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TZITZEICA CURVES WITH Q-FRAME IN THREE-DIMENSIONAL MINKOWSKI SPACE

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Abstract. In this work, both timelike and spacelike Tzitzeica, spherical, and spherical Tzitzeica curves are analyzed in 3-dimensional Minkowski space by using q-frame. Tzitzeica and spherical curves are characterized using spacelike and timelike q-frames within the context of Minkowski three-space, and the theorems concerning spherical Tzitzeica curves are established.

1. Introduction

At the start of the 20th century, a Romanian Mathematician, named Gheorgha Tzitzeica, defined a space curve called the Tzitzeica curve, where the constant value is the ratio of the torsion to the square of the distance from the curve's origin to the osculating plane at any arbitrary point on the curve [20], [21].

After this Tzitzeica curve was defined, many researchers have studied this subject. Karacan and Bükcü worked on two different hyperbolic cylindrical Tzitzeica curve in 2008 and gave the condition for cylindrical curve being a Tzitzeica curve dealing with third order ordinary differential equation in three-dimensional Minkowski space in 2009 [11], [12]. In 2010, Agnew et al. presented a thorough definition of Tzitzeica curves and surfaces, predating Bobe et al.'s work in 2012. The latter researchers established the connections between Tzitzeica curves and surfaces in Minkowski spaces and their counterparts originating from Euclidean space [4], [7], [9], [18], facilitated by the introduction of three novel centro-affine invariant functions [1], [5]. In [3], both Tzitzeica curves and rectifying curves were

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discussed and Tzitzeica conditions were given for both spacelike and timelike helices and pseudospherical curves in $\mathbb{E}_1^3$. Furthermore, calculations were performed using a different frame, Bishop frame, for fixed-width space curves in Euclidean 3-space [10] and spacelike curves in Minkowski 3-space [6].

In this study, a new frame called q-frame, found in [8], [13]- [15], [23], is used to examine the Tzitzeica, spherical, and spherical Tzitzeica curves in 3-dimensional Minkowski space. The conditions being Tzitzeica curve and spherical Tzitzeica curve are analyzed for the both spacelike and timelike curves.

2. Preliminaries

Consider a real vector space denoted as V. A bilinear form on this vector space can be defined as a function, denoted as $\langle , \rangle : V \times V \to \mathbb{R}$. In three-dimensional Minkowski space $\mathbb{E}_1^3$, this function of two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ is expressed

$$\langle u, v \rangle = u_1 v_1 + u_2 v_2 - u_3 v_3.$$

A scalar product space is called Lorentz space when $v = 1$ and $dimV \geq 2$ [17].

Definition 1. *A tangent vector $u \in V$ is*

- *spacelike if $\langle u, u \rangle > 0$ or $u = 0$,*
- *timelike if $\langle u, u \rangle < 0$,*
- *null if $\langle u, u \rangle = 0$ and $u \neq 0$ [16], [17].*

The norm of the vector u is given by $\|u\| = |\langle u, u \rangle|^{1/2}$.

Definition 2. *Let Γ be the set of all timelike vectors in a Lorentz vector space V. For $u \in \Gamma$,*

$$C(u) = \{v \in \Gamma | \langle u, v \rangle > 0\}$$

is the timecone of V containing u [17].

Proposition 1. *Let u and v be timelike vectors in a Lorentz vector space. Then, it holds that:*

- *$|\langle u, v \rangle| \geq |u||v|$, with equality if and only if u and v are collinear.*
- *If u and v belong to the same timecone in $C(u)$, there exists a unique non-negative number $\theta \geq 0$, known as the hyperbolic angle between u and v, such that:*

$$\langle u, v \rangle = -|u||v| \cosh \theta.$$

The cross product of u and v in three-dimensional Minkowski space $\mathbb{E}_1^3$ is defined as

$$u \times v = (u_3 v_2 - u_2 v_3, u_1 v_3 - u_3 v_1, u_1 v_2 - u_2 v_1)$$

[2].

Definition 3. *The Lorentzian unit circle in the Lorentz plane* $\mathbb{R}^2_1$ *is given by the set*

$$S^1_1 = \{u \in \mathbb{R}^2_1 | \langle u, u\rangle = 1\}.$$

The tangent vectors of this Lorentzian circle are always timelike type. Besides, the hyperbolic unit circle shown in Figure 1 in the Lorentz plane is given by the set

$$H^1_0 = \{u \in \mathbb{R}^2_1 | \langle u, u\rangle = -1\}.$$

The tangent vectors of this hyperbolic unit circle are always spacelike [22].

Similarly, the Lorentzian unit sphere and hyperbolic sphere shown in Figure 2 are given

$$S^2_1 = \{v \in \mathbb{R}^3_1 | \langle v, v\rangle = 1\},$$

$$H^2_0 = \{v \in \mathbb{R}^3_1 | \langle v, v\rangle = -1\},$$

respectively [22].

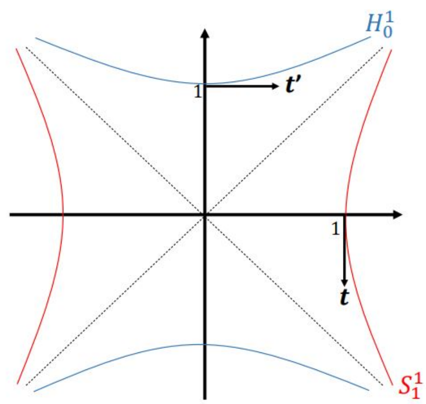

FIGURE 1. Lorentzian and hyperbolic unit circles

Definition 4. *The distance of a point* P *in space to a plane is called the length of the vector* $\overrightarrow{PS}$ *such that* S *is the foot of the perpendicular projection of* P *on the plane is* S.

$$l = d(P, S) = || \ \overrightarrow{PS} \ || = \frac{| < \overrightarrow{AP}, \overrightarrow{n} > |}{|| \ \overrightarrow{n} \ ||}$$

where A *be any point on the plane and* $\overrightarrow{n}$ *be the normal vector of the plane [24].*

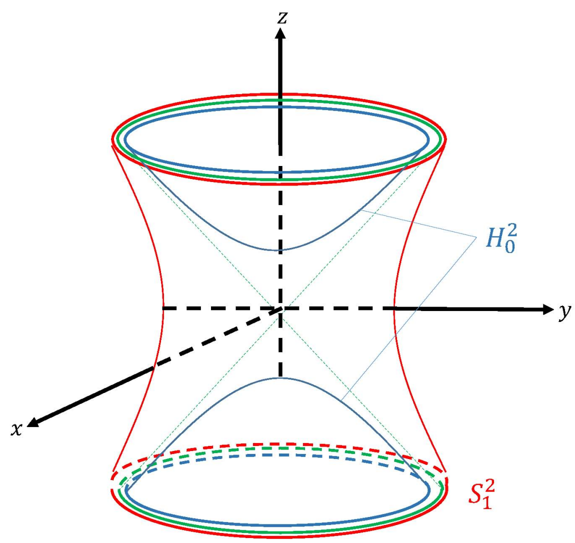

FIGURE 2. Lorentzian and hyperbolic unit spheres

3. Tzitzeica Curves in 3-Dimensional Minkowski Space

In this chapter, Tzitzeica and spherical curves are defined by using q-frame in Minkowski three-space. For these Tzitzeica and spherical curves, some results are given and they are characterized with respect to their curvatures. After defining spherical curve, the condition being Tzitzeica spherical curve is examined.

3.1. **Spacelike Tzitzeica Curves with q-frame in Minkowski 3-space.** In this part of our work, we deal with a spacelike curve that occurs when the projection vector $\mathbf{k}$ is timelike. For that spacelike curve, we examine both Tzitzeica and spherical curves, and then we work on Tzitzeica spherical curve. Lastly, investigations are shown on the Lorentz sphere.

Theorem 1. *The derivative formula of q-frame vectors for spacelike curve when* $\mathbf{t}$ *spacelike,* $\mathbf{k} = (0, 0, 1)$ *timelike,* $\mathbf{n_q}$ *spacelike, and* $\mathbf{b_q}$ *timelike is given*

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}'_\mathbf{q} \\ \mathbf{b}'_\mathbf{q} \end{bmatrix} = \begin{bmatrix} 0 & k_1 & -k_2 \\ -k_1 & 0 & -k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n_q} \\ \mathbf{b_q} \end{bmatrix}.$$

The q-curvatures are written

$$k_1 = \kappa \cosh \theta, \; k_2 = \kappa \sinh \theta \; ve \; k_3 = -d\theta - \tau$$

[19].

Definition 5. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}_1^3$ be a spacelike curve with arc-length parameter when $k_1 > 0$ and $k_3 \neq 0$. The curve α is called the Tzitzeica curve if the α satisfies the condition*

$$\frac{k_3}{d_{qos}^2} = a$$

with the distance d_{qos} of the curve from the origin of the q-osculator plane at the arbitrary point $\alpha(s)$. Here, $a \neq 0$ is a constant.
Using definition 4 and $\mathbf{b_q} = \mathbf{t} \times \mathbf{n_q}$, one can write

$$\begin{aligned} d(O, qos) &= d_{qos} \\ &= \left| \frac{\langle \alpha(s), \mathbf{t} \times \mathbf{n_q} \rangle}{\|\mathbf{t} \times \mathbf{n_q}\|} \right| \\ &= \left| \frac{\langle \alpha(s), \mathbf{b_q} \rangle}{\|\mathbf{b_q}\|} \right|. \end{aligned}$$

Since the timelike binormal vector b is a unit vector, from which the distance of the q-osculator plane to the origin is found in the form of

$$d_{qos} = |\langle \alpha(s), \mathbf{b_q} \rangle| \,. \tag{1}$$

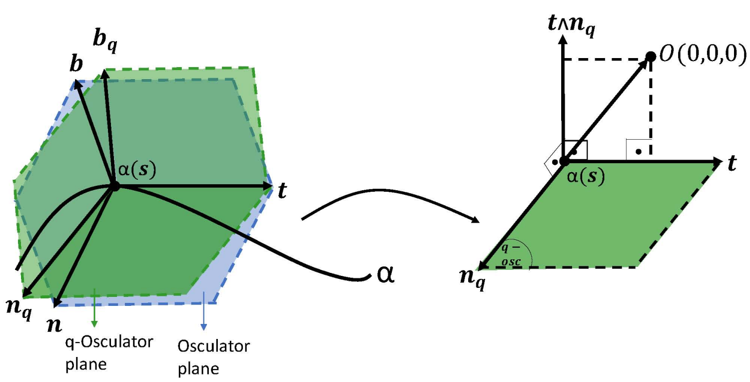

FIGURE 3. The distance d_{qos} of the q-osculator plane to the origin

Theorem 2. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}_1^3$ be a unit spacelike curve in $\mathbb{R}_1^3$. The curve α is called Tzitzeica curve when the following equality satisfies*

$$k_3' \langle \mathbf{b_q}, \alpha \rangle + 2k_2 k_3 \langle \mathbf{t}, \alpha \rangle + 2k_3^2 \langle \mathbf{n_q}, \alpha \rangle = 0.$$

Proof. Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}_1^3$ be a unit spacelike curve in $\mathbb{R}_1^3$. Assume that the curve α is Tzitzeica curve. Using definition 5 and equation (1), one can get

$$\frac{k_3}{\langle \mathbf{b_q}, \alpha \rangle^2} = a \neq 0. \tag{2}$$

Derivative of the last equation gives us

$$\frac{k_3' \langle \mathbf{b_q}, \alpha \rangle^2 - k_3(2 \langle \mathbf{b_q}, \alpha \rangle \langle \mathbf{b_q'}, \alpha \rangle)}{\langle \mathbf{b_q}, \alpha \rangle^4} = 0.$$

When the necessary simplifications are made, we can conclude

$$\frac{2k_3^2 \langle \mathbf{n_q}, \alpha \rangle + 2k_2 k_3 \langle \mathbf{t},\alpha \rangle + k_3' \langle \mathbf{b_q}, \alpha \rangle}{\langle \mathbf{b_q}, \alpha \rangle^3} = 0.$$

This gives us a proof of theorem. □

The spacelike spherical curve is written

$$\alpha(s) = s_1 \mathbf{t}(s) + s_2 \mathbf{n_q}(s) + s_3 \mathbf{b_q}(s)$$

with respect to q-frame vectors. One can write

$$\left\| \overrightarrow{O\alpha} \right\| = r$$

for the sphere with radius r. Using $\alpha(s) \in S_1^2$, the properties of symmetry of scalar product and the curve being unit speed, we obtain

$$s_1 = \langle \mathbf{t},\alpha \rangle = 0.$$

The first and second partial derivatives of this equation are as follows

$$k_1 \langle \mathbf{n_q},\alpha \rangle - k_2 \langle \mathbf{b_q},\alpha \rangle = 0 \tag{3}$$

and

$$\langle \mathbf{n_q},\alpha \rangle (k_1' + k_2 k_3) - \langle \mathbf{b_q},\alpha \rangle (k_2' + k_1 k_3) = 0 \tag{4}$$

respectively. When the equation (3) is multiplied by $-k_2' + k_1 k_3$ and the equation (4) is multiplied by k_2, and added together, the equation

$$\langle \mathbf{n}_q,\alpha \rangle = -\frac{1}{k_1 - \dfrac{k_2(k_1' + k_2 k_3)}{k_2' + k_1 k_3}}$$

is obtained. Similarly, multiplying the equation (3) by $-k_1' - k_2 k_3$ and the equation (4) by k_1, and adding together, we can get

$$\langle \mathbf{b_q},\alpha \rangle = \frac{1}{k_2 - k_1 \dfrac{(k_2' + k_1 k_3)}{k_1' + k_2 k_3}}.$$

Taking derivative of $\langle \mathbf{n_q},\alpha \rangle = s_2$ gives us

$$\langle \mathbf{b_q},\alpha \rangle = \frac{s_2'}{k_3} = s_3.$$

Since $\langle \mathbf{n}_q,\alpha \rangle = s_2$, one can get

$$s_2 = \frac{k_2' + k_1 k_3}{k_1 k_2' + k_1^2 k_3 - k_2 k_1' - k_2^2 k_3}.$$

Now, to find s_3, it is enough to take the square of the above equation, and divide by k_3. In these settings, we are able to find

$$\begin{aligned} s_3 \quad = \quad & \frac{1}{(k_1^2(\frac{k_2}{k_1})' + (k_1^2 - k_2^2)k_3)^2}[-k_1'(k_3(k_1^2 + k_2^2) + k_1 k_2' \\ & + \frac{k_1 k_2 k_3{}' + k_2 k_2''}{k_3}) + \frac{k_2' k_2 (2k_1 k_3^2 + k_1'' + k_2 k_3')}{k_3} \\ & - k_2((k_1')^2 - 2(k_2')^2 - k_1'' k_1 + k_2'' k_2)]. \end{aligned}$$

Corollary 1. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3_1$ be a unit speed spacelike curve in $\mathbb{R}^3_1$. α is a spherical Tzitzeica curve if and only if*

$$k_3' \langle \mathbf{b_q}, \alpha \rangle + 2k_3^2 \langle \mathbf{n_q}, \alpha \rangle = 0.$$

Theorem 3. *Let $M \subset \mathbb{R}^3_1$ be a spacelike curve with coordinate neighborhood (I, α). The geometric location of the centers of the spheres, which are the three common points of M and infinity, is*

$$a(s) = \alpha(s) + s_2(s)\mathbf{n_q}(s) + \lambda \mathbf{b_q}(s),$$

so that the q-vectors at the point $\alpha(s)$ corresponding to the point $s \in I$ are $\{\mathbf{t}(s), \mathbf{n_q}(s), \mathbf{b_q}(s)\}$ where $\lambda \in \mathbb{R}$ and $s_2 : I \longrightarrow \mathbb{R}$ is the same as the coefficient of $\mathbf{n_q}$ in the equation of the spherical curve.

Proof. Let $f : I \longrightarrow \mathbb{R},\ f(s) = \langle a - \alpha(s), a - \alpha(s) \rangle - r^2$. Since there are three common points with the spheres

$$S^2_1 = \{x \mid x \in \mathbb{R}^3_1, \langle x - a, x - a \rangle = r^2\}$$

of the point $a(s)$ of M, there must be

$$f(s) = f'(s) = f''(s) = 0.$$

Since $f(s) = 0$, the equality $\langle a - \alpha(s), a - \alpha(s) \rangle = r^2$ must be satisfied. Using this equality, we can get

$$\langle \mathbf{t}(s), a - \alpha(s) \rangle = 0.$$

With the help of derivation of the last equality and $f'(s) = 0$,

$$k_1(s) \langle \mathbf{n_q}(s), a - \alpha(s) \rangle - k_2(s) \langle \mathbf{b_q}(s), a - \alpha(s) \rangle + 1 = 0$$

is found. On the other hand, since $\langle a - \alpha(s), \mathbf{t}(s) \rangle = s_1(s)$, we have $s_1(s) = 0$. Similarly, we can get

$$\langle a - \alpha(s), \mathbf{n_q}(s) \rangle = s_2(s)$$

and

$$\langle a - \alpha(s), \mathbf{b_q}(s) \rangle = -s_3(s).$$

In this setting, we are able to obtain

$$s_1^2(s) + s_2^2(s) - s_3^2(s) = r^2.$$

Using the equalities of s_1 and s_2, it is easily found as

$$a = \alpha(s) + s_2(s)\mathbf{n_q}(s) + \lambda \mathbf{b_q}(s).$$

$\square$

Theorem 4. *Let $M \subset \mathbb{R}^3_1$ be a spacelike curve with coordinate neighborhood (I, α). For any $s \in I$, when $k_2 = 0$ at the point $\alpha(s)$, the radius of osculating sphere is constant if and only if the center of osculating sphere are the same such that $s_3 \neq 0$, $k_3 \neq 0$.*

Proof. The center of osculating sphere is written

$$a(s) = \alpha(s) + s_2(s)\mathbf{n_q}(s) + s_3(s)\mathbf{b_q}(s)$$

such that $\alpha(s) \in M$ and the radius

$$\begin{aligned} r &= \|\overrightarrow{\alpha a}\| \\ &= \|a - \alpha(s)\| \\ &= \sqrt{\langle s_2(s)\mathbf{n_q}(s) + s_3(s)\mathbf{b_q}(s), s_2(s)\mathbf{n_q}(s) + s_3(s)\mathbf{b_q}(s)\rangle} \\ &= \sqrt{s_2^2(s) - s_3^2(s)}. \end{aligned}$$

Taking derivative of $r^2 = s_2^2(s) - s_3^2(s)$, and using $s_3(s) = \dfrac{s_2'(s)}{k_3(s)}$, we can found

$$k_3(s)s_2(s) - s_3'(s) = 0. \tag{5}$$

From the equation $a(s) = \alpha(s) + s_2(s)\mathbf{n_q}(s) + s_3(s)\mathbf{b_q}(s)$, we have

$$D_{\dot{\alpha}} a(s) = (1 - s_2(s)k_1(s) - s_3(s)k_2(s))\mathbf{t}(s) + (-s_2(s)k_3(s) + s_3'(s))\mathbf{b_q}(s).$$

After using the equality of s_2, s_3 and $k_2 = 0$, one can get

$$D_{\dot{\alpha}} a(s) = (-s_2(s)k_3(s) + s_3'(s))\mathbf{b_q}(s).$$

In the light of equation (5), for any $s \in I$, that $a(s)$ is a constant is obtained. On the other hand, let $a(s)$ be a constant for any $s \in I$. Since $r = \|\overrightarrow{\alpha a}\|$, we have

$$\langle a(s) - \alpha(s), a(s) - \alpha(s)\rangle = r^2(s).$$

Taking derivative of this equation gives

$$\left\langle D_{\dot{\alpha}} a(s), a(s) - \alpha(s)\right\rangle = r(s) \left.\frac{dr}{ds}\right|_s.$$

We then have

$$r(s) \left.\frac{dr}{ds}\right|_s = 0.$$

Either $r(s) = 0$ or $\left.\dfrac{dr}{ds}\right|_s = 0$ is provided. Being $r(s) = 0$ contradicts both $s_2 = s_3 = 0$. Therefore, $\left.\dfrac{dr}{ds}\right|_s = 0$. We then conclude the proof by finding $r(s) = 0$ for any $s \in I$. $\square$

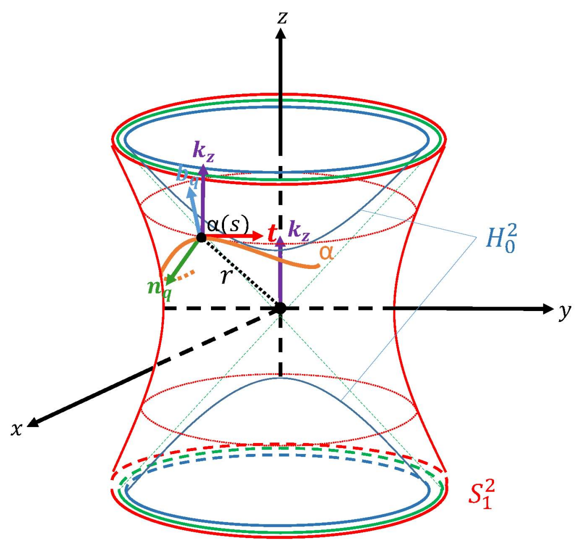

FIGURE 4. Spacelike spherical curve with q-frame in Minkowski 3-space

3.2. **Timelike Tzitzeica Curves with q-frame in Minkowski 3-space.** In this part, we work on the similar theorems in previous section for timelike curve when the projection vector k is spacelike. Since the proofs are also made in similar ways as in the case of the spacelike curve, we omit them in this case.

Theorem 5. *The derivative formula of q-frame vectors for timelike curve when* $\mathbf{t}$ *timelike,* $\mathbf{k} = (0, 1, 0)$ *spacelike,* $\mathbf{n_q}$ *spacelike and* $\mathbf{b_q}$ *spacelike is given*

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}'_\mathbf{q} \\ \mathbf{b}'_\mathbf{q} \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & k_3 \\ k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n_q} \\ \mathbf{b_q} \end{bmatrix}.$$

The q-curvatures are written

$$k_1 = \kappa \cos\theta, k_2 = -\kappa \sin\theta, k_3 = d\theta + \tau$$

[19].

Definition 6. *Let* $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3_1$ *be a timelike curve with arc-length parameter when* $k_1 > 0$ *and* $k_3 \neq 0$. *The curve* α *is called the Tzitzeica curve if the* α *satisfies the condition*

$$\frac{k_3}{d^2_{qos}} = a$$

with the distance d_{qos} of the curve from the origin of the q-osculator plane at the arbitrary point $\alpha(s)$. Here, $a \neq 0$ is a constant.
Using definition 6 and $\mathbf{b_q} = \mathbf{t} \times \mathbf{n_q}$, one can write

$$\begin{aligned} d(O, qos) &= d_{qos} \\ &= \left| \frac{\langle \alpha(s), \mathbf{t} \times \mathbf{n_q} \rangle}{\|\mathbf{t} \times \mathbf{n_q}\|} \right| \\ &= \left| \frac{\langle \alpha(s), \mathbf{b_q} \rangle}{\|\mathbf{b_q}\|} \right|. \end{aligned}$$

Since the spacelike binormal vector b is a unit vector, from which the distance of the q-osculator plane to the origin is found in the form of

$$d_{qos} = |\langle \alpha(s), \mathbf{b_q} \rangle| = |\langle \mathbf{b_q}, \alpha(s) \rangle|.$$

Corollary 2. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3_1$ be a unit speed timelike curve in $\mathbb{R}^3_1$. α is a spherical Tzitzeica curve if and only if*

$$k_3' \langle \mathbf{b_q}, \alpha \rangle - 2k_2k_3 \langle \mathbf{t}, \alpha \rangle + 2k_3^2 \langle \mathbf{n_q}, \alpha \rangle = 0.$$

The timelike spherical curve is given

$$\alpha(s) = s_1 \mathbf{t}(s) + s_2 \mathbf{n_q}(s) + s_3 \mathbf{b_q}(s)$$

with respect to q-frame vectors. In the light of recent calculations given above section, one can find

$$\begin{aligned} s_1 &= \langle \mathbf{t}, \alpha \rangle = 0, \\ s_2 &= \frac{k_2' + k_1k_3}{-k_1k_2' - k_1^2k_3 + k_2k_1' - k_2^2k_3}, \\ s_3 = \frac{s_2'}{k_3} &= \frac{1}{(k_2^2(\frac{k_1}{k_2})' - (k_1^2 + k_2^2)k_3)^2}\Big[\frac{k_2'k_2(2k_1k_3^2 - k_1'' + k_2k_3')}{k_3} \\ &\quad + k_1'(k_3(k_1^2 - k_2^2) + \frac{k_1k_2' + k_1k_2k_3{}' + k_2k_2''}{k_3}) \\ &\quad + k_2(2(k_2')^2 + (k_1')^2 - k_1''k_1 - k_2''k_2)\Big]. \end{aligned}$$

Corollary 3. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3_1$ be a unit speed timelike curve in $\mathbb{R}^3_1$. α is a spherical Tzitzeica curve if and only if*

$$k_3' \langle \mathbf{b_q}, \alpha \rangle + 2k_3^2 \langle \mathbf{n_q}, \alpha \rangle = 0.$$

Theorem 6. *Let $M \subset \mathbb{R}^3_1$ be a timelike curve with coordinate neighborhood (I, α). The geometric location of the centers of the spheres, which are the three common points of M and infinity, is*

$$a(s) = \alpha(s) + s_2(s)\mathbf{n_q}(s) + \lambda \mathbf{b_q}(s),$$

so that the q-vectors at the point $\alpha(s)$ corresponding to the point $s \in I$ are $\{\mathbf{t}(s), \mathbf{n_q}(s), \mathbf{b_q}(s)\}$ where $\lambda \in \mathbb{R}$ and $s_2 : I \longrightarrow \mathbb{R}$ is the same as the coefficient of $\mathbf{n_q}$ in the equation of the spherical curve.

Theorem 7. *Let $M \subset \mathbb{R}^3_1$ be a timelike curve with coordinate neighborhood (I, α). For any $s \in I$, when $k_2 = 0$ at the point $\alpha(s)$, the radius of osculating sphere is constant if and only if the centers of osculating spheres are the same such that $s_3 \neq 0$, $k_3 \neq 0$.*

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