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Lower and Upper Bounds for Some Degree-Based Indices of Graphs

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Abstract

Topological indices are mathematical measurements regarding the chemical structures of any simple finite graph. These are used for QSAR and QSPR studies. We get bounds for some degree based topological indices of a graph using solely the vertex degrees. We obtain upper and lower bounds for these indices and investigate for the complete graphs, path graphs and Fibonacci-sum graphs.

1. Introduction

Topological indices are important for the graph theory studies. Several significant topological indices such as Zagreb index, Randic index and Wiener index has been introduced to measure the characters of graphs.

Now, we recall the definitions of some topological indices we used in this study:

The multiplicative Randic index is defined in [1] as

$$MR(G) = \prod_{uv \in E(G)} \sqrt{\frac{1}{\deg(u)\deg(v)}}.$$

The reduced reciprocal Randic index was described in [1] as

$$RRR(G) = \sum_{uv \in E(G)} \sqrt{(\deg(u)-1)(\deg(v)-1)}.$$

The Narumi-Katayama index was introduced in [2] as

$$NK(G) = \prod_{i=1}^{n} \deg(v_i).$$

The symmetric division deg index was described in [3] as

$$SD(G) = \sum_{uv \in E(G)} \frac{\deg(u)^2 + \deg(v)^2}{\deg(u)\deg(v)}.$$

In literature, there are some studies including these indices such as [4, 5, 6, 7].

In [8], a Fibonacci-sum graph was defined as $G_n = (V, E)$, where $V = [n] = \{F_2 = 1, F_3 = 2, F_4 = 3, 4, 5, \cdots, n\}$ is the vertex set and $E = \{\{i, j\} : i, j \in V,\ i \neq j,\ i + j$ is a Fibonacci number$\}$ is the edge set.

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It is obvious from the definition that G_n is a simple graph.

Also, some properties of the Fibonacci-sum graphs were obtained in the following theorems [9]:

Lemma 1.1. *G_n is connected for each $n \geq 1$.*

Lemma 1.2. *Let $n \geq 2$, and t be any positive integer satisfy that $F_t \leq n < F_{t+1}$. Then the only neighbour of the vertex F_t is F_{t-1}.*

Lemma 1.3. *Let $n \geq 1$ and let $y \in [1,n]$. Let for $t \geq 2$, $F_t \leq y < F_{t+1}$ and for $l \geq t$, $F_l \leq y+n < F_{l+1}$. Then the degree of y is*

$$\deg_{G_n}(y) = \begin{cases} l-t, & \text{if } 2y \text{ is not a Fibonacci number,} \\ l-t-1, & \text{if } 2y \text{ is a Fibonacci number.} \end{cases}$$

Theorem 1.4. *Vertex 2 has maximum degree in the Fibonacci-sum graph G_n (for any $n \geq 2$). Also, if $n+2$ is a Fibonacci number, then $\deg_{G_n}(1) = \deg_{G_n}(2) - 1$; otherwise, $\deg_{G_n}(1) = \deg_{G_n}(2)$.*

As a result of the above theorem, in the Fibonacci-sum graph G_n, 2 has the maximum degree and one of the vertices with maximum degree less than the degree of 2 is 1. Also, by Lemma 1.2 $d(F_k) = 1$ for $F_k \leq n < F_{k+1}$. Thus, for any $i \in V(G_n)$, we have

$$d(2) \geq d(1) \geq d(i) \geq d(F_k) \tag{1.1}$$

where $F_k \leq n < F_{k+1}$. In this case, we get

$$F_{l_1} \leq 2+n < F_{l_1+1}, \text{ then } \deg(2) = l_1 - 3, \tag{1.2}$$

$$F_{l_2} \leq 1+n < F_{l_2+1}, \text{ then } \deg(1) = l_2 - 3. \tag{1.3}$$

In [10], the spectral properties of Fibonacci-sum and Lucas-sum graphs were examined and some bounds were obtained. Also, in [11] another type of graphs associated with Fibonacci numbers was studied.

The aim of this study is obtain upper and lower bounds of multiplicative Randic index, reduced reciprocal Randic index, Narumi-Katayama index and symmetric division index for the general graphs using vertex degree. Then, we obtain upper and lower bounds for these indices for some special graphs and Fibonacci-sum graphs. Finally, we compared the bounds on these indices for some graphs.

2. Main results

In this section all of the theorems are given for $n \geq 3$.

Theorem 2.1. *Let G be a simple connected graph with n vertices, k pendant vertices and m edges. Then we get*

$$\left(\frac{1}{n-1}\right)^m \leq MR(G) \leq \left(\frac{1}{2}\right)^{\frac{2m-k}{2}}.$$

The lower bound holds for $G \cong K_n$ and the upper bound holds for $G \cong P_n$.

Proof. Since the graph has k pendant vertices and the other vertices is of at least degree 2, we get the upper bound for the multiplicative Randic index of G as

$$MR(G) \leq \left(\frac{1}{\sqrt{2}}\right)^k \left(\frac{1}{2}\right)^{n-1-k} = \left(\frac{1}{2}\right)^{\frac{2m-k}{2}}.$$

Also, since the vertices have the maximum degree at most $n-1$, we have the lower bound for the multiplicative Randic index of G as

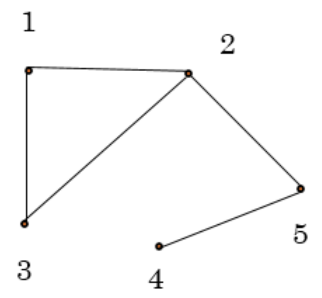
$$\left(\frac{1}{n-1}\right)^m \leq MR(G).$$

As a conclusion, we obtain

$$\left(\frac{1}{n-1}\right)^m \leq MR(G) \leq \left(\frac{1}{2}\right)^{\frac{2m-k}{2}}.$$

□

Figure 1: Simple connected graph



Example 2.2. *For the given graph in Figure 1 the bounds for the multiplicative Randic index are*

$$0.0009 \leq MR(G) = 0.044 \leq 0.024.$$

Corollary 2.3. *Let $G = K_{p,q}$. If $p < q$, then*

$$\left(\frac{1}{q}\right)^{pq} \leq MR(K_{p,q}) \leq \left(\frac{1}{p}\right)^{pq}.$$

If $p = q$, then

$$MR(K_{p,q}) = \left(\frac{1}{p}\right)^{p^2}.$$

Proof. Since the $K_{p,q}$ graph has pq edges, the proof can be seen easily. □

Theorem 2.4. *If G_n is a Fibonacci-sum graph, then*

$$\left(\frac{1}{\sqrt{(l_1-3)(l_2-3)}}\right)^{n-1} \leq MR(G_n) \leq \left(\frac{1}{\sqrt{2}}\right)^{n-r}$$

where l_1,l_2 are integers in (1.2), (1.3), respectively and r is the number of the vertices with degree 1 in G_n.

Proof. Since r is the number of the vertices with degree 1 in G_n, the degrees of the other vertices are at least 2. Thus, there are r vertices with degree 1 and $n-r$ vertices with degree at least 2. Hence, we get the upper bound for the multiplicative Randic index of G_n as

$$MR(G_n) \leq \left(\frac{1}{\sqrt{2}}\right)^{n-r}.$$

Also, since by Theorem 1.4, 2 has the maximum degree and one of the vertices with maximum degree less than the degree of 2 is 1, we have the lower bound for the multiplicative Randic index of G_n as

$$\left(\frac{1}{\sqrt{\deg(2)\deg(1)}}\right)^{n-1} \leq MR(G_n).$$

As a conclusion, we obtain

$$\left(\frac{1}{\sqrt{(l_1-3)(l_2-3)}}\right)^{n-1} \leq MR(G_n) \leq \left(\frac{1}{\sqrt{2}}\right)^{n-r}.$$

□

Theorem 2.5. *Let G be a simple connected graph with n vertices, k pendant vertices and m edges, then*

$$m-k \leq RRR(G) \leq m(n-2).$$

The lower bound holds for $G \cong P_n$ and the upper bound holds for $G \cong K_n$.

Proof. Since the graph has k pendant vertices and the other vertices is of at least degree 2, we have the lower bound as

$$m-k \leq RRR(G).$$

Also, since the vertices have the maximum degree at most $n-1$, we have the upper bound as

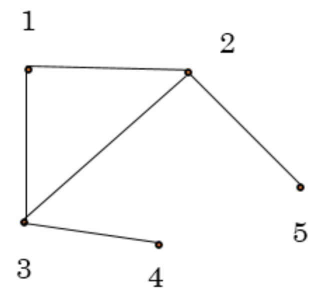
$$RRR(G) \leq m(n-2).$$

As a conclusion, we obtain

$$m-k \leq RRR(G) \leq m(n-2).$$

□

Figure 2: Simple connected graph



Example 2.6. *For the given graph in Figure 2 the bounds for the reduced reciprocal Randic index are*

$$3 \leq RRR(G) = 4.82 \leq 15.$$

Corollary 2.7. *Let $G = K_{p,q}$. If $p < q$, then*

$$RRR(K_{p,q}) = pq\sqrt{(p-1)(q-1)}.$$

If $p = q$

$$RRR(K_{p,q}) = p^2(p-1).$$

Proof. Since $m = pq$ in $K_{p,q}$,the proof is trivial. □

Theorem 2.8. *If G_n is a Fibonacci-sum graph, then*

$$m \leq RRR(G_n) \leq m\sqrt{(l_1-4)(l_2-4)}$$

where l_1,l_2 are the integers in (1.2), (1.3), respectively, and $m = |E(G_n)|$.

Proof. By Lemma 1.2, in the Fibonacci-sum graph G_n, F_t is adjacent to only F_{t-1} for $F_t \leq n < F_{t+1}$. Also, since the other neighbour of F_{t-1} is F_{t-2}, $\deg(F_{t-1}) = 2$. By the same way, $\deg(F_{t-2}) \geq 2$. Thus, we get the lower bound for the reduced reciprocal Randic index of G_n as

$$m\sqrt{\deg(F_{t-1}-1)\deg(F_{t-2}-1)} = m \leq RRR(G_n).$$

Since $1 \sim 2$ and by using (1.1), we get the upper bound for the reduced reciprocal Randic index of G_n as

$$RRR(G_n) \leq m\sqrt{(\deg(1)-1)(\deg(2)-1)}.$$

Hence, we obtain

$$m \leq RRR(G_n) \leq m\sqrt{(l_1-4)(l_2-4)}.$$

□

Theorem 2.9. *Let G be a simple connected graph with n vertices and k pendant vertices then*

$$2^{n-k} \leq NK(G) \leq (n-1)^n$$

The lower bound holds for $G \cong P_n$ and the upper bound holds for $G \cong K_n$.

Proof. Since the graph has k pendant vertices and the other vertices is of at least degree 2, we obtain the lower bound as

$$2^{n-k} \leq NK(G).$$

Also, since the vertices have the maximum degree at most $n-1$, we get the upper bound as

$$NK(G) \leq (n-1)^n.$$

□

Example 2.10. *For the given graph in Figure 2 the bounds for the Narumi-Katayama index are*

$$8 \leq NK(G) = 18 \leq 1024.$$

Corollary 2.11. *Let $G = K_{p,q}$ then*

$$NK(K_{p,q}) = p^q q^p.$$

Proof. Since there are q points of degree p and p points of degree q in the graph $K_{p,q}$, we obtain $NK(K_{p,q}) = p^q q^p$. □

Theorem 2.12. *For the Narumi-Katayama index of the Fibonacci-sum graph G_n, the following inequality holds:*

$$2^{n-r} \leq NK(G_n) \leq (l_1-3)(l_2-3)^{n-1}$$

where l_1,l_2 are the integers in (1.2), (1.3) , respectively and r is the number of the vertices with degree 1 in G.

Proof. Since r is the number of the vertices with degree 1 in G_n, then the degrees of the other vertices are at least 2. Thus, there are r vertices with degree 1 and $n-r$ vertices with degree at least 2. Hence, we get the lower bound for the Narumi-Katayama index of G_n as

$$2^{n-r} \leq NK(G_n).$$

Also, since by Theorem 1.4, 2 has the maximum degree and one of the vertices with maximum degree less than the degree of 2 is 1, we have the upper bound for the Narumi-Katayama index of G_n as

$$NK(G_n) \leq \deg(2)(\deg(1))^{n-1}.$$

As a result, we obtain

$$2^{n-r} \leq NK(G_n) \leq (l_1-3)(l_2-3)^{n-1}.$$

□

Theorem 2.13. *Let G be a simple connected graph with n vertices and m edges, then*

$$2m \leq SD(G) \leq m\frac{(n-1)^2+1}{n-1}.$$

Proof. If $\deg(u)$ is maximum and $\deg(v)$ is minimum, then the expression

$$\frac{\deg(u)^2+\deg(v)^2}{\deg(u)\deg(v)} \tag{2.1}$$

takes its maximum value. In G, $n-1$ is the maximum degree and if we take the pendant vertex which is adjacent to $n-1$, then the expression (2.1) takes its maximum value. Thus, we get

$$SD(G) = \frac{\deg(u)^2+\deg(v)^2}{\deg(u)\deg(v)} \leq m\frac{(n-1)^2+1}{n-1}.$$

In other way, when $\deg(u)$ and $\deg(v)$ are equal, then the expression (2.1) takes its minimum value. Thus, we get

$$2m \leq SD(G).$$

Hence, we obtain

$$2m \leq SD(G) \leq m\frac{(n-1)^2+1}{n-1}.$$

□

Example 2.14. *For the given graph in Figure 1 the bounds for the symmetric division index are*

$$10 \leq SD(G) = 11 \leq 21.25.$$

Corollary 2.15. *Let $G = K_{p,q}$ then*

$$SD(K_{p,q}) = p^2 + q^2.$$

Proof. Since $m = pq$ in $K_{p,q}$, the proof is trivial. □

Theorem 2.16. *If G_n is a Fibonacci-sum graph, then*

$$2m \leq SD(G_n) \leq m\,(l_1 - 2)$$

where l_1 is the integer in (1.2) and $m = |E(G_n)|$.

Proof. If $\deg(u)$ is maximum and $\deg(v)$ is minimum, then the expression

$$\frac{\deg(u)^2 + \deg(v)^2}{\deg(u)\deg(v)} \tag{2.2}$$

takes its maximum value. In G_n, 2 has the maximum degree and if we take the 1 degreed vertex which is adjacent to 2, then the expression (2.2) takes its maximum value. Thus we have

$$\frac{\deg(u)^2 + \deg(v)^2}{\deg(u)\deg(v)} \leq \deg(2) + 1.$$

Hence, we get the upper bound for the symmetric division index of G_n as

$$SD(G_n) \leq m\,(l_1 - 2).$$

In other way, when $\deg(u)$ and $\deg(v)$ are equal, then the expression (2.2) takes its minimum value. Thus we have

$$2 \leq \frac{\deg(u)^2 + \deg(v)^2}{\deg(u)\deg(v)}.$$

Hence, we get

$$2m \leq SD(G_n).$$

In conclusion, we obtain

$$2m \leq SD(G_n) \leq m\,(l_1 - 2).$$

□

Declarations

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