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Hardy Type Inequalities for Convex Functions

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Abstract

In this article, we aim to extend the scope of Hardy type inequalities by exploring their applicability to convex functions. We present various types of new Hardy integral inequalities for convex functions, which can be applied in diverse scenarios. Additionally, we provide several practical applications of these inequalities.

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1. Introduction

The Hardy integral inequality is a fundamental result in analysis that has wide-ranging applications in various branches of mathematics, physics, and engineering. It provides a powerful tool for studying the behaviour of integrable functions and is a crucial component in the development of numerous mathematical techniques. The inequality is named after the British mathematician G.H. Hardy, who first formulated it in the early 20th century. Classical Hardy's integral inequality to stated without proof in 1920 by G. H. Hardy [3]:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(s)ds\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty (f(x))^p dx, \tag{1.1}$$

where $p>1$, $x>0$, f is a nonnegative measurable function on $(0,\infty)$ and the constant $\left(\frac{p}{p-1}\right)^p$ is the best possible. This interesting result was later proved by Hardy himself in 1925 and 1928, respectively [4]and [5] like as the following integral inequalities:
Let f non-negative measurable function on $(0,\infty)$,
Classical Hardy's Inequality:

$$F(x)=\begin{cases} \int_0^x f(t)dt & for \quad m>1, \\ \int_x^\infty f(t)dt & for \quad m<1, \end{cases}$$

then

$$\int_0^\infty x^{-m}F^p(x)dx \le \left(\frac{p}{|m-1|}\right)^p \int_0^\infty x^{-m}\left(xf(x)\right)^p dx, \quad \text{for } p\ge 1. \tag{1.2}$$

The Hardy inequalities have been extensively studied and generalized in various directions, leading to a rich theory with many important ramifications. It has found numerous applications in areas such as harmonic analysis, partial differential equations, probability theory, and spectral theory, among others. Over the last twenty years a large number of papers have been appeared in the literature which deals with the simple proofs, various generalizations and discrete analogues of Hardy's inequality and its generalizations, see [1], [2], [6]-[15].

The purpose of this paper is to establish some generalizations of Hardy type inequalities for convex functions. Then, new Hardy integral inequalities of different kinds, which will be recovered in different cases, will be given for convex functions. Some applications will also be given.

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2. Main Results

Now, we present the main results:

Theorem 2.1. *Let f be a nonnegative convex function on $[a,b]\subset(0,\infty)$, and $p>1$. Also assume w be a nonnegative integrable function on $[a,b]$. Then, we have the following inequalities*

$$\int_a^b w(t)\,T_1^p f(t)\,dt \le \frac{1}{2^p}\int_a^b w(t)(t-a)^p (f(a)+f(t))^p\,dt \tag{2.1}$$

and

$$\int_a^b w(t)\,T_2^p f(t)\,dt \le \frac{1}{2^p}\int_a^b w(t)(b-t)^p (f(b)+f(t))^p\,dt \tag{2.2}$$

where

$$T_1 f(t)=\int_a^t f(s)\,ds \quad \text{and} \quad T_2 f(t)=\int_t^b f(s)\,ds.$$

Proof. Since f is convex functions on $[a,t],\ [t,b]\subset[a,b]$, then

$$\begin{aligned} T_1 f(t) &= \int_a^t f(s)\,ds=\int_a^t f\left(\frac{t-s}{t-a}a+\frac{s-a}{t-a}t\right)ds \\ &\le \int_a^t \left(\frac{t-s}{t-a}f(a)+\frac{s-a}{t-a}f(t)\right)ds \\ &= \frac{t-a}{2}(f(a)+f(t)) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} T_2 f(t) &= \int_t^b f(s)\,ds=\int_t^b f\left(\frac{b-s}{b-t}t+\frac{s-t}{b-t}b\right)ds \\ &\le \int_t^b \left(\frac{b-s}{b-t}f(t)+\frac{s-t}{b-t}f(b)\right)ds \\ &= \frac{b-t}{2}(f(b)+f(t)). \end{aligned} \tag{2.4}$$

By using (2.3) and (2.4) we get

$$\int_a^b w(t)\,T_1^p f(t)\,dt \le \frac{1}{2^p}\int_a^b w(t)(t-a)^p (f(a)+f(t))^p\,dt$$

and

$$\int_a^b w(t)\,T_2^p f(t)\,dt \le \frac{1}{2^p}\int_a^b w(t)(b-t)^p (f(b)+f(t))^p\,dt$$

which give the required inequalities. □

Remark 2.2. *In Theorem 2.1*
i) if $w(t)=\frac{1}{t^p}$ on $(0,\infty)$and $f(0)=0$, then the inequality (2.1) reduces to

$$\int_0^\infty \left(\frac{1}{t}\int_0^t f(s)ds\right)^p dt \le \frac{1}{2^p}\int_0^\infty f^p(t)\,dt \tag{2.5}$$

which is the Hardy inequality such that $\frac{1}{2^p}<\left(\frac{p}{p-1}\right)^p$ for $p>1$.
ii) if $w(t)=\frac{1}{(b-t)^p}$ on $[a,b]$, then the inequality (2.2) reduces to

$$\int_a^b \left(\frac{1}{b-t}\int_t^b f(s)ds\right)^p dt \le \frac{1}{2^p}\int_a^b (f(b)+f(t))^p\,dt.$$

Theorem 2.3. *With the assumptations in Theorem 2.1. Then, we have the following inequalitiy*

$$\int_a^b w(t)\left[T_1^p f(t)+T_2^p f(t)\right]dt \le \frac{(b^p-a^p)}{2}\left[f^p(a)+f^p(b)\right]\int_a^b w(t)\,dt+(b^p-a^p)\int_a^b w(t)\,f^p(t)\,dt. \tag{2.6}$$

Proof. By adding (2.1) and (2.2), we have

$$\begin{aligned}\int_a^b w(t)\left[T_1^p f(t)+T_2^p f(t)\right]dt &\le \frac{1}{2^p}\int_a^b w(t)\left[(t-a)^p+(b-t)^p\right]\left[(f(a)+f(t))^p+(f(b)+f(t))^p\right]dt\\ &\le \frac{(b^p-a^p)}{2}\int_a^b w(t)\left[(f^p(a)+f^p(b)+2f^p(t))\right]dt\\ &= \frac{(b^p-a^p)}{2}\left[f^p(a)+f^p(b)\right]\int_a^b w(t)\,dt+(b^p-a^p)\int_a^b w(t)f^p(t)\,dt.\end{aligned}$$

Here, we use

$(A-B)^p \le A^p - B^p$

and

$(A+B)^p \le 2^{p-1}(A^p+B^p)$

for any$A > B \ge 0$ and $p > 1$. This completes the proof. □

Remark 2.4. *In Theorem 2.3*
i) if $w(t)=\frac{1}{t^p}$ on $[a,b]$, then the inequality (2.6) reduces to

$$\int_a^b\left[\left(\frac{1}{t}\int_a^t f(s)\,ds\right)^p+\left(\frac{1}{t}\int_t^b f(s)\,ds\right)^p\right]dt \le \frac{(b^p-a^p)\left(b^{p-1}-a^{p-1}\right)}{2(p-1)(ab)^{p-1}}\left[f^p(a)+f^p(b)\right]+(b^p-a^p)\int_a^b\left(\frac{f(t)}{t}\right)^p dt. \tag{2.7}$$

Theorem 2.5. *With the assumptations in Theorem 2.1. Then, we have the following inequalitiy*

$$\int_a^b w(t)T_1^p f(t)\,dt \le \frac{pf(a)}{2^{p-1}}\int_a^b W(t)(t-a)^{p-1}(f(a)+f(t))^{p-1}dt+\frac{p}{(b-a)2^{p-1}}\left[f(b)-f(a)\right]\int_a^b W(t)(t-a)^p(f(a)+f(t))^{p-1}dt \tag{2.8}$$

where

$$W(t)=\int_t^b w(s)\,ds.$$

Proof. Integrating by parts, we get

$$\int_a^b w(t)T_1^p f(t)\,dt = -W(t)T_1^p f(t)\Big|_a^b+p\int_a^b W(t).T_1^{p-1}f(t).f(t)\,dt.$$

Using (2.3) and convexity of f on $[a,b]$, we conclude

$$\begin{aligned}\int_a^b w(t)T_1^p f(t)\,dt &\le \frac{p}{2^{p-1}}\int_a^b W(t)(t-a)^{p-1}(f(a)+f(t))^{p-1}\left(\frac{b-t}{b-a}f(a)+\frac{t-a}{b-a}f(b)\right)dt\\ &= \frac{pf(a)}{(b-a)2^{p-1}}\int_a^b W(t)(t-a)^{p-1}(b-t)(f(a)+f(t))^{p-1}dt\\ &\quad+\frac{pf(b)}{(b-a)2^{p-1}}\int_a^b W(t)(t-a)^p(f(a)+f(t))^{p-1}dt\\ &= \frac{pf(a)}{2^{p-1}}\int_a^b W(t)(t-a)^{p-1}(f(a)+f(t))^{p-1}dt\\ &\quad+\frac{p}{(b-a)2^{p-1}}\left[f(b)-f(a)\right]\int_a^b W(t)(t-a)^p(f(a)+f(t))^{p-1}dt.\end{aligned}$$

This completes the proof of the inequality (2.8). □

Theorem 2.6. *With the assumptations in Theorem 2.1. Then, we have the following inequalitiy*

$$\int_a^b w(t)T_2^p f(t)\,dt \le \frac{pf(b)}{2^{p-1}}\int_a^b H(t)(b-t)^{p-1}(f(b)+f(t))^{p-1}dt+\frac{p}{(b-a)2^{p-1}}\left[f(a)-f(b)\right]\int_a^b H(t)(b-t)^p(f(b)+f(t))^{p-1}dt \tag{2.9}$$

where

$$H(t)=\int_a^t w(s)\,ds.$$

Proof. Integrating by parts, we get

$$\int_a^b w(t)\, T_2^p f(t)\, dt = H(t)\, T_2^p f(t)\big|_a^b + p\int_a^b H(t)\, .T_2^{p-1} f(t)\, .f(t)\, dt$$

Using (2.3) and convexity of f on $[a,b]$, we conclude that

$$\int_a^b w(t)\, T_2^p f(t)\, dt \leq \frac{p}{2^{p-1}}\int_a^b H(t)(b-t)^{p-1}(f(b)+f(t))^{p-1}\left(\frac{b-t}{b-a}f(a)+\frac{t-a}{b-a}f(b)\right)dt$$

$$= \frac{pf(a)}{(b-a)\,2^{p-1}}\int_a^b H(t)(b-t)^{p}(f(b)+f(t))^{p-1}\, dt$$

$$+ \frac{pf(b)}{(b-a)\,2^{p-1}}\int_a^b H(t)(b-t)^{p-1}(t-a)(f(b)+f(t))^{p-1}\, dt$$

$$= \frac{pf(b)}{2^{p-1}}\int_a^b H(t)(b-t)^{p-1}(f(b)+f(t))^{p-1}\, dt + \frac{p}{(b-a)\,2^{p-1}}\left[f(a)-f(b)\right]\int_a^b H(t)(b-t)^{p}(f(b)+f(t))^{p-1}\, dt.$$

This completes the proof. □

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