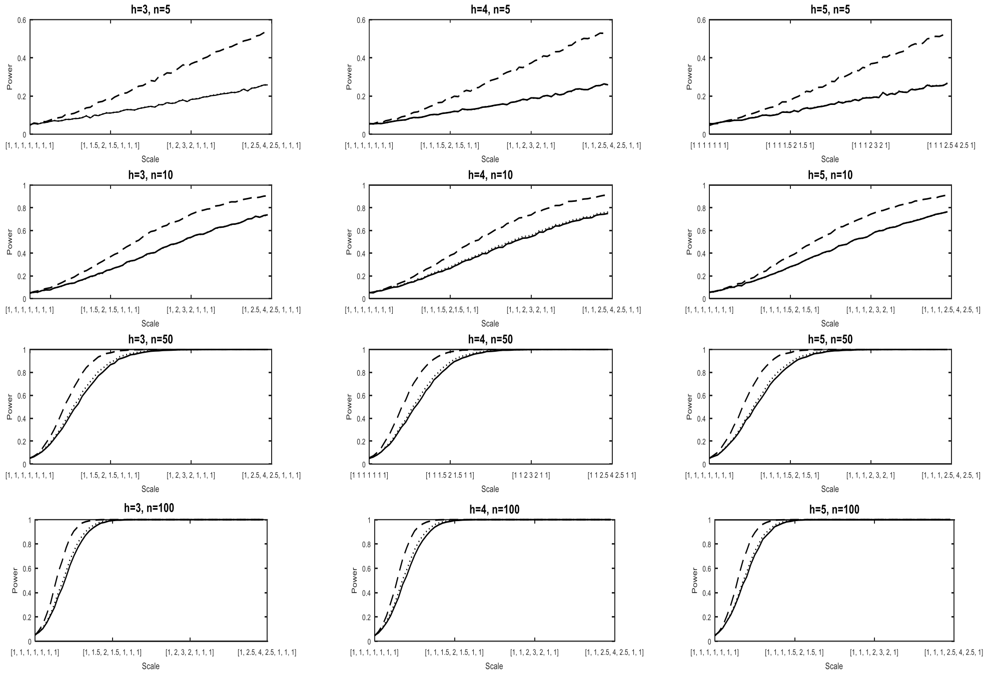

Figure 3. *The Monte Carlo estimation of α and 1-β for k=7, n=5, 10, 50, 100 and h=3, 4, 5 when data is generated from two-parameter exponential distribution. ('- - - ' , '. . .' and ' —— ' represent the power functions of* T_1, T_2 *and* T_3*, respectively)*

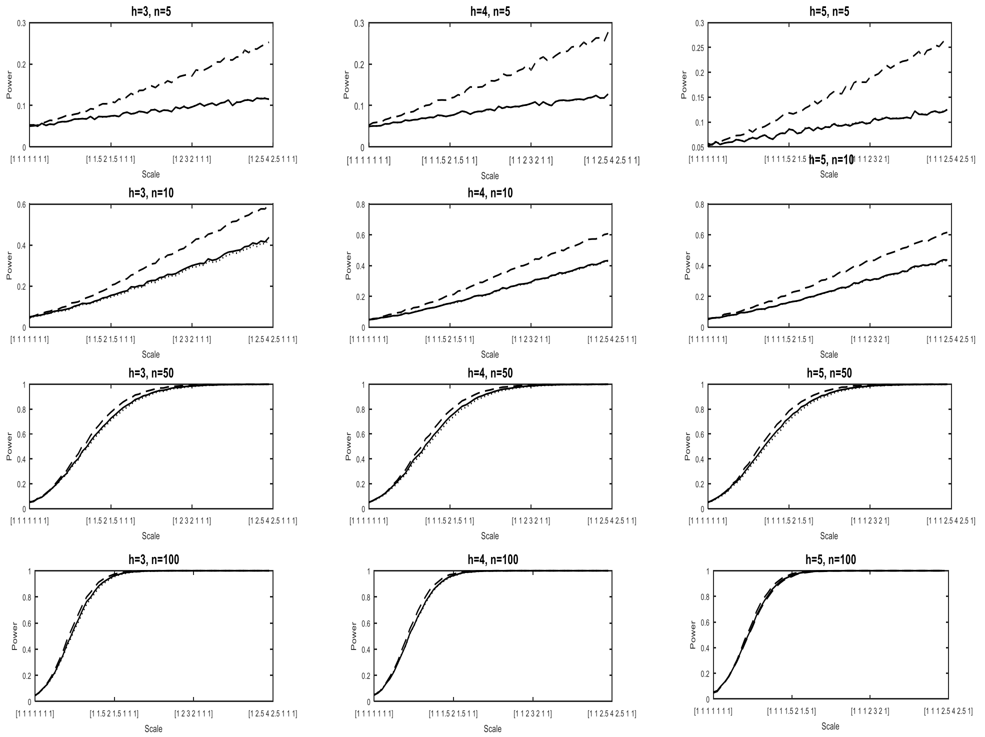

Figure 4. *The Monte Carlo estimation of α and 1-β for k=7, n=5, 10, 50, 100 and h=3, 4, 5 when data is generated from normal distribution. ('- - -' , '. . .' and ' —— ' represent the power functions of* T_4, T_5 *and* T_6*, respectively)*

5. RESULTS

According to the simulation results, the Monte Carlo estimation of α for T_2 and T_3 statistics based on Trimmed Variances and Winsorized Variances, which are robust estimators for scale parameters, and T_1 statistics based on S_i statistics are almost equal to nominal α for testing the null hypothesis against to umbrella alternative hypothesis. For given values of k, h and n, it is observed that T_1 statistic has more power than T_2 and T_3 statistics and as the sample size increases, the Monte Carlo estimation of power values increase and the power values of T_1, T_2 and T_3 statistics are similar to each other when the data is generated from exponential distribution. For different h values, the Monte Carlo estimation of power values are observed to be almost similar when k and n are constants. When the data is generated from normal distribution, the results are similar to the ones calculated from exponential distribution. Although the power values increase as n increases, the decrease in the difference among the power values of T_4, T_5 and T_6 statistics is faster than the decrease in the difference among the power values of T_1, T_2 and T_3 statistics.

CONFLICT OF INTEREST

No conflict of interest was declared by the author.

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