Turk. J. Math. Comput. Sci.
16(1)(2024) 137–142

DOI : 10.47000/tjmcs.1366843

On the Norm in the Plane $\mathbb{R}^2_{\pi 3}$

Ziya Akça

Eskişehir Osmangazi University, Faculty of Science, Department of Mathematics and Computer Science, 26480, Eskişehir, Turkey.

Received: 26-09-2023 • Accepted: 25-03-2024

Abstract. In this study, norm in the Plane $\mathbb{R}^2_{\pi 3}$ is produced naturally from a different vector norm. Its triangle inequality, Schwarz inequality properties and geometrical interpretation in the Plane $\mathbb{R}^2_{\pi 3}$ are given.

2020 AMS Classification: 51F99

Keywords: Iso-taxicab distance, non-Euclidean geometry, norm.

1. Introduction and Preliminaries

Recall that the unit circle, where distances are calculated using the common Euclidean norm, is the location of all points in the plane $\mathbb{R}^2$ that are one unit away from the origin. The trigonometric functions $\sin\theta$ and $\cos\theta$ are just the unit circle's parametrization with respect to arc length. It is known that the Lp norm is induced by an inner product if and only if $p = 2$. The norms are induced by inner products, Stirling numbers, Bell polynomials, Lagrange inversion, gamma functions, and generalized π values, [13].

The norms generalize the notion of length from Euclidean space. A norm on a vector space V is a function $\|.\| : V \to \mathbb{R}$ that satisfies

(i)$\|v\| \geq 0$, with equality if and only if $v = 0$

(ii) $\|\alpha v\| = |\alpha| \, \|v\|$

(iii) $\|u + v\| \leq \|u\| + \|v\|$ (the triangle inequality)

for all $u, v \in V$ and all $\alpha \in F$. A vector space endowed with a norm is called a normed vector space, or simply a normed space.

An important fact about norms is that they induce metrics, giving a notion of convergence in vector spaces.

A Minkowski or normed plane is a 2−dimensional vector space with a norm. This norm is induced by its unit ball U, which is a compact, convex set centered at the origin.

The geometries in which the Euclidean distance between two points is replaced by d_T and d_C are called taxicab and Chinese checker geometries [5, 14, 16]. In [3, 4, 7–9], the lengths and norm in taxicab and CC plane geometry were given.

Iso-taxicab geometry is a non-Euclidean geometry defined by K.O. Sowell in 1989 in [15]. In this geometry presented by Sowell three distance functions arise depending upon the relative positions of the points A and B. There are three axes at the origin; the x-axis, the y-axis and the y' -axis, having 60^0 angle which each other. These tree axes separate the plane into six regions.The iso-taxicab trigonometric functions in iso-taxicab plane with three axes were given in [10, 11]. A family of distances, $d_{\pi n}$, that includes Taxicab, Chinese-Checker and Iso-taxi distances, as special

Email address: zakca@ogu.edu.tr (Z. Akça)

cases introduced and the group of isometries of the plane with $d_{\pi n}$ metric is the semi-direct product of D_{2n} and $T(2)$ was shown in [6]. The trigonometric Functions in $\mathbb{R}^2_{\pi 3}$ and the versions in the plane $\mathbb{R}^2_{\pi 3}$of some Euclidean theorems were given in [1, 2, 12].

The definition of $d_{\pi n}-$distances family is given as follows;

Definition 1.1. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be any two points in $\mathbb{R}^2$, a family of $d_{\pi n}$ distances is defined by;

$$d_{\pi n}(A,B) = \frac{1}{\sin\frac{\pi}{n}}\left(\left|\sin\frac{k\pi}{n} - \sin\frac{(k-1)\pi}{n}\right||x_1 - x_2| + \left|\cos\frac{(k-1)\pi}{n} - \cos\frac{k\pi}{n}\right||y_1 - y_2|\right)$$

$$\begin{cases} 1 \le k \le \left[\frac{n-1}{2}\right], k \in \mathbb{Z}, & \tan\frac{(k-1)\pi}{n} \le \left|\frac{y_2 - y_1}{x_2 - x_1}\right| \le \tan\frac{k\pi}{n} \\ k = \left[\frac{n+1}{2}\right], & \tan\frac{\left[\frac{n-1}{2}\right]\pi}{n} \le \left|\frac{y_2 - y_1}{x_2 - x_1}\right| < \infty \text{ or } x_1 = x_2 . \end{cases}$$

For $n = 3$ and accordingly $k = 1$, $k = 2$, we obtain the formula of $d_{\pi 3}$-distance between the points A and B according to the inclination in the plane $\mathbb{R}^2_{\pi 3}$

$$d_{\pi 3}(A,B) = \frac{1}{\sin\frac{\pi}{3}}\left(\left|\sin\frac{k\pi}{3} - \sin\frac{(k-1)\pi}{3}\right||x_1 - x_2| + \left|\cos\frac{(k-1)\pi}{3} - \cos\frac{k\pi}{3}\right||y_1 - y_2|\right)$$

$$\begin{cases} k = 1 \quad , & 0 \le \left|\frac{y_2 - y_1}{x_2 - x_1}\right| \le \tan\frac{\pi}{3} \\ k = 2, & \tan\frac{\pi}{3} \le \left|\frac{y_2 - y_1}{x_2 - x_1}\right| < \infty \text{ or } x_1 = x_2 \end{cases}$$

or

$$d_{\pi 3}(A,B) = \begin{cases} |x_1 - x_2| + \frac{1}{\sqrt{3}}|y_1 - y_2|, & 0 \le \left|\frac{y_2 - y_1}{x_2 - x_1}\right| \le \sqrt{3} \\ \frac{2}{\sqrt{3}}|y_1 - y_2|, & \sqrt{3} \le \left|\frac{y_2 - y_1}{x_2 - x_1}\right| < \infty \text{ or } x_1 = x_2. \end{cases}$$

Definition 1.2. These values of $\sin_{\pi 3}\theta$, $\cos_{\pi 3}\theta$, $\tan_{\pi 3}\theta$ can be calculated in similar ways for other regions. The calculated $\sin_{\pi 3}\theta$, $\cos_{\pi 3}\theta$ values for all regions are shown as;

$$\sin_{\pi 3}\theta = \begin{cases} \frac{2\sin\theta}{|\sin\theta| + \sqrt{3}|\cos\theta|}, & I - III - IV - VI \\ 1 & , \quad II \\ -1 & , \quad V \end{cases}$$

$$\cos_{\pi 3}\theta = \begin{cases} \frac{\sqrt{3}\cos\theta - \sin\theta}{|\sin\theta| + \sqrt{3}|\cos\theta|}, & I - III - IV - VI \\ \frac{\sqrt{3}\cos\theta - \sin\theta}{2|\sin\theta|} & , \quad II - V \end{cases}.$$

2. Defining Angle Measurement Through Inner Product in the Plane $\mathbb{R}^2_{\pi 3}$

In this section, one of the common ways to measure an angle, called the angle between a vector and the positive $x-$axis using dot product, will be defined. Before diving into this definition, a proposition will be presented to assist us in making this definition, which includes a method for determining the norm of a vector and offers a new perspective.

Proposition 2.1. *If a vector space $\mathbb{R}^2_{\pi 3}$ is equipped with a norm*

$$\left\|\overrightarrow{u}\right\|_{\pi 3} = \begin{cases} |x| + \frac{1}{\sqrt{3}}|y| & , \quad 0 \le \left|\frac{y}{x}\right| \le \sqrt{3} \\ \frac{2}{\sqrt{3}}|y| & , \quad \sqrt{3} \le \left|\frac{y}{x}\right| < \infty, \end{cases}$$

then $d_{\pi 3}$ is a metric on $\mathbb{R}^2_{\pi 3}$.

Proof. Consider a position vector $\overrightarrow{u} = \overrightarrow{OA}$ with its endpoint $A = (x, y)$ coordinates. The norm of this vector can be calculated using the coordinates of the starting and ending points

$$\left\|\overrightarrow{u}\right\|_{\pi 3} = \begin{cases} |x| + \frac{1}{\sqrt{3}}|y| & , \quad 0 \leq \left|\frac{y}{x}\right| \leq \sqrt{3} \\ \frac{2}{\sqrt{3}}|y| & , \quad \sqrt{3} \leq \left|\frac{y}{x}\right| < \infty. \end{cases}$$

Additionally, if the vector $\overrightarrow{u}$ lies on the region determined by vectors $\overrightarrow{v}_k$ and $\overrightarrow{v}_{k+1}$, the norm of the vector $\overrightarrow{u}$, denoted as $\left\|\overrightarrow{u}\right\|_{\pi 3}$, can be expressed as

$$\left\|\overrightarrow{u}\right\|_{\pi 3} = \overrightarrow{u}_k.\overrightarrow{u}$$

Here, the vectors $\overrightarrow{u}_k$ can be determined by equations that provide the values, serving as corner vectors that separate each region of the unit circle

$$\begin{aligned} \overrightarrow{u}_k &= \left(\frac{\sin\frac{k\pi}{3} - \sin\frac{(k-1)\pi}{3}}{\sin\frac{\pi}{3}}, \frac{\cos\frac{(k-1)\pi}{3} - \cos\frac{k\pi}{3}}{\sin\frac{\pi}{3}} \right) \quad , \\ \overrightarrow{v}_k &= \left(\cos\frac{(k-1)\pi}{3}, \sin\frac{(k-1)\pi}{3} \right) \quad , \quad k = \{1, 2, 3, ..., 6\}. \end{aligned}$$

By substituting these equations according to the values of k

$$\begin{aligned} \overrightarrow{u}_1 &= \left(1, \frac{1}{\sqrt{3}}\right) \quad , \quad \overrightarrow{u}_2 = \left(0, \frac{2}{\sqrt{3}}\right) \quad , \quad \overrightarrow{u}_3 = \left(-1, \frac{1}{\sqrt{3}}\right) \\ \overrightarrow{u}_4 &= \left(-1, -\frac{1}{\sqrt{3}}\right) \quad , \quad \overrightarrow{u}_5 = \left(0, -\frac{2}{\sqrt{3}}\right) \quad , \quad \overrightarrow{u}_6 = \left(1, -\frac{1}{\sqrt{3}}\right) \end{aligned}$$

and

$$\begin{aligned} \overrightarrow{v}_1 &= (1, 0) \quad , \quad \overrightarrow{v}_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \quad , \quad \overrightarrow{v}_3 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \\ \overrightarrow{v}_4 &= (-1, 0) \quad , \quad \overrightarrow{v}_5 = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \quad , \quad \overrightarrow{v}_6 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \end{aligned}$$

values are obtained

$\|.\|_{\pi 3}$ satisfies the norm properties. Let $\overrightarrow{u}$ be a vector with slope m ;

i) If the slope of vector $\overrightarrow{u}$ is such that $0 \leq |m| \leq \sqrt{3}$, then $|x| + \frac{1}{\sqrt{3}}|y| \geq 0$.

$|x| + \frac{1}{\sqrt{3}}|y| = 0 \Leftrightarrow x = 0, y = 0$. This means that $\overrightarrow{u} = 0$

If the slope of vector $\overrightarrow{u}$ is such that $\sqrt{3} \leq |m| \leq \infty$ then $\frac{2}{\sqrt{3}}|y| \geq 0$.

$\frac{2}{\sqrt{3}}|y| = 0 \Leftrightarrow y = 0$. This means that $\overrightarrow{u} = 0$.

ii) If the slope of vector $\overrightarrow{u}$ is such that $0 \leq |m| \leq \sqrt{3}$ and $\alpha \in \mathbb{R}$, then

$$\begin{aligned} \left\|\alpha\overrightarrow{u}\right\|_{\pi 3} &= |\alpha x| + \frac{1}{\sqrt{3}}|\alpha y| \\ &= |\alpha|\left(|x| + \frac{1}{\sqrt{3}}|y|\right) \\ &= |\alpha|\left\|\overrightarrow{u}\right\|_{\pi 3}. \end{aligned}$$

If the slope of vector $\overrightarrow{u}$ is such that $\sqrt{3} \leq |m| \leq \infty$, then

$$\begin{aligned}\left\|\alpha \overrightarrow{u}\right\|_{\pi 3} &= \frac{2}{\sqrt{3}}|\alpha y| \\ &= \frac{2}{\sqrt{3}}|\alpha|\,|y| \\ &= |\alpha|\left\|\overrightarrow{u}\right\|_{\pi 3}.\end{aligned}$$

The proof of the triangle inequality is given as follows:
iii) For the vectors $\overrightarrow{u}$ and $\overrightarrow{v}$, $\left\|\overrightarrow{u}+\overrightarrow{v}\right\|_{\pi 3} \leq \left\|\overrightarrow{u}\right\|_{\pi 3} + \left\|\overrightarrow{v}\right\|_{\pi 3}$

This inequality can be obtained from the convexity of the closed unit circle $\left\{\overrightarrow{u} \in \mathbb{R}^2 \;:\; \left\|\overrightarrow{u}\right\|_{\pi 3} \leq 1\right\}$ and the norm function on $\mathbb{R}^2_{\pi 3}$. In the plane $\mathbb{R}^2_{\pi 3}$, the set of vectors x that lie on the unit circle satisfies the equation $u_k \cdot x = 1$. Additionally, the coordinates of the vertices of this hexagon are known as:

$$\overrightarrow{v}_k = \left(\cos\frac{(k-1)\pi}{3}, \sin\frac{(k-1)\pi}{3}\right), k = \{1, 2, 3, ..., 6\}.$$

When considering the vector $\overrightarrow{v} = OB$ (O, origin), if the vector $\overrightarrow{v}$ lies within the region determined by $\overrightarrow{v}_k$ and $\overrightarrow{v}_{k+1}$, then similarly,

$$\left\|\overrightarrow{v}\right\|_{\pi 3} = u_k \cdot v$$

can be written. Correspondingly, with $\overrightarrow{t}_k$ and $\overrightarrow{t}_{k+1}$ being non-negative numbers,

$$\overrightarrow{v} = \overrightarrow{t}_k \overrightarrow{v}_k + \overrightarrow{t}_{k+1} \overrightarrow{v}_{k+1}$$

can be written and

$$\left\|\overrightarrow{v}\right\|_{\pi 3} = \overrightarrow{t}_k + \overrightarrow{t}_{k+1}.$$

Furthermore, vectors inside a unit circle have a norm less than 1, and vectors outside the unit circle have a norm greater than 1.

Now, for the final part of the proof, consider a position vector $\overrightarrow{OP}$ with endpoint coordinates $P = (x_3, y_3)$, then

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OV} + \overrightarrow{VP} \\ &= \overrightarrow{OV} + t\overrightarrow{VU} \\ &= \overrightarrow{OV} + t(\overrightarrow{OU} - \overrightarrow{OV}) \\ &= (1-t)\overrightarrow{OV} + t\overrightarrow{OU} \\ &= t\overrightarrow{u} + (1-t)\overrightarrow{v}.\end{aligned}$$

The vectors $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{p}$ on the unit circle, and for $0 \leq t \leq 1$, the convexity of the unit sphere implies that the vector $t\overrightarrow{u} + (1-t)\overrightarrow{v}$ is either on or inside the unit circle. Thus,

$$\left\|t\overrightarrow{u} + (1-t)\overrightarrow{v}\right\|_{\pi 3} \leq 1.$$

To obtain the triangle inequality for $t = \dfrac{a}{a+b}$, where a and b are both greater than 0,

$$\frac{\left\|a\overrightarrow{u} + b\overrightarrow{v}\right\|_{\pi 3}}{a+b} = \left\|\frac{a}{a+b}\overrightarrow{u} + (1 - \frac{a}{a+b})\overrightarrow{v}\right\|_{\pi 3} \leq 1$$

and

$$\left\|a\overrightarrow{u} + b\overrightarrow{v}\right\|_{\pi 3} \leq a\left\|\overrightarrow{u}\right\|_{\pi 3} + b\left\|\overrightarrow{v}\right\|_{\pi 3}$$

is obtained. Thus, the triangle inequality holds for arbitrary nonzero vectors $a\overrightarrow{u}$ and $b\overrightarrow{v}$. Here, if the vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ are in the same region, then

$$\left\|\overrightarrow{u} + \overrightarrow{v}\right\|_{\pi 3} \leq \left\|\overrightarrow{u}\right\|_{\pi 3} + \left\|\overrightarrow{v}\right\|_{\pi 3}$$

which completes the proof. □

Proposition 2.2. $d_{\pi 3}(A, 0) = \|A\|_{\pi 3}$.

Proof. Consider a position vector $\overrightarrow{u} = \overrightarrow{OA}$ with its endpoint $A = (x, y) \in \mathbb{R}^2_{\pi 3}$ coordinates. Using the Definition 1.1, we have

$$d_{\pi 3}(A, 0) = \begin{cases} |x| + \dfrac{1}{\sqrt{3}}|y| & , \quad 0 \le \left|\dfrac{y}{x}\right| \le \sqrt{3} \\ \dfrac{2}{\sqrt{3}}|y| & , \quad \sqrt{3} \le \left|\dfrac{y}{x}\right| < \infty \end{cases} = \|A\|_{\pi 3}.$$

□

Proposition 2.3. *(Schwarz Inequality) If* $A = (x_1, y_1)$ *and* $B = (x_2, y_2) \in \mathbb{R}^2_{\pi 3}$. *Then,*

$$|\langle A, B\rangle| \le \|A\|_{\pi 3} \, . \, \|B\|_{\pi 3}.$$

Proof. This follows easily from the fact that norm of a vector in the plane $\mathbb{R}^2_{\pi 3}$ is always larger than or equal to its Euclidean length. □

3. Geometrical Interpretation

It is weel known that

$$|\langle A, B\rangle| \le \|A\|_{\pi 3} \, . \, \|B\|_{\pi 3} \cos\theta, \qquad 0 \le \theta \le \pi$$

in Euclidean plane. Now, consider the Schwarz inequality

$$|\langle A, B\rangle| \le \|A\|_{\pi 3} \, . \, \|B\|_{\pi 3}$$

in the plane $\mathbb{R}^2_{\pi 3}$. If A and B are nonzero vectors one gets

$$\frac{|\langle A, B\rangle|}{\|A\|_{\pi 3} \, . \, \|B\|_{\pi 3}} \le 1$$

from the Schwarz inequality. The last inequality can be expressed as

$$-1 \le \frac{|\langle A, B\rangle|}{\|A\|_{\pi 3} \, . \, \|B\|_{\pi 3}} \le 1$$

which also to define Iso-taxicab $\cos_{\pi 3}\theta$, as follows:

$$|\langle A, B\rangle| \le \|A\|_{\pi 3} \, . \, \|B\|_{\pi 3} \, .cos_{\pi 3}\theta$$

and consequently, the relationship between the inner product and lengths and angles in the plane $\mathbb{R}^2_{\pi 3}$can be interpreted as in Euclidean plane, by related norm.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed the published version of the manuscript.

References

[1] Akça, Z., Nazlı, S., *On the trigonometric functions in* $\mathbb{R}^2_{\pi 3}$, Konuralp Journal of Mathematics, **8**(2020), 429–437.

[2] Akça, Z., Nazlı, S., *On the versions in the plane* $\mathbb{R}^2_{\pi 3}$ *of some Euclidean theorems*, New Trends in Mathematical Sciences, **10**(2022), 20–27.

[3] Akça, Z., Bayar, A., Ekmekçi, S., *The norm in CC plane geometry*, The Pi Mu Epsilon Journal, **6**(2007), 321–324.

[4] Akça, Z., Kaya, R., *On the norm in higher dimensional taxicab spaces*, Hadronic J. Suppl, **19**(4)(2004), 491–501.

[5] Bayar, A., Ekmekçi, S., *The Chinese Checker Sine and Cosine functions*, International Journal of Mathematics and Analysis, **1**(3)(2006), 249–259.

[6] Bayar, A., Kaya, R., *On isometries of* $\mathbb{R}^2_{\pi n}$, Hacettepe Journal of Mathematics and Statistics, **40**(5)(2011), 673–679.

[7] Ekmekçi, S., Akça, Z., Altıntaş, A., *On trigonometric functions and norm in the generalized taxicab metric*, Mathematical Sciences And Applications E-Notes, **3**(2)(2015), 27–33.

[8] Ekmekçi, S., Bayar, A., Akça, Z., *On the plane geometry with generalized absolute value metric*, Math. Probl. Eng., **1**(0)(2008), 1–8.

[9] Ekici, C., Kocayusufoğlu, İ., Akça, Z., *The norm in taxicab geometry*, Tr.J.of Mathematics, **22**(3)(1998), 295–307.

[10] Kocayusufoglu, İ., Ada, T., *On the iso-taxicab trigonometry*, Applied Sciences, **8**(2006), 101–111.

[11] Kocayusufoglu, İ., *Trigonometry on iso-taxicab geometry*, Mathematical and Computational Applications, **5**(3)(2002), 201–212.

[12] Nazlı, S., *On the plane geometry* $\mathbb{R}^2_{\pi 3}$, *Doctoral Dissertation*, Eskişehir Osmangazi University, (2020), 1–118.
[13] Moore, C., Hatfield, A., Klette, R., Chebolu, S., Warden, E., *Trigonometric functions in the p-norm*, https://arxiv.org/pdf/2109.14036.pdf.
[14] Özcan, M., Ekmekçi, S., Bayar, A., *A note on the variation of the taxicab lengths under rotations*, The Pi Mu Epsilon Journal, **11**(7)(2002), 381–384.
[15] Sowell, K. O.,*Taxicab geometry-a new slant*, Mathematics Magazine, **62**(1989), 238–248.
[16] Turan, M., Özcan, M., *General equation for Chinese Checker conics and focus-directrix Chinese Checker conics*, International Journal of Pure and Applied Mathematics, **30**(3)(2006), 393–401.