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Gray Wolf Optimizer for Knot Placement in B-Spline Curve Fitting

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ABSTRACT—The formation of a realistic representation constitutes the main theme of the surface modeling by the transfer of a real object to digital media. This transfer process is performed by first scanning the objects and then obtaining curve and surface models by fitting the data representing the object from the data points obtained later. The data fitting for curve problem has an important area in research topics in geometric modeling, computer-aided design (CAD), computer-aided manufacturing (CAM), and computerized production. Reverse engineering is used to obtain curve and surface models from data points. B-Spline curves are very flexible curves, especially for surface modeling. Several optimization algorithms have been used in the literature for B-Spline curve fitting. In this study, B-Spline curve fitting is carried out by Gray Wolf Optimizer (GWO). The estimation of the knot locations and the number of knots are randomly selected in the curve estimation by the GWO method and the curve estimation with the smallest error is aimed. For the curve fitting, six different functions are used which are frequently used in the literature. The experimental studies show that the proposed algorithm obtains the results with low error rates for more than one functions.

Keywords — *B-Spline Curve Fitting; Gray Wolf Optimizer (GWO); Knot placement; Optimization; Reverse Engineering*

B-Spline Eğri Tahmininde Düğüm Yerleşimi İçin Gri Kurt Optimizasyon Algoritması

ÖZET— Gerçek bir nesnenin dijital ortama aktarılmasıyla gerçekçi bir temsilinin oluşturulması yüzey modellemenin ana temasıdır. Bu aktarım işlemi öncelikle bu nesnelerin taranması daha sonra elde edilen veri noktalarından nesneyi temsil edecek verileri uydurarak eğri ve yüzey modelleri elde edilmesidir. Veri uydurma problemi geometrik modelleme, bilgisayarlı tasarım(CAD), bilgisayarlı modelleme(CAM) ve bilgisayarlı üretim alanlarındaki araştırma konuları içerisinde önemli bir yer tutmaktadır. Veri noktalarından eğri ve yüzey modellerinin elde etmek için tersine mühendislik kullanılmaktadır. B-Spline eğrileri özellikle yüzey modelleme ve eğri oluşturma için kullanımı çok esnek eğrilerdir. B-Spline eğri tahmini için literatürde birden fazla optimizasyon algoritmaları kullanmıştır. Bu çalışmada, Gri Kurt Optimizasyon(GWO) Algoritması ile B-Spline eğri tahmini yapılmıştır. GWO yöntemi ile yapılan eğri tahmininde düğüm yerlerinin tespiti ve düğüm sayısı gelişigüzel seçilerek en küçük hata ile eğri tahmini hedeflenmiştir. Eğri tahmini için literatürde sıklıkla kullanılan 6 farklı fonksiyonu kullanılmıştır. GWO ile birden fazla fonksiyon düşük hata oranı ile başarılı bir şekilde elde edilmiştir.

Anahtar Kelimeler *— B-Spline Eğri Uydurma, Gri Kurt Optimizasyon(GWO), Düğüm Yerleştirme, Optimizasyon, Ters Mühendislik*

1. Introduction

In order to be able to represent objects in the real world in a digital media, these objects must first be scanned. Then, with the help of the data points obtained, curve and surface models to represent the object need to be obtained. Reverse engineering (Varady and RR.Martin 2002) is used to obtain curve and surface models from data points. In addition to improvements in computerized design, modeling (Pottmann, Leopoldseder et al. 2005), it can be said that the method of curve fitting to data points is basis of many innovations in industrial breakout designs, ship hull designs and in the medical sector. Similarly, data fitting has an important place in research topics in geometric modeling(Hoschek, Lasser et al. 1993), computer-aided design (CAD), computer modeling (CAM) (Barnhill 1992, Patrikalakis 2002), and computerized production. Mathematical functions need to be used in order to obtain the object again from the data points obtained from real objects. Different functions can be used for this operation. Different mathematical functions need to be used, especially as the complexity of the shape increases. For complex shapes, free-form polynomial functions such as Bezier, B-Spline, NURBS (Ma and Kruth 1995, Piegl 1997, Varady, Martin et al. 1997, Ma and Kruth 1998, De Boor 2001, Echevarria, Iglesias et al. 2002, Galvez and Iglesias 2012, Galvez, Iglesias et al. 2012) are used. The most commonly used ones among these functions are B-Splines because B-Splines have more mathematical superiority and geometric flexibility. A change in the local points in the B-splines does not affect other points. The most important point in B-splines is the knot vector. In particular, knot selection significantly affects the shape of the curve (Farin 2002., Goldenthal and Bercovier 2004). An appropriate detection of the B-Spline parameters is necessary to obtain a good curve estimation (Piegl 1997). It is seen that the ideal solution of B-spline knot vector is realized by many artificial intelligence techniques (Ulker and Arslan 2009, Ulker 2013). Different artificial intelligence techniques have been used in the literature for the B-Spline curve fitting problem. Yoshimoto et al. used genetic algorithm in automatic node placement in data fitting problem (Yoshimoto, Moriyama et al. 1999). Kumar et al. proposed an approach based on Genetic Algorithm (GA) for parameter optimization in Non Uniform B-spline (NURBS) curve fitting (Kumar, Kalra et al. 2003). In another study, using the artificial immune system, B-spline knot placement was performed(Ulker and Arslan 2009). Elitist Clonal Selection algorithm was used for B-spline automatic knot placement (Galvez, Iglesias et al. 2015). Finally, (inik 2016) proposed the Gravitational Search Algorithm (GSA) for B-spline curve estimation. However, for the first time, the GWO is handled for B-spline curve fitting in this paper.

The remainder of this paper is organized as follows. General information on B-Spline curves is given in Chapter 2. In Chapter 3, the GWO algorithm is described. How to estimate a B-Spline curve is described step by step in Chapter 4. Chapter 5 gives the experimental results obtained by the GWO method in estimating the B-Spline curve. Finally, Section 6 describes the conclusion.

2. B-Spline Curves

When the B-Splines were first proposed by De Boor (De Boor 1978), they gained popularity, especially on the industrial areas (Ulker and Arslan 2009). B-spline curves have generally been developed by the development of Bezier curves. A B-spline curve consists of a combination of at least one or more polynomial segments. If the B-spline curve consists of a single segment, this curve is also the Bézier curve. The most important feature of B-spline curves is that if any one point of the data points moves, only the relevant part of the curve changes. However, in Bézier curves even if only any one of the points in the data set moves, the whole curve from the first point to the last point are affected by this situation.

B-Spline curves and surfaces are defined by corner points called control points. Although the curves and surfaces obtained using these points do not pass through the control points, the form of the curve or surface is completely shaped according to the positions of these points. The polygon generated by these control points is named as control polygon. These points behave like a magnet, allowing the curve to follow the shape of the control polygon. As a result, a characteristic and smooth curve is obtained within the borders of the control polygon (Sarıöz 2005).

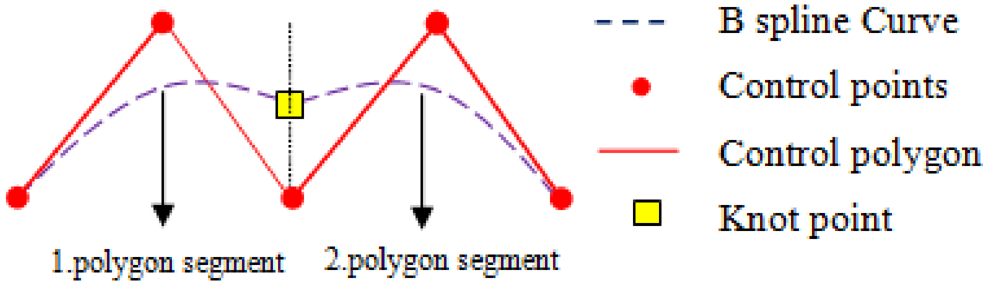

Figure 1. *B-Spline curve and control polygon*

The definition of the B-spline curves is as in (1).

$$P(\mathrm{t}) = \sum_{\mathrm{i}=0}^{\mathrm{n}} \mathrm{P_i N_{i,k}(t)} \tag{1}$$

P_i is one of $n+1$ polygon edges. The expression of B-spline basis function defined for each control point is given as in (2) and (3).

$$N_{i,1}(t) = \begin{cases} 1 & x_i \le t < x_{i+1} \\ 0 & otherwise \end{cases} \tag{2}$$

$$N_{i,k}(t) = \frac{(t-x_i)}{x_{i+k-1}-x_i} N_{i,k-1}(t) + \frac{(x_{i+k}-t)}{x_{i+k}-x_{i+1}} N_{i+1,k-1}(t) \tag{3}$$

3. Grey Wolf Optimizer (GWO)

The GWO algorithm is proposed by mimicking hunting behavior and social behavior of gray wolves. Regarding social hierarchy, gray wolves are classified as *alpha*, *beta*, *delta* and *omega*. The *alpha* group is a dominant type within the whole wolf population because the other wolf groups obey its rules. The *beta* class refers to secondary wolves that help *alpha* in their decisions. *Omega* represents the lowest gray wolfs. If a wolf does not belong to any of the species mentioned above, it is called a *delta*. Group hunting is an interesting social behavior of gray wolves, as well as social interactions of wolves. The main parts of the GWO are the stages of encircling prey, hunting, attacking prey and searching for prey (Mirjalili, Mirjalili et al. 2014).

a. Social Hierarchy

Candidate solutions are structured taking into account the social hierarchy of the wolves. The solutions with the best fitness value are regarded as worms named *alpha*, *beta*, *delta* and *omega* respectively.

b. Encircling prey

The gray wolf may randomly update its position around the prey using (4) and (5). The encircling behavior of gray wolves can be presented as follows:

$$z = \left|y.W_p(t) - W(t)\right| \tag{4}$$

$$W(t+1) = \left|W_p(t) - x.z\right| \tag{5}$$

Here, *Wp* is the position vector of the prey, and *W* expresses the location vector of a grey wolf. *x* and *y* values are coefficient vectors and they are calculated as in (6) and (7) respectively:

$$x = |2a.r_1 - a| \tag{6}$$

$$y = |2a.r_2| \tag{7}$$

The components of a are reduced linearly from 2 to 0 during the iterations. r_1 and r_2 are random variables which take values in the range [0, 1].

c. Hunting

Alpha, *beta* and *delta* species have remarkable about the current position of the hunt. Therefore, the best first three solutions obtained are saved and the other wolves are allowed to update their positions according to their position of the best search agents. Equations (8)-(14) can be used in this context (Jayakumar, Subramanian et al. 2016).

$$z_\propto = |y_1.W_\propto - W| \tag{8}$$

$$z_\beta = \left|y_2.W_\beta - W\right| \tag{9}$$

$$z_\delta = |y_3.W_\delta - W| \tag{10}$$

$$W_1 = |W_\propto - x_1.z_\propto| \tag{11}$$

$$W_2 = \left|W_\beta - x_2.z_\beta\right| \tag{12}$$

$$W_3 = |W_\delta - x_3.z_\delta| \tag{13}$$

$$W(t+1) = \frac{(W_1 + W_2 + W_3)}{3} \tag{14}$$

d. Attacking prey

At this stage, *a* value is reduced and therefore the range of change of *x* is reduced. When *x* has random values in the range [-1, 1], the next location of the search agent will be anywhere between the current location and the location of the prey.

e. Search for prey

Gray wolfs usually search by *alpha*, *beta*, and *delta* locations. They leave each other to search for their prey and come together at the moment of attacking their prey. To model the distribution mathematically, a parameter *x* with random values greater than 1 or less than -1 is used. This process makes exploration important and supports the global search of the GWO algorithm. In addition, the flow chart of GWO algorithm is given in Fig. 2.

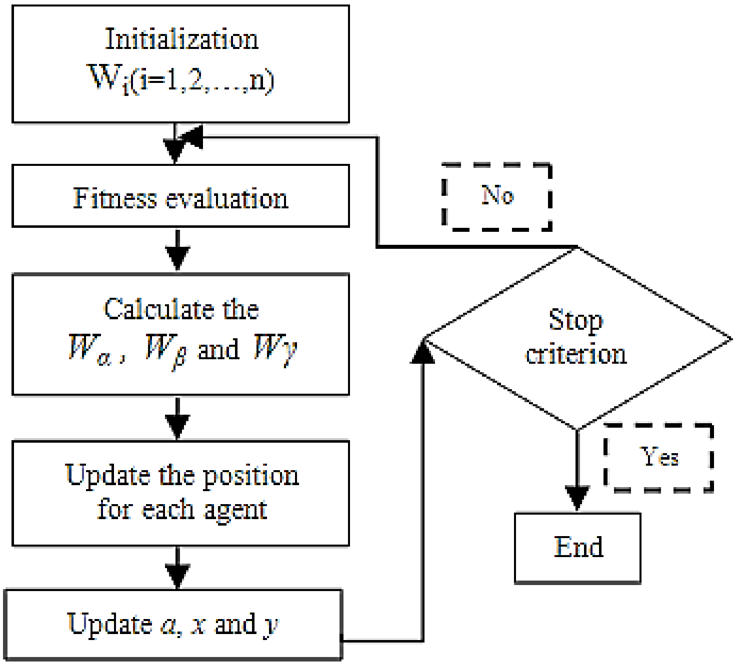

Figure 2. *Flow chart of GWO algorithm (Guha, Roy et al. 2016)*

4. Obtaining B-Spline Curves by GWO

In Section II, the required parameters to obtain a B-spline curve was stated to be the control points, the node vector and the spline grade, respectively. If there is only a structure consisting of points, the control points and knot points can be obtained again from this point set. Firstly, the knot vector must be found from this point cloud. In this study, the knot vector is obtained by Centripetal (Lee 1989) method. The steps of the method are as follows.

1. The points to be fitted to the curve are taken as an F variable.

2. The original data sequence F is preserved, the F sequence is transferred to another Q sequence, and the point operations are performed via this Q sequence.

3. The initial step of the GWO method is applied to generate the initial population by generating random numbers between 0 and 1 as much as the Q array size at the beginning.

4. Which points will be knot points within the whole points are randomly selected.

5. After finding the centripetal nodes, the knots of the estimated B-spline are calculated with the help of (15) and (16).

$$\mathrm{U} = \{0,0, \dots 0, \mathrm{u}_{\mathrm{p\_esas}+1}, \dots, \mathrm{u}_{\mathrm{m}}, 1,1, \dots .,1\} \quad (15)$$

$$\mathrm{U}_{\mathrm{j}+\mathrm{p}_{esas}} = \frac{1}{\mathrm{p}_{\mathrm{esas}}} \sum_{\mathrm{i}=\mathrm{j}}^{\mathrm{j}+\mathrm{p}_{\mathrm{esas}}-1} \mathrm{u}_{\mathrm{i}} \text{ , j} = 1, \dots \mathrm{m} - \mathrm{p\_esas} \quad (16)$$

6. According to the formula of B-spline curve, $Q = P * R$ matrix representation receivable. Where *R* matrix is produced by B-spline blending functions with *N* values. Here the *P* control points are calculated as $P = QxR^{-1}$.
7. After calculating the control points and the node vector, the curve is estimated.

The total error is calculated by summing the error amounts between the actual curve and the estimated (calculated) curve. The model with the least amount of errors is the ideal model. Different error calculation methods are available in the literature. In this article, the ideal solutions are tried to be found by means of MSE (Mean Squared Error) error calculation method as in (17).

$$MSE = \frac{1}{n} \sum_{i=1}^{n} \sqrt{(X_{1i} - X_{2i})^2 - (Y_{1i} - Y_{2i})^2} \quad (17)$$

Here, n is the number of points of the curve to be plotted, X_{1i} and X_{2i} are respectively *X* coordinate values of i^{th} point of actual and estimated curve and Y_{1i} and Y_{2i} are respectively *Y* coordinate values of i^{th} point of actual and estimated curve. The Akaike Information Criterion (AIC) (Akaike 1973, Akaike 1974) and the Bayesian Information Criterion (BIC) (Schwarz 1978) are calculated alongside MSE and RMSE. The calculation of these models is carried out according to (18) and (19).

$$AIC = n * Ln(MSE) + 2(2 * Nod + p) \quad (18)$$

$$BIC = n * Ln(MSE) + Ln(n) * 2 * (2 * Nod + p) \quad (19)$$

where *n* is the number of points, *Nod* is the number of knots, and *p* is degree of B-spline.

5. Experimental Results

Six different functions are used in this study. These functions are chosen because they are frequently used in the literature in the problem of knot placement and curve fitting. The definition of the functions and the variable ranges are given in Table 1. In order to obtain points by means of the functions, a special software has been developed, which are shown its interface in Figure 3. Only the points of *function 1* (Titanium Heat Data) are obtained from MATLAB R2014a (8.3.0.532) software. The constant variables for curve estimation are as in Table 1, and the curve grade is set to 3 (cubic), the number of population is 50, the number of iteration is 100, and each experimental study is run independently 5 times. The B-Spline curve estimation for 6 functions is obtained by running on a computer with the Intel Core i5 4690k CPU @ 3.50Ghz 16Gb Ram.

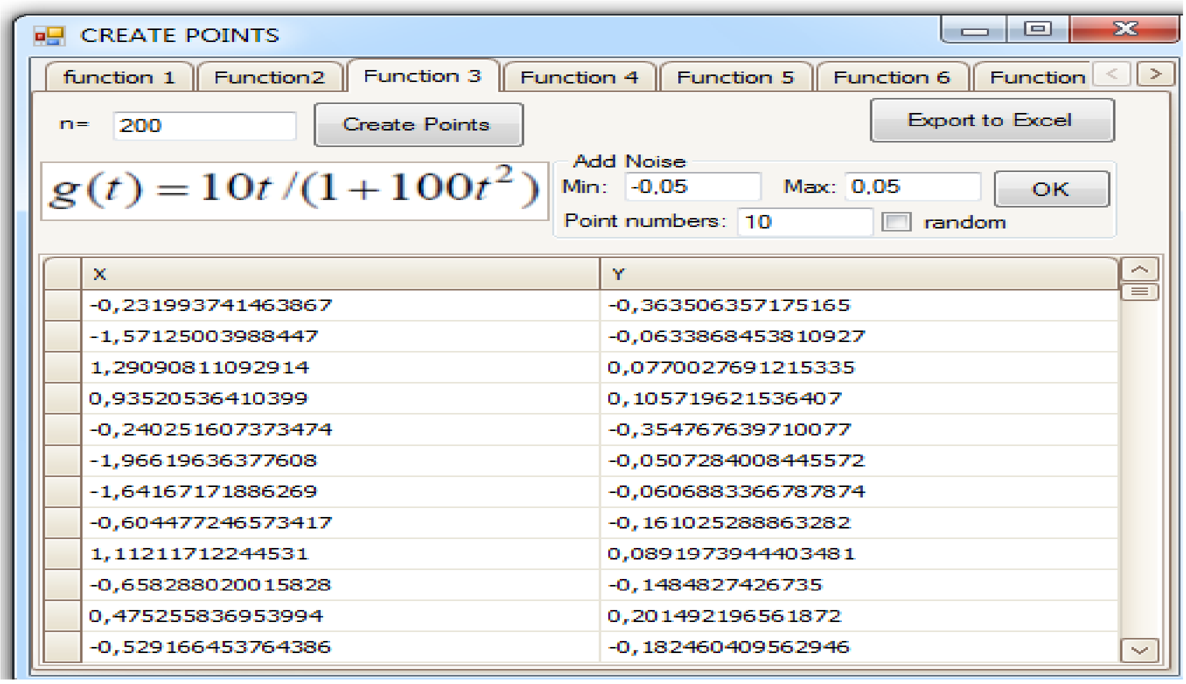

Figure 3. *Interface of software developed to obtain function points*

Table 1. *Benchmark functions*

Function Index	Description	Variable Range
2	$F(x) = \frac{10x}{(1+100x^2)}$	$x \in [-2,+2]$
3	$F(x) = 0.2e^{-0.5x}\sin 5x + 4$	$x \in [0,4\pi]$
4	$F(x) = \frac{100}{e^{\|10x-5\|}} + \frac{(10x-5)^5}{500}$	$x \in [0,1]$
5	$F(x) = sin(x) + 2e^{-30x^2}$	$x \in [-2,2]$
6	$F(x) = \sin(2x) + 2e^{-16x^2} + 2$	$x \in [-2,2]$

The experimental results for all functions are given in Table 2. From Table 2, it can be seen that the number of point for *function 1* is 49. Number of knot is 16, *Min MSE* is found as 0.02496, *Max MSE* is 0.03339 and *Mean MSE* is found as 0.02528. Also, *Bayesian Information Criterion (BIC)* is 143.814, *Akaike Information Criterion (AIC)* is 73.814, and *Computation time* (s) is 0.26563 in seconds for iteration. Graphics of actual and estimated curves by GWO are given in Figure 4-9. Looking at the figures, almost all of the curves are precisely estimated visually.

When looking at the results of *Function 1* in Table 3, the proposed algorithm achieves a higher error value than the results obtained by other researchers. The range of values used to obtain *Function 1* is in the range of 595-1075 while the other researchers are 0-1. The higher error may be due to the difference in this value range.

Table 2. *The experimental results for all functions*

	Func. 1	***Func. 2***	***Func .3***	***Func. 4***	***Func. 5***	***Func. 6***
Curve Grade(P)	3	3	3	3	3	3
Population Size	50	50	50	50	50	50
Number of Iteration	100	100	100	100	100	100
Run	5	5	5	5	5	5
Number of Point	49	90	200	201	201	201
Number of Knot	16	53	77	40	46	37
Min MSE	0.02496	0.01067	0.00868	1.3957	0.03255	0.02640
Max MSE	0.03339	0.01835	0.01236	1.77553	0.05269	0.03544
Mean MSE	0.02528	0.01131	0.00930	1.41977	0.03476	0.02693
BIC	143	610	995	704	603	477
AIC	73	204	325	429	268	230
Computation time(s)	0.26563	0.4375	2.0313	4.0469	3.9531	4.125

Table 3. *Comparison of the results obtained by the proposed algorithm for Function 1 with other studies*

	(De Boor 1968)	(Jupp 1978)	(Yuan, Chen et al. 2013).	**Proposed Algorithm**
Knot Value	5	5	6	**16**
Error	0.01305	0.01227	0.01174	**0,02496**

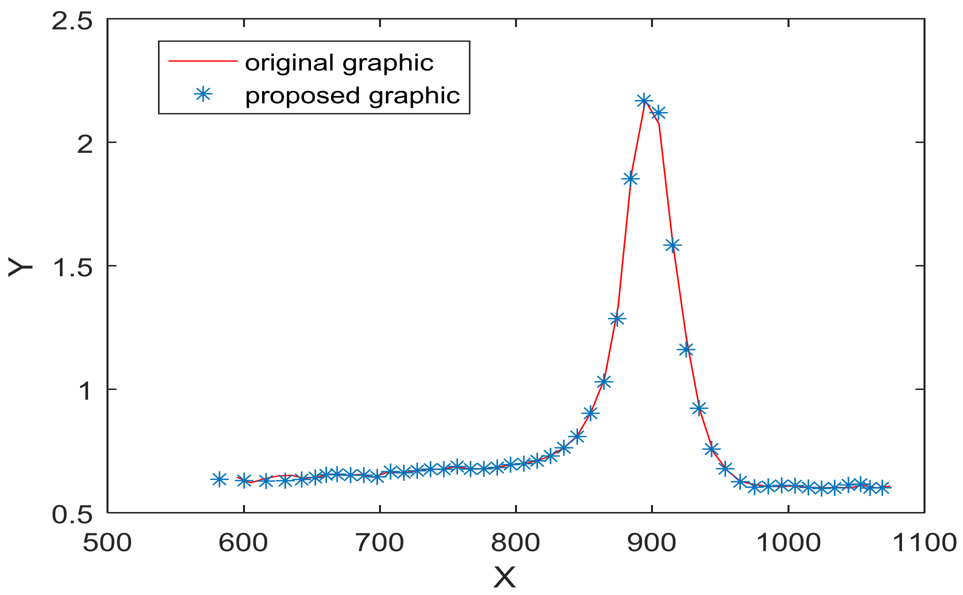

Figure 4. *Actual and estimated curves for Function 1*

In Table 4, In previous studies Schwetlick and Schutze(Schwetlick and Schutze 1995) obtained function 2 with error 0.0739568 and Yuan Yuan et al(Yuan, Chen et al. 2013) 0.067471. In the study performed, function 2 was obtained with an error of 0.01067. The proposed algorithm for *Function 2* yields a better result than the other algorithms with lower error.

Table 4. *Comparison of the results obtained by the proposed algorithm for Function 2 with other studies*

	(Schwetlick and Schutze 1995)	(Yuan, Chen et al. 2013).	**Proposed Algorithm**
MSE	0.0739568	0.067471	**0,01067**

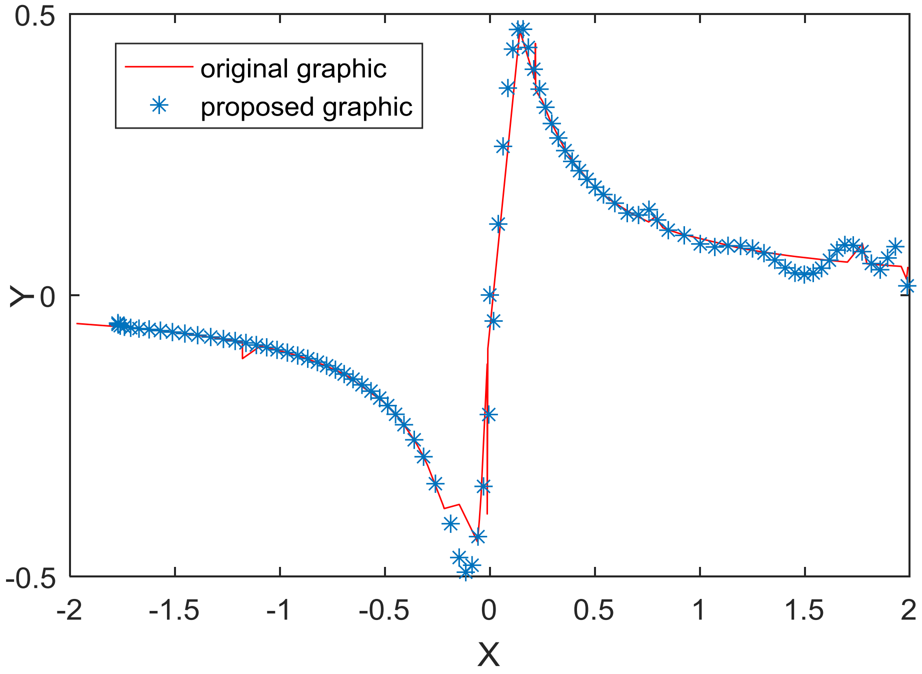

Figure 5. *Actual and estimated curves for Function 2*

The proposed algorithm for *Function 3* in Table 5 shows that Min MSE is 0.00868, Max MSE is 0.01236 and Mean MSE is 0.00930 were obtained. when compared to the previous study, although the Min *MSE* error value is high, the mean and Max *MSE* values are lower. The result is that the proposed algorithm works more stable.

Table 5. *Comparison of the results obtained by the proposed algorithm for Function 3 with other studies*

(Valenzuela, Pasadas et al. 2013).			
Knot	Min MSE	Max MSE	Mean MSE
10	0.00241	0.0915	0.0208
Proposed Algorithm			
Knot	Min MSE	Max MSE	Mean MSE
77	**0,00868**	**0,01236**	**0,00930**

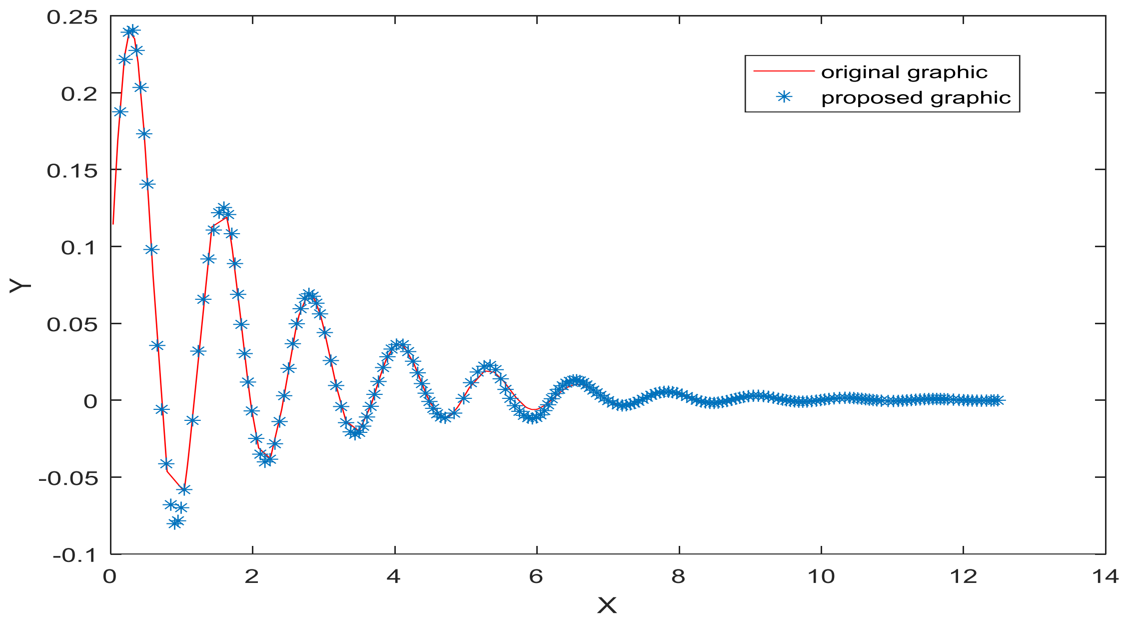

Figure 6. *Actual and estimated curves for Function 3*

As shown in Table 6 the number of iterations is 100, the BIC value is 704, and the calculation time is 4.0469 seconds obtained by *Function 4*. It is seen that the proposed algorithm for *Function 4* is lower than the other *BIC* values.

Table 6. *Comparison of the results obtained by the proposed algorithm for Function 4 with other studies*

	(Yoshimoto, Harada et al. 2003).	(Galvez and Iglesias 2011).	**Proposed Algorithm**
Number of iterations	200-300	10	**100**
Error(BIC)	1150-1170	1012	**704**
Computation time	Tens of seconds	0.1 –1 s	**4.0469 s**

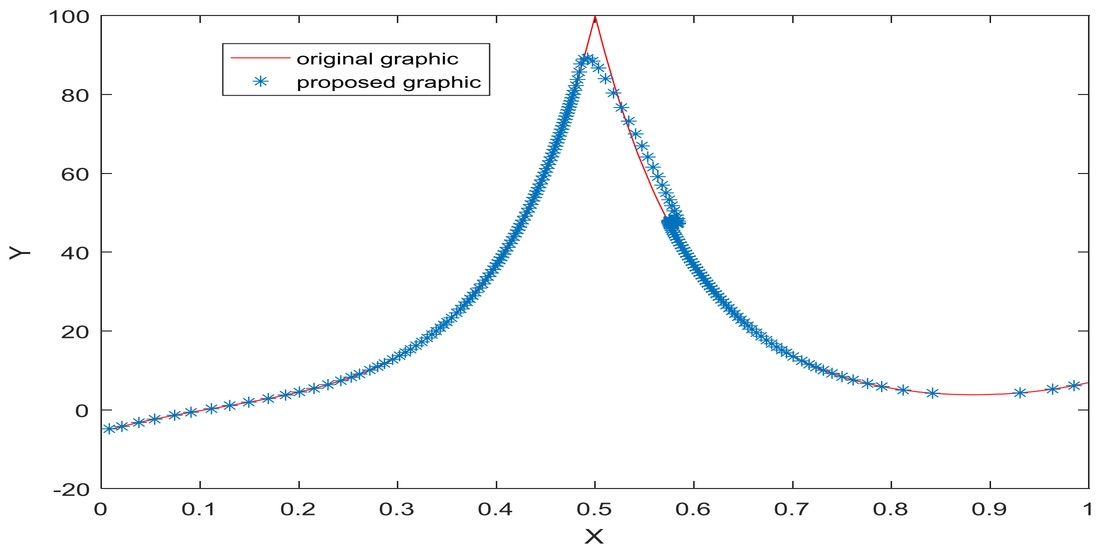

Figure 7. *Actual and estimated curves for Function 4*

Looking at Table 7 and 8 respectively for *Function 5* and *6*, it can be said that the performance of the GWO algorithm is slightly low than those of the other algorithms.

Table 7. *Comparison of the results obtained by the proposed algorithm for Function 5 with other studies*

	(Yoshimoto, Moriyama et al. 1999)	(Yoshimoto, Harada et al. 2003).	(Galvez and Iglesias 2011).	**Proposed Algorithm**
Iteration	200	200-300	10	**100**
BIC	-46	-193	-279	**603**
Computation time	5-15s	Tens of seconds	0.1-1s	**3.9531s**

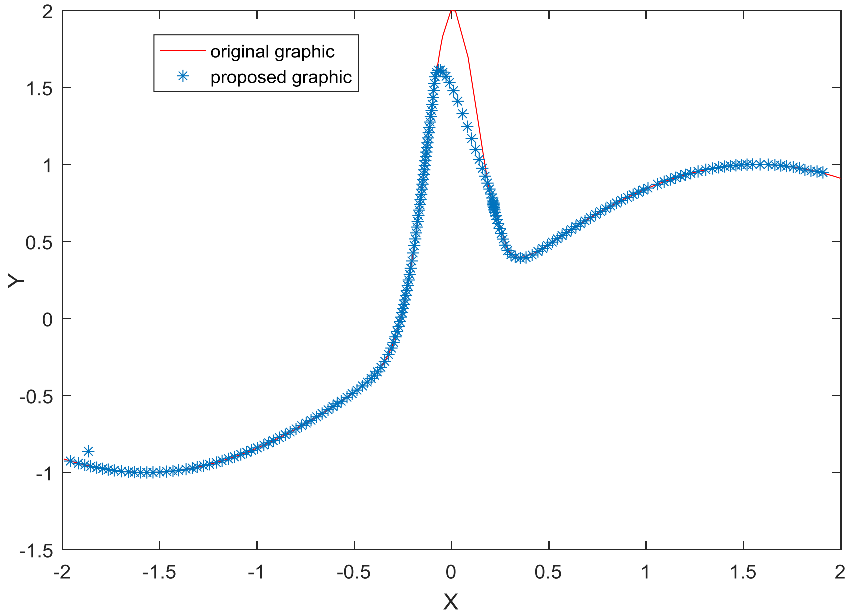

Figure 8. *Actual and estimated curves for Function 5*

Table 8. *Comparison of the results obtained by the proposed algorithm for Function 6 with other studies*

	(Yoshimoto, Moriyama et al. 1999).	(Yoshimoto, Harada et al. 2003).	(Ulker and Arslan 2009)	(Galvez and Iglesias 2011).	**Proposed Algorithm**
Iteration	200	200-300	500	10	**100**
BIC	134	49	362	-63	**477**
Computati on time	5-15s	Tens of seconds	Tens of seconds–minutes	0.1-1s	**4.125s**

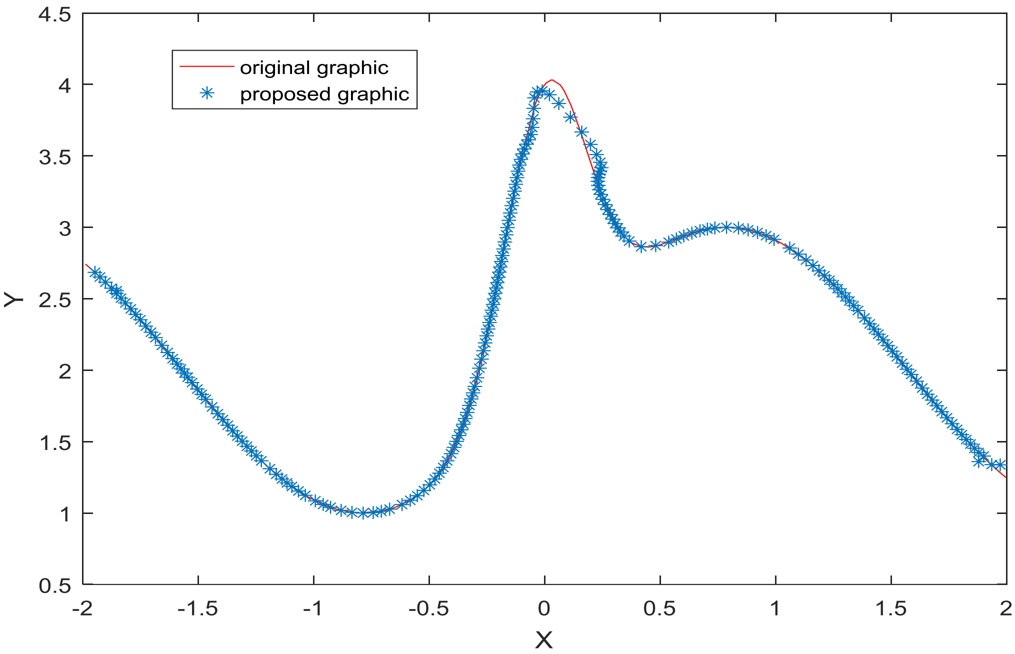

Figure 9. *Actual and estimated curves for Function 6*

6. Conclusion

In this study, the GWO algorithm for knot estimation in the problem of B-Spline curve fitting are tested on 6 different functions and the proposed method is seen to be successful in general. Seven different values are obtained in order to make more flexible comparison with the other studies in the literature. Reverse engineering is used to estimate the curve and *Centripetal* method is used for knot estimation. The GWO algorithm dynamically acquires location and number of knots. When the experimental studies are examined, it can be stated that the GWO algorithm is an alternative technique for curve fitting problem. For further studies, GWO can be used for solving other binary problems as well as curve fitting problem.

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