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Generic Riemannian Submersions from Almost Product Riemannian Manifolds

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Abstract

In this paper, we define and study generic Riemannian submersions from almost product Riemannian manifolds onto Riemannian manifolds. We give an example, investigate the geometry of foliations which are arisen from the definition of a Riemannian submersion. We also find necessary and sufficient conditions for a generic Riemannian submersion to be totally geodesic.

1. INTRODUCTION

The theory of smooth maps between Riemannian manifolds has been widely studied in Riemannian geometry. Such maps are useful for comparing geometric structures between two manifolds. In this point of view, the study of Riemannian submersions between Riemannian manifolds was initiated by O'Neill [12] and Gray [6], see also [5] and [19]. Riemannian submersions have several applications in mathematical physics. Indeed, Riemannian submersions have their applications in the Yang-Mills theory ([3], [18]), Kaluza-Klein theory ([4], [7]), supergravity and superstring theories ([8], [11]), etc. Later such submersions were considered between manifolds with differentiable structures, see [5]. Furthermore, we have the following submersions: semi-Riemannian submersion and Lorentzian submersion [5], Riemannian submersion [6], almost Hermitian submersion [17], contact-complex submersion [10], quaternionic submersion [9], etc.

Recently, B. Şahin [14] introduced the notion of anti-invariant Riemannian submersions which are Riemannian submersions from almost Hermitian manifolds such that the vertical distributions are anti-invariant under the almost complex structure of the total manifold and as a generalization of anti-invariant Riemannian submersions and almost Hermitian submersions, B. Şahin [15] introduced the notion of semi-invariant Riemannian submersions when the base manifold is an almost Hermitian manifold. (For recent developments on the geometry of almost Hermitian manifolds, see also:[16]). He showed that such submersions have rich geometric properties and they are useful for investigating the geometry of the total space. On the other hand, as a generalization of semi-invariant submersions, Ali and Fatima [1] introduced the notion of generic Riemannian submersions. They showed that such submersions have rich geometric properties and they are useful for investigating the geometry of the total space. The present work, we define and study the notion of generic Riemannian submersion from almost product Riemannian manifolds. The paper is organized as follows. In Section 2 we recall some notions needed for this paper. In section 3 we define generic Riemannian submersions from an almost product Riemannian manifold onto a Riemannian manifold. We also investigate the geometry of leaves of the distributions. Finally, we give necessary and sufficient conditions for such submersions to be totally geodesic.

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2. PRELIMINARIES

In this section, we define almost product Riemannian manifolds, recall the notion of Riemannian submersions between Riemannian manifolds and give a brief review of basic facts of Riemannian submersions.

Let M be a m-dimensional manifold with a tensor F of a type (1,1) such that

$$F^2 = I, (F \neq I).$$

Then, we say that M is an almost product manifold with almost product structure F. We put

$$P = \frac{1}{2}(I + F), \; Q = \frac{1}{2}(I - F).$$

Then we get

$$P + Q = I, \; P^2 = P, \; Q^2 = Q, \; PQ = QP = 0, \; F = P - Q.$$

Thus P and Q define two complementary distributions P and Q. We easily see that the eigenvalues of F are +1 or -1. If an almost product manifold M admits a Riemannian metric g such that

$$g(FX, FY) = g(X, Y) \tag{1}$$

for any vector fields X and Y on M. Then M is called an almost product Riemannian manifold, denoted by (M, g, F). Denote the Levi-Civita connection on M with respect to g by $\overset{M}{\nabla}$. Then, M is called a locally product Riemannian manifold [19] if F is parallel with respect to $\overset{M}{\nabla}$, i.e.,

$$\overset{M}{\nabla}_X F = 0, \; X \in \Gamma(TM) \tag{2}$$

Let (M, g) and $(N, g^{'})$ be two Riemannian manifolds. A surjective $C^\infty -$map $\pi : M \to N$ is a $C^\infty -$ submersion if it has maximal rank at any point of M. Putting $\mathcal{V}_x = ker\pi_{*x}$, for any $x \in M$, we obtain an integrable distribution $\mathcal{V}$, which is called vertical distribution and corresponds to the foliation of M determined by the fibres of π. The complementary distribution $\mathcal{H}$ of $\mathcal{V}$, determined by the Riemannian metric g, is called horizontal distribution. A $C^\infty -$submersion $\pi : M \to N$ between two Riemannian manifolds (M, g) and $(N, g^{'})$ is called a Riemannian submersion if, at each point x of M, π_{*x} preserves the length of the horizontal vectors. A horizontal vector field X on M is said to be basic if X is $\pi-$ related to a vector field $X^{'}$ on N. It is clear that every vector field $X^{'}$ on N has a unique horizontal lift X to M and X is basic.

We recall that the sections of $\mathcal{V}$, respectively $\mathcal{H}$, are called the vertical vector fields, respectively horizontal vector fields. A Riemannian submersion $\pi : M \to N$ determines two $(1,2)$ tensor fields T and A on M, by the formulas:

$$T(E, F) = T_E F = \mathcal{H}\overset{M}{\nabla}_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\overset{M}{\nabla}_{\mathcal{V}E}\mathcal{H}F \tag{3}$$

and

$$A(E, F) = A_E F = \mathcal{V}\overset{M}{\nabla}_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\overset{M}{\nabla}_{\mathcal{H}E}\mathcal{H}F \tag{4}$$

for any $E, F \in \Gamma(TM)$, where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections (see [5]). From (3) and (4), one can obtain

$$\overset{M}{\nabla}_V W = T_V W + \hat{\nabla}_V W; \tag{5}$$

$$\overset{M}{\nabla}_V X = T_V X + \mathcal{H}(\overset{M}{\nabla}_V X); \tag{6}$$

$$\overset{M}{\nabla}_X V = \mathcal{V}(\overset{M}{\nabla}_X V) + A_X V; \tag{7}$$

$$\overset{M}{\nabla}_X Y = A_X Y + \mathcal{H}(\overset{M}{\nabla}_X Y), \tag{8}$$

for any $X, Y \in \Gamma((ker\pi_*)^{\perp})$, $V, W \in \Gamma(ker\pi_*)$. Moreover, if X is basic then

$$\mathcal{H}(\overset{M}{\nabla}_V X) = \mathcal{V}(\overset{M}{\nabla}_X V) = A_X V. \tag{9}$$

We note that for $U, V \in \Gamma(ker\pi_*)$, $T_U V$ coincides with the second fundamental form of the immersion of the fibre submanifolds and for $X, Y \in \Gamma((ker\pi_*)^{\perp})$, $A_X Y = \frac{1}{2}\mathcal{V}[X, Y]$ reflecting the complete integrability of the horizontal distribution $\mathcal{H}$. It is known that A is alternating on the horizontal distribution: $A_X Y = -A_Y X$, for $X, Y \in \Gamma((ker\pi_*)^{\perp})$ and T is symmetric on the vertical distribution: $T_U V = T_V U$, for $U, V \in \Gamma(ker\pi_*)$.

We now recall the following result which will be useful for later.

Lemma 2.1 (see [5],[12]). If $\pi : M \to N$ is a Riemannian submersion and X, Y basic vector fields on M, $\pi-$related to X' and Y' on N, then we have the following properties

1. $\mathcal{H}[X, Y]$ is a basic vector field and $\pi_*\mathcal{H}[X, Y] = [X', Y'] \circ \pi$;
2. $\mathcal{H}(\overset{M}{\nabla}_X Y)$ is a basic vector field $\pi-$related to $(\overset{N}{\nabla}_{X'} Y')$, where $\overset{M}{\nabla}$ and $\overset{N}{\nabla}$ are the Levi-Civita connection on M and N;
3. $[E, U] \in \Gamma(ker\pi_*)$, for any $U \in \Gamma(ker\pi_*)$ and for any basic vector field E.

Let (M, g_M) and (N, g_N) be Riemannian manifolds and $\pi : M \to N$ is a smooth map. Then the second fundamental form of π is given by

$$(\nabla\pi_*)(X, Y) = \nabla_{\pi_* X}\pi_* Y - \pi_*(\nabla_X Y) \tag{10}$$

for $X, Y \in \Gamma(TM)$, where we denote conveniently by ∇ the Levi-Civita connections of the metrics g_M and g_N. Recall that π is said to be harmonic if $trace(\nabla\pi_*) = 0$ and π is called a totally geodesic map if $(\nabla\pi_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM)$ [2]. It is known that the second fundamental form is symmetric.

Let g be a Riemannian metric tensor on the manifold $M = M_1 \times M_2$ and assume that the canonical foliations D_{M_1} and D_{M_2} intersect perpendicularly everywhere. Then g is the metric tensor of a usual product of Riemannian manifolds if and only if D_{M_1} and D_{M_2} are totally geodesic foliations [13].

3. GENERIC RIEMANNIAN SUBMERSIONS

In this section, we define and study generic Riemannian submersions from an almost product Riemannian manifold onto a Riemannian manifold, investigate the integrability of distributions and obtain a necessary and sufficient condition for such submersions to be totally geodesic map.

Definition 3.1 *Let (M, g, F) be an almost product Riemannian manifold and $(N, g^{'})$ a Riemannian manifold. A Riemannian submersion $\pi : (M, g, F) \to (N, g^{'})$ is called a generic Riemannian submersion if there is a distribution $D_1 \subset ker\pi_*$ such that*

$$ker\pi_* = D_1 \oplus D_2, F(D_1) = D_1,$$

where D_2 is the orthogonal complement of D_1 in $\Gamma(ker\pi_*),$ and is purely real distribution on the fibres of the submersion π.

Now, we will give an example in order to guarantee the existence of generic Riemannian submersions in locally product Riemannian manifolds and demonstrate that the method presented in this paper is effective. Note that given an Euclidean space $\mathbb{R}^8$ with coordinates $(x_1,..., x_8)$, we can canonically choose an almost product structure F on $\mathbb{R}^8$ as follows:

$$F(a_1,..., a_8) = \frac{1}{\sqrt{2}}(-a_2 - a_8, -a_1 - a_7, -a_4 + a_6, -a_3 + a_5, a_6 + a_4, a_5 + a_3, a_8 - a_2, a_7 - a_1),$$

where $a_1,..., a_8 \in \mathbb{R}.$

Example 3.1 *Let π be a submersion defined by*

$$\begin{array}{cccc} \pi : & \mathbb{R}^8 & \to & \mathbb{R}^2 \\ & (x_1,..., x_8) & & (\frac{x_1 + x_3}{\sqrt{2}}, \frac{x_1 - x_3}{\sqrt{2}}). \end{array}$$

Then it follows that

$$ker\pi_* = span\{Z_1 = \partial x_2, Z_2 = \partial x_4, Z_3 = \partial x_5, Z_4 = \partial x_6, Z_5 = \partial x_7, Z_6 = \partial x_8\}$$

and

$$(ker\pi_*)^{\perp} = span\{H_1 = \frac{1}{\sqrt{2}}\partial x_1 + \frac{1}{\sqrt{2}}\partial x_3, H_2 = \frac{1}{\sqrt{2}}\partial x_1 - \frac{1}{\sqrt{2}}\partial x_3\}.$$

Hence we have

$$FZ_1 = -\frac{1}{\sqrt{2}}Z_5 - \frac{1}{2}X_1 - \frac{1}{2}X_2, FZ_2 = \frac{1}{\sqrt{2}}Z_3 - \frac{1}{2}X_1 + \frac{1}{2}X_2,$$

$$FZ_3 = \frac{1}{\sqrt{2}}Z_2 + \frac{1}{\sqrt{2}}Z_4, FZ_4 = \frac{1}{\sqrt{2}}Z_3 + \frac{1}{2}X_1 - \frac{1}{2}X_2,$$

$$FZ_5 = -\frac{1}{\sqrt{2}}Z_1 + \frac{1}{\sqrt{2}}Z_6 \text{ and } FZ_6 = \frac{1}{\sqrt{2}}Z_5 - \frac{1}{2}X_1 - \frac{1}{2}X_2.$$

Thus it follows that $D_1 = span\{Z_3, Z_5\}$ and $D_2 = span\{Z_1, Z_2, Z_4, Z_6\}$. Also by direct computations, we get

$$g_1(H_1, H_1) = g_2(\pi_* H_1, \pi_* H_1) \text{ and } g_1(H_2, H_2) = g_2(\pi_* H_2, \pi_* H_2),$$

where g_1 and g_2 denote the standard metrics (inner products) of $\mathbb{R}^8$ and $\mathbb{R}^2$. This show that π is a Riemannian submersion. Thus π is a generic Riemannian submersion.

Let π be a generic Riemannian submersion from an almost product Riemannian manifold (M,g,F) onto a Riemannian manifold $(N,g^{'})$. Then for $V\in\Gamma(\ker\pi_*)$, we write

$$FV=\phi V+\omega V \tag{11}$$

where $\phi V\in\Gamma(D_1)$ and $\omega V\in\Gamma((\ker\pi_*)^{\perp})$. We denote the complementary distribution to ωD_2 in $(\ker\pi_*)^{\perp}$ by μ. Then we have

$$(\ker\pi_*)^{\perp}=\omega D_2\oplus\mu \tag{12}$$

and μ is invariant under F. Thus, for any $X\in\Gamma((\ker\pi_*)^{\perp})$, we have

$$FX=\mathcal{B}X+\mathcal{C}X \tag{13}$$

where $\mathcal{B}X\in\Gamma(D_2)$ and $\mathcal{C}X\in\Gamma(\mu)$. From (11), (12) and (13) we have

$$\phi D_1=D_1,\omega D_1=0,\phi D_2\subset D_2,\mathcal{B}((\ker\pi_*)^{\perp})=D_2,$$

$$\phi^2+\mathcal{B}\omega=-id,\mathcal{C}^2+\omega\mathcal{B}=-id,\omega\phi+\mathcal{C}\omega=0,\mathcal{B}\mathcal{C}+\phi\mathcal{B}=0.$$

Then by using (5), (6), (11) and (13) we get

$$(\overset{M}{\nabla}_U\phi)V=\mathcal{B}T_UV-T_U\omega V \tag{14}$$

$$(\overset{M}{\nabla}_U\omega)V=\mathcal{C}T_UV-T_U\phi V \tag{15}$$

for $U,V\in\Gamma(\ker\pi_*)$, where

$$(\overset{M}{\nabla}_U\phi)V=\hat{\nabla}_U\phi V-\phi\hat{\nabla}_UV$$

and

$$(\overset{M}{\nabla}_U\omega)V=A_{\omega V}U-\omega\hat{\nabla}_UV.$$

Next, we easily have the following lemma:

Lemma 3.1 *Let (M,g,F) be a locally product Riemannian manifold and $(N,g^{'})$ a Riemannian manifold. Let $\pi:(M,g,F)\to(N,g^{'})$ be a generic Riemannian submersion. Then we have*

1. $$A_X\mathcal{B}Y+\mathcal{H}\overset{M}{\nabla}_X\mathcal{C}Y=\mathcal{C}\mathcal{H}\overset{M}{\nabla}_XY+\omega A_XY$$
   $$\mathcal{V}\overset{M}{\nabla}_X\mathcal{B}Y+A_X\mathcal{C}Y=\mathcal{B}\mathcal{H}\overset{M}{\nabla}_XY+\phi A_XY,$$
2. $$T_U\phi V+A_{\omega V}U=\mathcal{C}T_UV+\omega\hat{\nabla}_UV$$
   $$\hat{\nabla}_U\phi V+T_U\omega V=\mathcal{B}T_UV+\phi\hat{\nabla}_UV,$$
3. $$A_X\phi U+\mathcal{H}\overset{M}{\nabla}_U\omega V=\mathcal{C}A_XU+\omega\mathcal{V}\overset{M}{\nabla}_XU$$
   $$\mathcal{V}\overset{M}{\nabla}_X\phi U+A_X\omega U=\mathcal{B}A_XU+\phi\mathcal{V}\overset{M}{\nabla}_XU,$$

for $X,Y \in \Gamma((\ker \pi_*)^{\perp})$ and $U,V \in \Gamma(\ker \pi_*)$.

Theorem 3.1 *Let π be a generic Riemannian submersion from a locally product Riemannian manifold (M,g,F) onto a Riemannian manifold $(N,g^{'})$. Then the distribution D_1 is integrable if and only if we have*

$$\omega(\hat{\nabla}_{U_1}V_1 - \hat{\nabla}_{V_1}U_1) = \mathcal{C}(T_{V_1}U_1 - T_{U_1}V_1)$$

for $U_1, V_1 \in \Gamma(D_1)$.

Proof. For $U_1, V_1 \in \Gamma(D_1)$ and $Z \in \Gamma((ker\pi_*)^{\perp})$, since $[U_1,V_1] \in \Gamma(ker\pi_*)$, from (5), (11) and (13) we get

$$g(F[U_1,V_1],Z) = g(F\overset{M}{\nabla}_{U_1}V_1 - F\overset{M}{\nabla}_{V_1}U_1, Z) = g(FT_{U_1}V_1 + F\hat{\nabla}_{U_1}V_1 - FT_{V_1}U_1 - F\hat{\nabla}_{V_1}U_1, Z)$$

$$= g(\mathcal{B}T_{U_1}V_1 + \mathcal{C}T_{U_1}V_1 + \phi\hat{\nabla}_{U_1}V_1 + \omega\hat{\nabla}_{U_1}V_1$$

$$-\mathcal{B}T_{V_1}U_1 - \phi\hat{\nabla}_{V_1}U_1 - \mathcal{C}T_{V_1}U_1 - \omega\hat{\nabla}_{V_1}U_1, Z)$$

$$= g(\mathcal{C}T_{U_1}V_1 + \omega\hat{\nabla}_{U_1}V_1 - \mathcal{C}T_{V_1}U_1 - \omega\hat{\nabla}_{V_1}U_1, Z).$$

Hence, we have the result.

Theorem 3.2 *Let π be a generic Riemannian submersion from a locally product Riemannian manifold (M,g,F) onto a Riemannian manifold $(N,g^{'})$. Then the purely real distribution D_2 is integrable if and only if we have*

$$\hat{\nabla}_{U_2}\phi V_2 - \hat{\nabla}_{V_2}\phi U_2 + T_{U_2}\omega V_2 - T_{V_2}\omega U_2 \in \Gamma(D_2)$$

for $U_2, V_2 \in \Gamma(D_2)$.

Proof. We note that the purely real distribution D_2 integrable if and only if $g([U_2,V_2],FZ) = g_1([U_2,V_2],W) = 0$ for $U_2,V_2 \in \Gamma(D_2), Z \in \Gamma(D_1)$ and $W \in \Gamma((\ker\pi_*)^{\perp})$. Since $\ker\pi_*$ is integrable $g([U_2,V_2],W) = 0$. Thus D_2 integrable if and only if $g([U_2,V_2],FZ) = 0$. Moreover, by using (1), (2) and (13) we have

$$g([U_2,V_2],FZ) = g(F(\overset{M}{\nabla}_{U_2}\phi V_2 + \overset{M}{\nabla}_{U_2}\omega V_2),FZ) - g(F(\overset{M}{\nabla}_{V_2}\phi U_2 + \overset{M}{\nabla}_{V_2}\omega U_2),FZ).$$

Using (5), (6) and (9), we get

$$g([U_2,V_2],FZ) = g(\mathcal{B}(T_{U_2}\phi V_2 - T_{V_2}\phi U_2 + A_{\omega V_2}U_2 - A_{\omega U_2}V_2)$$

$$+\phi(\hat{\nabla}_{U_2}\phi V_2 - \hat{\nabla}_{V_2}\phi U_2 + T_{V_2}\omega V_2 - T_{V_2}\omega U_2),FZ).$$

Since $\mathcal{B}(T_{U_2}\phi V_2 - T_{V_2}\phi U_2 + A_{\omega V_2}U_2 - A_{\omega U_2}V_2) \in \Gamma(D_2)$,

$$g([U_2,V_2],FZ) = g_{(}\phi(\hat{\nabla}_{U_2}\phi V_2 - \hat{\nabla}_{V_2}\phi U_2 + T_{U_2}\omega V_2 - T_{V_2}\omega U_2),FZ),$$

which proves assertion.

Now, we investigate the geometry of the leaves of the distributitons D_1 and D_2.

Theorem 3.3 *Let π be a generic Riemannian submersion from a locally product Riemannian manifold (M,g,F) onto a Riemannian manifold $(N,g^{'})$. Then D_1 defines a totally geodesic foliation on M_1 if and only if*

$$(\nabla\pi_*)(U_1,FV_1)\in\Gamma(\mu)$$

and

$$g(\hat{\nabla}_{U_1}FV_1,\mathrm{B}X)=g^{'}((\nabla\pi_*)(U_1,FV_1),\pi_*\mathrm{C}X)$$

for $U_1,V_1\in\Gamma(D_1),U_2\in\Gamma(D_2)$ and $X\in\Gamma((ker\pi_*)^{\perp})$.

Proof. From the definition of a generic Riemannian submersion, it follows that the distribution D_1 defines a totally geodesic foliation on M if and only if $g(\overset{M}{\nabla}_{U_1}V_1,U_2)=0$ and $g(\overset{M}{\nabla}_{U_1}V_1,X)=0$ for $U_1,V_1\in\Gamma(D_1),U_2\in\Gamma(D_2)$ and $X\in\Gamma((ker\pi_*)^{\perp})$. Since π is a generic Riemannian submersion, from (2), we get

$$g(\overset{M}{\nabla}_{U_1}V_1,U_2)=-g^{'}((\nabla\pi_*)(U_1,FV_1),\pi_*\omega U_2). \tag{16}$$

On the other hand, by using (13) we have

$$g(\overset{M}{\nabla}_{U_1}V_1,X)=g_1(\overset{M}{\nabla}_{U_1}FV_1,\mathrm{B}X)+g_1(\overset{M}{\nabla}_{U_1}FV_1,\mathrm{C}X).$$

Since π is a generic Riemannian submersion, using (10) and (5) we get

$$g(\overset{M}{\nabla}_{U_1}V_1,X)=g(\hat{\nabla}_{U_1}FV_1,\mathrm{B}X)-g^{'}((\nabla\pi_*)(U_1,FV_1),\pi_*\mathrm{C}X). \tag{17}$$

Thus proof follows from (16) and (17).

For the leaves of D_2 we have the following result.

Theorem 3.4 *Let π be a generic Riemannian submersion from a locally product Riemannian manifold (M,g,F) onto a Riemannian manifold $(N,g^{'})$. Then D_2 defines a totally geodesic foliation on M if and only if*

$$(\nabla\pi_*)(U_2,FV_1)\in\Gamma(\mu)$$

and

$$g^{'}((\nabla\pi_*)(U_2,\mathrm{C}X),\pi_*\omega V_2)=g(\hat{\nabla}_{U_2}\mathrm{B}X,\phi V_2)+g(T_{U_2}\mathrm{B}X,\omega V_2)+g(T_{U_2}\mathrm{C}X,\phi V_2)$$

for $U_1\in\Gamma(D_1)$, $U_2,V_2\in\Gamma(D_2)$ and $X\in\Gamma((ker\pi_*)^{\perp})$.

Proof. The distribution D_2 defines a totally geodesic foliation on M if and only if $g(\overset{M}{\nabla}_{U_2}V_2,U_1)=0$ and $g(\overset{M}{\nabla}_{U_2}V_2,X)=0$ for $U_1\in\Gamma(D_1)$, $U_2,V_2\in\Gamma(D_2)$ and $X\in\Gamma((ker\pi_*)^{\perp})$. Since we have $g(\overset{M}{\nabla}_{U_2}V_2,U_1)=-g_1(\overset{M}{\nabla}_{U_2}U_1,V_2)$, from (10) we have

$$g(\overset{M}{\nabla}_{U_2}V_2,U_1)=-g^{'}((\nabla\pi_*)(U_2,FU_1),\pi_*\omega V_2). \tag{18}$$

In a similar way, by using (13) we obtain

$$g(\overset{M}{\nabla}_{U_2}V_2, X) = -g(\overset{M}{\nabla}_{U_2}\mathcal{B}X, FV_2) - g(\overset{M}{\nabla}_{U_2}\mathcal{C}X, FV_2).$$

Using (5), (6), (11) and if we take into account that π is a generic Riemannian submersion, we obtain

$$g(\overset{M}{\nabla}_{U_2}V_2, X) = -g(\hat{\nabla}_{U_2}\mathcal{B}X, \phi V_2) - g(T_{U_2}\mathcal{B}X, \omega V_2) - g(T_{U_2}\mathcal{C}X, \phi V_2) \quad (19)$$

$$+ g'((\nabla \pi_*)(U_2, \mathcal{C}X), \pi_* \omega V_2).$$

Thus proof follows from (18) and (19).

From Theorem 3.3 and Theorem 3.4, we have the following result.

Theorem 3.5 *Let* $\pi : (M, g, F) \to (N, g')$ *be a generic Riemannian submersion from a locally product Riemannian manifold* (M, g, F) *onto a Riemannian manifold* (N, g'). *Then the fibers of* π *are the locally product Riemannian manifold of leaves of* D_1 *and* D_2 *if and only if*

$$(\nabla \pi_*)(U_1, FV_1) \in \Gamma(\pi_* \mu)$$

and

$$(\nabla \pi_*)(U_2, FV_1) \in \Gamma(\pi_* \mu)$$

for any $U_1, V_1 \in \Gamma(D_1)$ and $U_2 \in \Gamma(D_2)$.

For the geometry of leaves of the horizontal distribution $((ker\pi_*)^{\perp})$, we have the following theorem.

Theorem 3.6 *Let* $\pi : (M, g, F) \to (N, g')$ *be a generic Riemannian submersion from a locally product Riemannian manifold* (M, g, F) *onto a Riemannian manifold* (N, g'). *Then the distribution* $(ker\pi_*)^{\perp}$ *defines a totally geodesic foliation on* M *if and only if*

$$A_{X_1}\mathcal{B}X_2 + \mathcal{H}\overset{M}{\nabla}_{X_1}\mathcal{C}X_2 \in \Gamma(\mu), \mathcal{V}\overset{M}{\nabla}_{X_1}\mathcal{B}X_2 + A_{X_1}\mathcal{C}X_2 \in \Gamma(D_2)$$

for any $X_1, X_2 \in \Gamma((ker\pi_*)^{\perp})$.

Proof. Since M is a locally product Riemannian manifold, from (1) and (2) we have $\overset{M}{\nabla}_{X_1}X_2 = F\overset{M}{\nabla}_{X_1}FX_2$ for $X_1, X_2 \in \Gamma((ker\pi_*)^{\perp})$. Using (13), (7) and (8)

$$\overset{M}{\nabla}_{X_1}X_2 = F(A_{X_1}\mathcal{B}X_2 + \mathcal{V}\overset{M}{\nabla}_{X_1}\mathcal{B}X_2) + F(\mathcal{H}\overset{M}{\nabla}_{X_1}\mathcal{C}X_2 + A_{X_1}\mathcal{C}X_2).$$

Then by using (11) and (13) we get

$$\overset{M}{\nabla}_{X_1}X_2 = \mathcal{B}A_{X_1}\mathcal{B}X_2 + \mathcal{C}A_{X_1}\mathcal{B}X_2 + \phi\mathcal{V}\overset{M}{\nabla}_{X_1}\mathcal{B}X_2$$

$$+ \omega\mathcal{V}\overset{M}{\nabla}_{X_1}\mathcal{B}X_2 + \mathcal{B}\mathcal{H}\overset{M}{\nabla}_{X_1}\mathcal{C}X_2 + \mathcal{C}\mathcal{H}\overset{M}{\nabla}_{X_1}\mathcal{C}X_2 + \phi A_{X_1}\mathcal{C}X_2 + \omega A_{X_1}\mathcal{C}X_2.$$

Hence, we have $\overset{M}{\nabla}_{X_1}X_2 \in \Gamma((ker\pi_*)^{\perp})$ if and only if

$$\mathcal{B}(A_{X_1}\mathcal{B}X_2 + \mathcal{H}\overset{M}{\nabla}_{X_1}\mathcal{C}X_2) + \phi(\mathcal{V}\overset{M}{\nabla}_{X_1}\mathcal{B}X_2 + A_{X_1}\mathcal{C}X_2) = 0.$$

Thus $\overset{M}{\nabla}_{X_1}X_2 \in \Gamma((ker\pi_*)^{\perp})$ if and only if

$$\mathcal{B}(A_{X_1}\mathcal{B}X_2 + \mathcal{H}\overset{M}{\nabla}_{X_1}\mathcal{C}X_2) = 0, \phi(\mathcal{V}\overset{M}{\nabla}_{X_1}\mathcal{B}X_2 + A_{X_1}\mathcal{C}X_2) = 0,$$

which completes proof.

In the sequel we are going to investigate the geometry of leaves of the vertical distribution $ker\pi_*$.

Theorem 3.7 *Let* $\pi:(M,g,F)\to(N,g')$ *be a generic Riemannian submersion from a locally product Riemannian manifold* (M,g,F) *onto a Riemannian manifold* (N,g'). *Then the distribution* $(ker\pi_*)$ *defines a totally geodesic foliation on* M *if and only if*

$$T_{Z_1}\phi Z_2 + A_{\omega Z_2}Z_1 \in \Gamma(\omega D_2), \hat{\nabla}_{Z_1}\phi Z_2 + T_{Z_1}\omega Z_2 \in \Gamma(D_1)$$

for any $Z_1, Z_2 \in \Gamma(ker\pi_*)$.

Proof. For any $Z_1, Z_2 \in \Gamma(ker\pi_*)$, using (2), (5), (6) and (11) we get

$$\overset{M}{\nabla}_{Z_1}Z_2 = F\overset{M}{\nabla}_{Z_1}FZ_2 = F(\overset{M}{\nabla}_{Z_1}\phi Z_2 + \overset{M}{\nabla}_{Z_1}\omega Z_2) = F(T_{Z_1}\phi Z_2 + \hat{\nabla}_{Z_1}\phi Z_2 + A_{\omega Z_2}Z_1 + T_{Z_1}\omega Z_2)$$

$$= \mathcal{B}T_{Z_1}\phi Z_2 + \mathcal{C}T_{Z_1}\phi Z_2 + \phi\hat{\nabla}_{Z_1}\phi Z_2 + \omega\hat{\nabla}_{Z_1}\phi Z_2$$

$$+ \mathcal{B}A_{\omega Z_2}Z_1 + \mathcal{C}A_{\omega Z_2}Z_1 + \phi T_{Z_1}\omega Z_2 + \omega T_{Z_1}\omega Z_2.$$

From above equation, it follows that $(ker\pi_*)$ defines a totally geodesic foliation if and only if

$$\mathcal{C}(T_{Z_1}\phi Z_2 + A_{\omega Z_2}Z_1) + \omega(\hat{\nabla}_{Z_1}\phi Z_2 + T_{Z_1}\omega Z_2) = 0.$$

Thus $\overset{M}{\nabla}_{Z_1}Z_2 \in \Gamma(ker\pi_*)$ if and only if

$$\mathcal{C}(T_{Z_1}\phi Z_2 + A_{\omega Z_2}Z_1) = 0, \omega(\hat{\nabla}_{Z_1}\phi Z_2 + T_{Z_1}\omega Z_2) = 0,$$

which completes proof.

From Theorem 3.5 and Theorem 3.6, we have the following result.

Theorem 3.8 *Let* $\pi:(M,g,F)\to(N,g')$ *be a generic Riemannian submersion from a locally product Riemannian manifold* (M,g,F) *onto a Riemannian manifold* (N,g'). *Then the total space* M *is a generic product manifold of the leaves of* D_1, D_2 *and* $(ker\pi_*)^\perp$, *i.e.,* $M = M_{D_1} \times M_{D_2} \times M_{(ker\pi_*)^\perp}$, *if and only if*

$$(\nabla\pi_*)(U_1, FV_1) \in \Gamma(\pi_*\mu),$$

$$(\nabla\pi_*)(U_2, FV_1) \in \Gamma(\pi_*\mu)$$

and

$$A_{X_1}\mathcal{B}X_2 + \mathcal{H}\overset{M}{\nabla}_{X_1}\mathcal{C}X_2 \in \Gamma(\mu), \mathcal{V}\overset{M}{\nabla}_{X_1}\mathcal{B}X_2 + A_{X_1}\mathcal{C}X_2 \in \Gamma(D_2)$$

for any $U_1, V_1 \in \Gamma(D_1)$, $U_2 \in \Gamma(D_2)$ and $X_1, X_2 \in \Gamma((ker\pi_*)^\perp)$, where M_{D_1}, M_{D_2} and $M_{(ker\pi_*)^\perp}$ are leaves of the distributions D_1, D_2 and $(ker\pi_*)^\perp$, respectively.

From Theorem 3.6 and Theorem 3.7, we have the following result.

Theorem 3.9 *Let* $\pi:(M,g,F)\to(N,g')$ *be a generic Riemannian submersion from a locally product Riemannian manifold* (M,g,F) *onto a Riemannian manifold* (N,g'). *Then the total space* M *is a generic product manifold of the leaves of* $(ker\pi_*)^{\perp}$ *and* $ker\pi_*$, *i.e.,* $M = M_{(ker\pi_*)^{\perp}}\times M_{ker\pi_*}$, *if and only if*

$$A_{X_1}\mathcal{B}X_2+\mathcal{H}\nabla^M_{X_1}\mathcal{C}X_2\in\Gamma(\mu),\ \mathcal{V}\nabla^M_{X_1}\mathcal{B}X_2+A_{X_1}\mathcal{C}X_2\in\Gamma(D_2)$$

and

$$T_{Z_1}\phi Z_2+A_{\omega Z_2}Z_1\in\Gamma(\omega D_2),\ \hat{\nabla}_{Z_1}\phi Z_2+T_{Z_1}\omega Z_2\in\Gamma(D_1)$$

for any $X_1,X_2\in\Gamma((ker\pi_*)^{\perp})$ and $Z_1,Z_2\in\Gamma(ker\pi_*)$, where $M_{(ker\pi_*)^{\perp}}$ and $M_{ker\pi_*}$ are leaves of the distributions $(ker\pi_*)^{\perp}$ and $ker\pi_*$, respectively.

Now, we give necessary and sufficient conditions for a generic Riemannian submersion to be totally geodesic. The Riemannian submersion map π is called totally geodesic map if the map π_* is parallel with respect to ∇, i.e., $\nabla\pi_*=0$. A geometric interpretation of a totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

Theorem 3.10 *Let* $\pi:(M,g,F)\to(N,g')$ *be a generic Riemannian submersion from a locally product Riemannian manifold* (M,g,F) *onto a Riemannian manifold* (N,g'). π *is a totally geodesic map if and only if*

$$\mathcal{C}T_{U_1}FV_1+\omega\hat{\nabla}_{U_1}FV_1=0,$$

$$\mathcal{C}(T_U\phi W+A_{\omega W}V)+\omega(\hat{\nabla}_V\phi W+T_V\omega W)=0,$$

$$\mathcal{C}(T_V\mathcal{B}X+A_{\mathcal{C}X}V)+\omega(\hat{\nabla}_V\mathcal{B}X+T_V\mathcal{C}X)=0,$$

for any $U_1,V_1\in\Gamma(D_1)$, $W\in\Gamma(D_2)$, $U\in\Gamma(ker\pi_*)$ and $X\in\Gamma((ker\pi_*)^{\perp})$.

Proof. For $X_1,X_2\in\Gamma((ker\pi_*)^{\perp})$, since π is a Riemannian submersion, from (10) we obtain

$$(\nabla\pi_*)(X_1,X_2)=0.$$

For $U_1,V_1\in\Gamma(D_1)$, using (2) and (10) we have

$$(\nabla\pi_*)(U_1,V_1)=-\pi_*(F\nabla_{U_1}FV_1).$$

Then from (5) we arrive at

$$(\nabla\pi_*)(U_1,V_1)=-\pi_*(F(T_{U_1}FV_1+\hat{\nabla}_{U_1}FV_1)).$$

Using (11) and (13) in above equation we obtain

$$(\nabla\pi_*)(U_1,V_1)=-\pi_*(\mathcal{B}T_{U_1}FV_1+\mathcal{C}T_{U_1}FV_1+\phi\hat{\nabla}_{U_1}FV_1+\omega\hat{\nabla}_{U_1}FV_1).$$

Since $\mathcal{B}T_{U_1}FV_1+\phi\hat{\nabla}_{U_1}FV_1\in\Gamma(ker\pi_*)$, we derive

$$(\nabla\pi_*)(U_1,V_1)=\pi_*(\mathcal{C}T_{U_1}FV_1+\omega\hat{\nabla}_{U_1}FV_1).$$

Then, since π is a linear isomorphism between $(ker\pi_*)^\perp$ and TM , $(\nabla\pi_*)(U_1,V_1)=0$ if and only if

$$\mathcal{C}T_{U_1}FV_1+\omega\hat{\nabla}_{U_1}FV_1=0. \tag{20}$$

For $U\in\Gamma(ker\pi_*), W\in\Gamma(D_2)$, using (2), (10) and (11), we have

$$(\nabla\pi_*)(U,W)=\nabla^\pi_U\pi_*W-\pi_*(\overset{M}{\nabla}_U W)=-\pi_*(F\overset{M}{\nabla}_U FW)=-\pi_*(F\overset{M}{\nabla}_U(\phi W+\omega W)).$$

Then from (6) we arrive at

$$(\nabla\pi_*)(U,W)=-\pi_*(F(T_U\phi W+\hat{\nabla}_V\phi W)+F(A_{\omega W}V+T_V\omega W)).$$

Using (11) and (13) in above equation we obtain

$$(\nabla\pi_*)(U,W)=-\pi_*((\mathcal{B}T_U\phi W+\mathcal{C}T_U\phi W)+(\phi\hat{\nabla}_V\phi W+\omega\hat{\nabla}_V\phi W)$$
$$+(\mathcal{B}A_{\omega W}V+\mathcal{C}A_{\omega W}V)+(\phi T_V\omega W+\omega T_V\omega W)).$$

Thus $(\nabla\pi_*)(V,W)=0$ if and only if

$$\mathcal{C}(T_U\phi W+A_{\omega W}V)+\omega(\hat{\nabla}_V\phi W+T_V\omega W)=0. \tag{21}$$

On the other hand using (2), (5), (6) and (13) for any $V\in\Gamma(ker\pi_*)$ and $X\in\Gamma((ker\pi_*)^\perp)$, we get

$$(\nabla\pi_*)(V,X)=\nabla^\pi_V\pi_*X-\pi_*(\overset{M}{\nabla}_V X)=-\pi_*(F\overset{M}{\nabla}_V FX)=-\pi_*(F\overset{M}{\nabla}_V(\mathcal{B}X+\mathcal{C}X))$$
$$=-\pi_*(\mathcal{B}T_V\mathcal{B}X+\mathcal{C}T_V\mathcal{B}X+\phi\hat{\nabla}_V\mathcal{B}X+\omega\hat{\nabla}_V\mathcal{B}X$$
$$+\mathcal{B}A_{\mathcal{C}X}V+\mathcal{C}A_{\mathcal{C}X}V+\phi T_V\mathcal{C}X+\omega T_V\mathcal{C}X).$$

Thus $(\nabla\pi_*)(V,X)=0$ if and only if

$$\mathcal{C}(T_V\mathcal{B}X+A_{\mathcal{C}X}V)+\omega(\hat{\nabla}_V\mathcal{B}X+T_V\mathcal{C}X)=0. \tag{22}$$

The result then follows from (20), (21) and (22).

CONFLICT OF INTEREST

No conflict of interest was declared by the authors

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