CHAOS
Theory and Applications
in Applied Sciences and Engineering

e-ISSN: 2687-4539
EDITORIAL
Vol.5 / No.3 / 2023 / pp.133-140

# Are Chaotic Attractors just a Mathematical Curiosity or Do They Contribute to the Advancement of Science?

**René Lozi** *,[1]
*LJAD, CNRS, Université Côte d'Azur, F-06000 Nice, France.

**ABSTRACT** Since the seminal work of Henri Poincaré on the three-body problem, and more recent research dating back to the second half of the 20th century on chaotic dynamical systems, many applications have emerged in different domains (economics, electronic, cryptography, physics, etc). We try to describe the evolution of the last 50 years on the subject and to find out whether applications have compromised the purity and beauty of theoretical research.

**KEYWORDS**
Chaotic attractors
Optimization
Cryptography
Memristor
Economy

## INTRODUCTION

Since the very beginning of their appearance in the history of humanity, research in mathematics has been guided by two different currents: theory and applications or in other words by beauty and utility. Around 5,000 years ago people in the Mesopotamia and Egypt began using arithmetic, algebra and geometry for commerce, trade, taxation and social activities. Later, in the 6th century BC, Greeks introduced mathematics as a "demonstrative discipline" (Heath 1931) (see (Høyrup J. 2011) for comparison between both approaches). This double current of research still functions today in competition-cooperation mode.

I had the immense privilege of being student of Jean Alexandre Dieudonné, one of the founding members of the Bourbaki group. For him, the only need to research mathematics for humanity was "for the honor of the human spirit" (as the great mathematician Karl Gustav Jacobi 1804-1851 said before him).

As a young student, I was imbued with this idea, but I was also attracted by research in physics and ultimately my university career was that of professor of numerical analysis. The subject of my doctoral thesis concerned the numerical analysis of bifurcations, which quickly led me to study chaotic dynamic systems from both aspects: theory and application. I was fortunate to see the birth of a new field of research in mathematics in the mid-1970s, that of chaotic attractors. I had the privilege of inventing one that, surprisingly, is still the subject of intensive research 50 years later. This is the reason why I often ask myself the question of the place of these attractors not only in mathematics, but also for the advancement of science.

It is widely accepted that the beginning of modern research on nonlinear dynamical systems is due to the initial work of Henri Poincaré on the three-body problem. Even if a real astronomical problem (will the Earth continue to orbit around the sun forever?) is at the origin of his reflection, no practical application of his "Méthodes nouvelles de la mécanique céleste" has guided his mind.

The "butterfly effect" reveled by Edward Lorenz in 1963 (Lorenz, E. N. 1963) and the "sexier" word "chaos" coined by James A. Yorke in 1975 (Li, T. Y. and Yorke, J. A. 1975) have brought global awareness of these concepts often not actually understood by the public. However, it is only at the beginning of 90' that the applications of chaotic properties of dynamical systems were introduced with the pioneering idea of synchronization of two chaotic attractors of Louis M. Pecora and Thomas L. Carroll (Pecora, L. M. and Carroll, T. L. 1990). Such concept was soon used (and improved) to transmit encrypted messages.

Since then, many applications have emerged in electronics (Chua circuit and memristors), optimization for meta-heuristic algorithms (particle swarm optimization (PSO), differential evolution (DE), Self-Organizing Migrating Algorithm (SOMA),...), cryptography based chaos, generation of pseudo-random number, economy, etc.

**Manuscript received:** 28 September 2023,
**Accepted:** 28 September 2023.

[1] Rene.LOZI@univ-cotedazur.fr (**Corresponding author**)

Have these applications compromised the purity and beauty of theoretical research? We attempt to describe the evolution of the last 50 years on the subject from the perspective of this question.

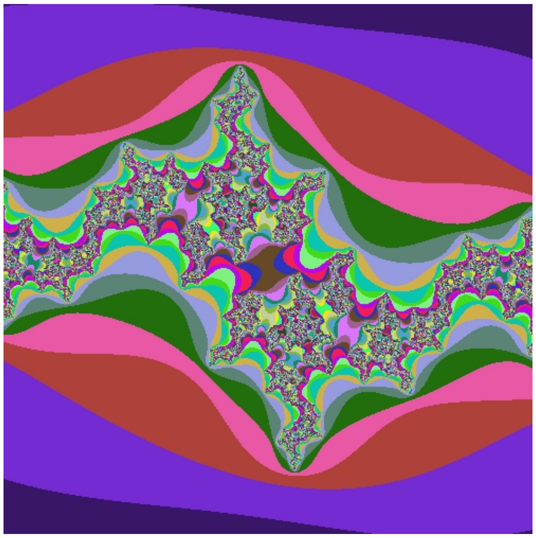


**Figure 1** Example of Julia set.

## THE DAWN OF CHAOTIC DYNAMICAL SYSTEMS

The study of the frighteningly complicated solutions discovered by Poincaré continued quietly for almost 80 years in several directions including conservative and dissipative dynamical systems, differential equations and difference equations. We can cite among many, the pioneer works of Pierre Fatou (1878-1929) and Gaston Julia (1893-1978) related to one-dimensional maps with a complex variable (see Figure 1), near a century ago; those of Cristian Mira and Igor Gumowski, who began their mathematical research in 1958 (the Gumowski-Mira map (1), see Figure 2), the fractals introduced in 1967 by Benoît Mandelbrot (1924-2010) (Mandelbrot 1967), and of course the continuous attractors of Lorenz (1963) (2) (Figure 3) and Rössler (1976) (Rössler 1976) (Rössler 2020), (3) (Figure 4); and the discrete attractors of Hénon (1976), Belykh (1976) (Belykh, V. N. *et al.* 2023) and Lozi (1977), among many others.

$$\begin{cases} x_{n+1} = f(x_n) + by_n \quad with \quad f(x_n) = ax + 2(1-a)\frac{x^2}{1+x^2}, \\ y_{n+1} = f(x_{n+1}) - x_n. \end{cases} \tag{1}$$

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = rx - y - xz, \\ \dot{z} = xy - bz. \end{cases} \tag{2}$$

$$\begin{cases} \dot{x}_1 = -x_2 - x_3, \\ \dot{x}_2 = x_1 + ax_2, \\ \dot{x}_3 = b + x_3(x_1 - c). \end{cases} \tag{3}$$

**Figure 2** Gumowski-Mira attractor for $a = 0.93333$, $b = 0.92768$.

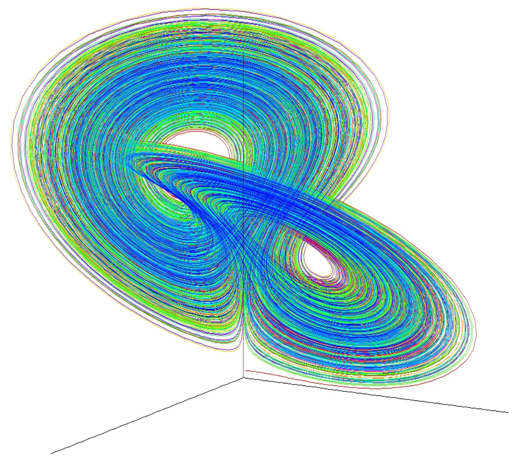


**Figure 3** Lorenz attractor for $\sigma = 10$, $b = 8/3$ and $r = 27$.

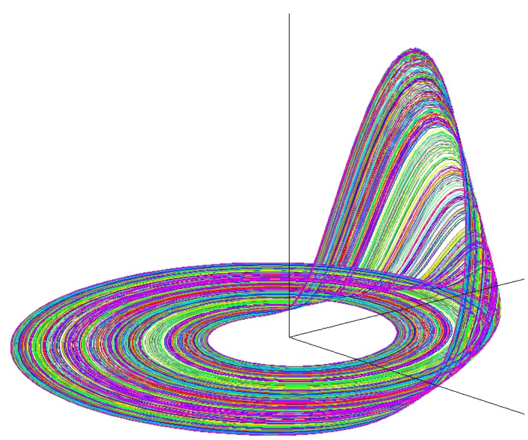


**Figure 4** Rössler attractor for $a = 0.2$, $b = 0.2$ and $c = 5.7$.

The images produced by these fractal sets 40 years ago, astonished not only mathematicians accustomed to geometric figures drawn only with rulers and compas, but also the general public. Heinz-Otto Peitgen published a book containing dozens of figures generated by complex dynamic systems, coining the name "computer art" (Peitgen, H.-O. and Richter, P. H. 2011). Today, no one is surprised by the use of chaotic systems in cinema or advertising.

## FIRST APPLICATIONS OF CHAOTIC DYNAMICAL SYSTEMS

### Electric circuits

In Japan the Hayashi's School (with disciples like Ikeda, Ueda and Kawakami) in the same period, were motivated by simulation of chaotic dynamics by electric and electronic circuits. Chaotic mappings were used as models of behavior of electric circuits (the Ikeda map (4), see Figure 5).

$$\begin{cases} x_{n+1} = 1 + u(x_n cos(t_n) - y_n sin(t_n) \quad with \quad t_n = 0.4 - \frac{6}{1+x_n^2+y_n^2}, \\ y_{n+1} = u(x_n sin(t_n) + y_n cos(t_n). \end{cases} \tag{4}$$

In 1983, Leon Chua invented a very simple electric circuit producing chaos (5). The advantage of this circuit (see Figure 6 a)) was that the variables of the mathematical equations corresponded to voltage and current and could be viewed on the screen of an oscilloscope (see Figure 6 (b)).

$$\begin{cases} \dot{x} = \alpha(y - \Phi(x)), \\ \dot{y}_2 = x - y + z, \\ \dot{x}_3 = -\beta y. \end{cases} \tag{5}$$

with $\Phi(x) = m_1 x + \frac{1}{2}(m_0 - m_1)\left[|x+1| - |x-1|\right]$.

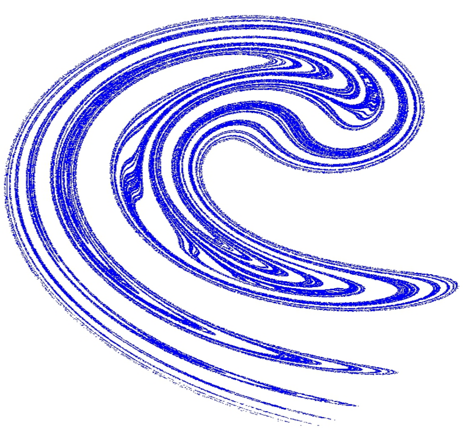
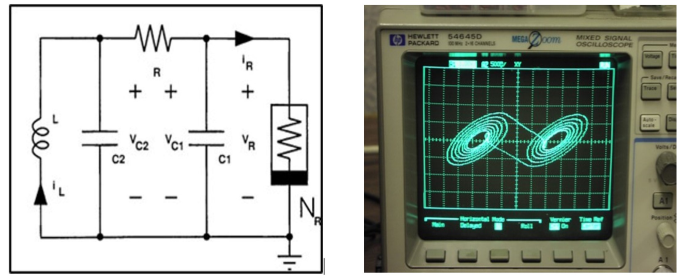


**Figure 5** Ikeda attractor for $u = 0.9$.

Before 1990 computers were not as efficient as they are today. It is why many experimenters still used analog electrical systems to explore the behavior of chaotic maps. Rodriguez-Vasquez et al. (Rodriguez-Vazquez, A. *et al.* 1987) in 1987 presented a special-purpose analog computer made of switched-capacitor circuit for analyzing chaos and bifurcation phenomena in nonlinear discrete dynamical systems modeled by discrete maps. They published results for four maps: the logistic map, a piece-wise linear map, the Hénon map and the Lozi map (6). For this last map, they built a rather complicated circuit realization (see Figure 25 of (Rodriguez-Vazquez, A. *et al.* 1987)) and compared the attractor measured from this circuit with the corresponding numerical simulation and found good agreement between them. Even if this example is not strictly speaking an application of the Lozi map for electric purposes, it constitutes one of the first examples of solid realization. However, these works cannot be considered as real applications.



(a) (b)

**Figure 6** (a) Chua circuit. (b) Chua attractor on oscilloscope.

$$\mathcal{L}_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - a|x| + y, \\ bx. \end{pmatrix} \tag{6}$$

### Secure communications

It was the discovery of the synchronization of chaotic electrical circuits by Pecora and Carroll (Pecora, L. M. and Carroll, T. L. 1990) in 1990 that sparked research into secure communications.

A first reported experimental secure communication system via chaotic synchronization using two Chua's circuits (one as master and one as slave) was built two years after. However, the signal recovered from this system which used the Chua circuit, contained some inevitable noise that degraded the fidelity of the original message. The system was soon improved in 1993, by cascading the output of the receiver in the original system, into an identical copy of this receiver (Lozi, R. and Chua, L. O. 1993) (see Figure 7). This cascading process was extended to multiple copies and analyzed using filtering theory (Lozi, R. 1995) in the case of a multi-tone signal.

In 2000, Dmitriev et al. (Dmitriev, A .S. *et al.* 2000) discussed a principle of multiple access, in satellite communication systems or cellular telephony based on fine structure of chaotic attractors, using control of special chaotic trajectories. They demonstrated the experimental verification of the proposed approach for asynchronous packet data transmission. In their approach they considered that a chaotic attractor can be treated as a number of countable sets of special trajectories: unstable periodic orbits (UPO) and transitions between these orbits. Instability of the periodic orbits and transient trajectories between them gives rise to irregular chaotic behavior. They used the set of the unstable

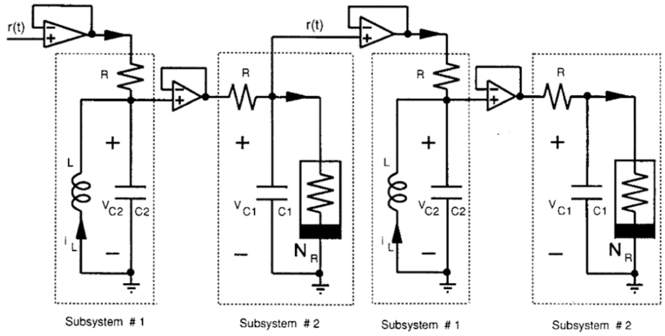



**Figure 7** Cascade of Chua circuits.

"skeleton" periodic trajectories, constituting the structure of the strange attractor (or a part of this set), as a "reservoir" of potential codes for multi-user communication systems. They observed that the multitude of the codes from a certain "reservoir" for communications is practically infinite, i.e., the number of users provided with individual code sets is unlimited.

As an example of the realization of their method, they considered twenty period-16 (UPO) of the Lozi map (6) for $a = 1.7$ and $b = 0.5$. They displayed the switching between them in the Figure 2 of (Dmitriev, A .S. *et al.* 2000) and showed from this diagram that the forming of all successive cycles (10-times repeating) is practically instantaneous. Improving their initial method, they remarked that unstable periodic orbits can be utilized for not only encoding the entire transmitted information, but also for attributing it to this or that group of users, i.e., they play the role of "chaotic markers". The idea to use the system of unstable periodic orbits as markers was applied to the problem of asynchronous packet transmission of data from several users through a single common communication channel. They concluded that the generating and controlling of UPO may be realized in rather high frequency band, provided in by modern digital methods.

### Memristors

In 1971, L.O. Chua predicted the existence of a missing fourth passive circuit element, in addition to the three classical ones: resistor, inductor and capacitor (Chua 1971). He called this new element "memristor" meaning it is a resistor with memory. It is characterized by a nonlinear constitutive relationship between the charge $q$ and the flux $\varphi$. Such a physical device would not be reported until 2008, when a physical model of a two-terminal *hp* device behaving like a memristor was announced (Strukov, D. B. *et al.* 2008) sparkling intense research with thousands of papers published to date. A general Ohm's law for theorizing this device was published ten years ago (Abdelouahab, M.- S. *et al.* 2008).

Nowadays, discrete memristor model is known as a research hotspot. Many researchers have devoted themselves to the analysis of chaotic phenomena in discrete memristors. Recently, hidden attractors have also been discovered in some discrete memristors based maps (Zhang, L. P. *et al.* 2022). Wang et al. (Wang, J. *et al.* 2022) included a discrete-time memristor to create a memristive Lozi map. This new 3-D memristor-based Lozi map was established by coupling a discrete memristor to the original 2-D Lozi map (6).

$$\begin{cases} x_{n+1} = 1 - a\,|x_n| + y_n, \\ y_{n+1} = bx_n + ky_n sin(z_n), \\ z_{n+1} = y_n + z_n, \end{cases} \tag{7}$$

where $k$ is a real valued control parameter coupling gain between the discrete-time memristor and the Lozi map. Since there are no fixed points but hyperchaos can emerge, the memristor-based Lozi map is a hidden hyperchaotic map.

For some specific control parameters, the 3-D memristor-based Lozi map can show heterogeneous and homogeneous hidden multistability. It should be noted that heterogeneous hidden multistability implies the coexisting behavior of multiple hidden attractors of different stability types, while homogeneous hidden multistability indicates the coexisting behavior of multiple hidden attractors of the same stability type but only in different dynamic intervals. In addition to the coexistence of these heterogeneous hidden attractors, the memristor-based Lozi map is very likely to produce the coexistence of homogeneous hidden hyperchaotic attractors, i.e., homogeneous hidden multistability. Therefore, the homogeneous hidden hyperchaotic attractors from the 3-D memristor based Lozi map can be robustly controlled by the memristor's initial conditions.

Additionally, Wang et al. implemented this memristor in a digital circuit based on a high-performance micro-controller. They physically obtained an image of the hyperchaotic hidden attractors using a digital oscilloscope. Eventually, a digital platform was exploited, and its experimental phase portraits were obtained to confirm the numerical portraits.

## APPLICATIONS IN OTHER DOMAINS

### Optimization

Most engineering problems can be defined as optimization problems, e.g. the finding of an optimal trajectory for a robot arm, the optimal thickness of steel in pressure vessels, the optimal set of parameters for controllers, optimal relations or fuzzy sets in fuzzy models, etc. Solutions to such problems are usually difficult to find their parameters which usually include variables of different types, such as floating point or integer variables.

Applications of chaotic maps in the now flourishing field of optimization took longer to appear than applications in electrical devices. The main reason comes from a paradigm shift in optimization algorithms: instead of using deterministic algorithms like gradient method or the steepest descent which are not efficient in high-dimensional problems optimization involving hundred or thousand of variables, heuristic algorithms based on an imitation of Darwin's theory of the evolution of species, were introduced a few decades ago. Such algorithms require easy access to random or chaotic numbers. This is why interest has only recently focused on chaotic attractors. It took three decades for this paradigm shift in the study of the chaotic maps (logistic, symmetric tent, Belykh, Hénon, Lozi, etc.). Instead of focusing on the theoretical study of their mathematical properties or on finding generalizations, Araujo and Coelho (Araujo and Coelho 2008) used them as a core for particle swarm optimization (PSO) (see Figure 8).

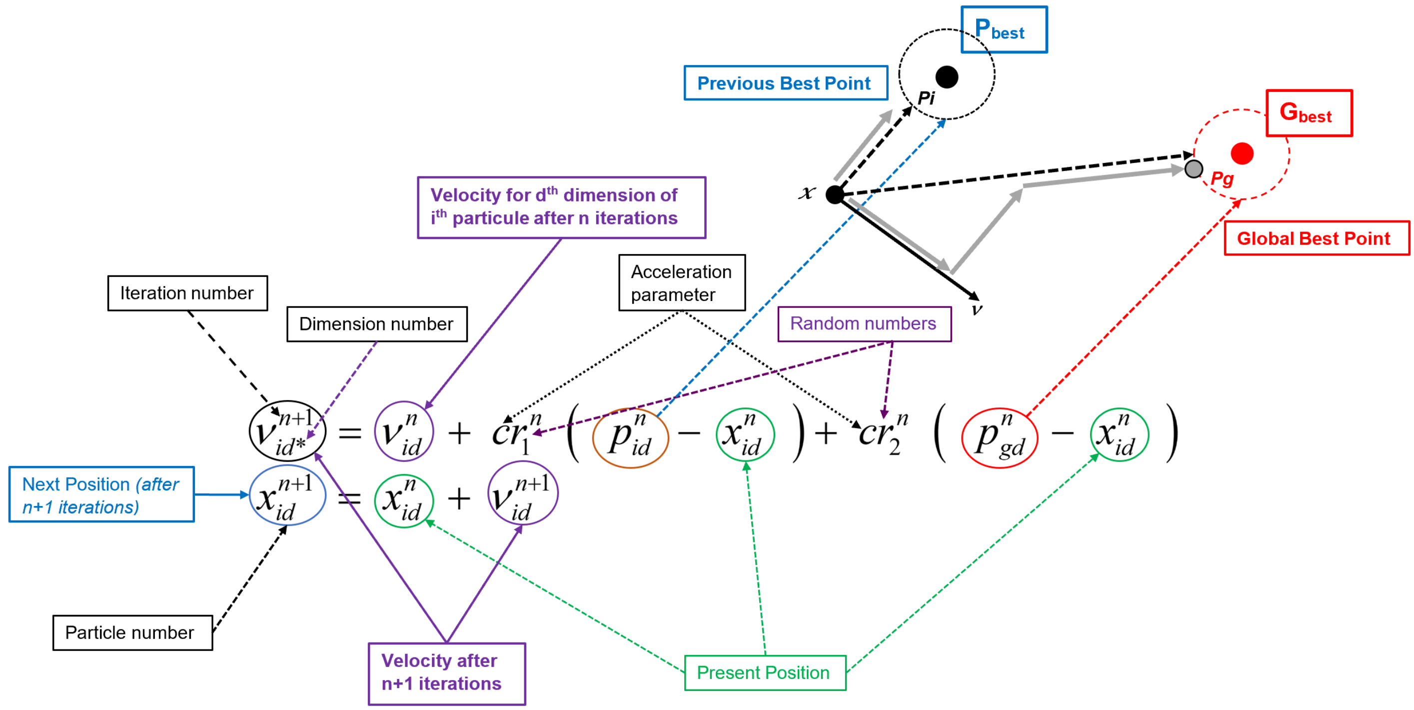



**Figure 8** Geometric core of Particle Swarm Optimization (PSO) algorithm.

Optimization algorithms based on the chaos theory are methodologies for searching optimal solutions that differ from any of the existing traditional stochastic optimization techniques. Due to the wandering of chaos, it can carry out overall searches in the solution space at higher velocities when compared to stochastic ergodic searches, which has its computing based on probabilities. This remark has been done in the pioneering work of Caponetto et al. (Caponetto, R. *et al.* 2003), who, four years before Araujo and Coelho found that chaotic sequences improved the performance of evolutionary algorithms.

PSO method was used for many purpose like the control of the thermal-vacuum system used at the Brazilian National Institute for Space Research (INPE). The original controller was designed to control the temperature on the shroud (set of pipes) of a chamber where satellites are tested (Marinke, R. *et al.* 2005). This method was used by Pluhacek et al. (Pluhacek, M. *et al.* 2012) who considered a Partial-Integral-Derivative (PID) controller for a Direct-Current (DC) motor system in order to obtain optimal settings. A DC motor is any of a class of rotary electrical motors that converts direct current electrical energy into mechanical energy. Proportional-Integral-Derivative (PID) control is the most common control algorithm used in industry and has been universally accepted in industrial control.

The optimization process involving PSO algorithm was applied to minimize errors of the output transfer function that can indicate the quality of regulation of such controller.

Another evolutionary optimization algorithm called Differential Evolution (DE) was used by Davendra et al. (Davendra, D. *et al.* 2010) in the same goal, and by Senkerik et al. (Senkerik, R. *et al.* 2013) in the task of optimization of batch chemical reactor geometry.

In 2004, Zelinka in (Zelinka, I. 2004), introduced SOMA (Self-Organizing Migrating Algorithm), a new class of stochastic optimization algorithms. Evolutionary algorithms work on populations of candidate solutions that are evolved in generations (two parents create one new individual – the offspring) in which only the best-suited – or fittest – individuals are likely to survive. Instead SOMA which can also works on a population of individuals, is based on the self-organizing behavior of groups of individuals in a "social environment", e.g. a herd of animals looking for food.

A group of animals such as wolves or other predators may be a good example. If they are looking for food, they usually cooperate and compete so that if one member of the group is successful (it has found some food or shelter) then the other animals of the group change their trajectories towards the most successful member. If a member of this group is more successful than the previous best one (is has found more food, etc.) then again all members change their trajectories towards the new successful member. It is repeated until all members meet around one food source. This principle from the real world is of course strongly simplified. Yet even so, it can be said it is that competitive-cooperative behavior of intelligent agents that allows SOMA to carry out very successful searches.

Recently Zelinka et al. used SOMA (Zelinka, I. *et al.* 2023) for the design of quantum computing circuits for the future quantum computers.

Of course, we cannot present, within the limited extend of this editorial, all the dozens of algorithms using chaotic attractors (see (Lozi, R. 2023) for a survey).

### Cryptography

Cryptography is the primary means of protecting communications in the cyber world in which mankind lives today. Modern technologies involve fast communication links between potentially billions of devices via complex networks (satellite, mobile phone, Internet, etc.). The primary concern posed by these complex and tangled networks is their protection against passive and active attacks that could compromise public safety and privacy. Cryptography has been around for over two thousand years with the famous Caesar code used by Emperor Julius Caesar. Today, the properties of chaotic attractors are recognized as being the basis of part of the methods of cryptography.

Among many algorithms based on chaotic dynamical systems, we can mention the image encryption algorithms, like the optical color image encryption scheme based on fingerprint key and three-step phase-shifting digital holography which was proposed by Su et al. (Su, Y. *et al.* 2021). In this scheme the fingerprint is served as secret key directly. The random phase masks generated from the fingerprint using secure hash algorithm (SHA-256) and the chaotic Lozi map are just used as interim variables. The fingerprint is served as secret key directly. With the help of the fingerprint-based random phase masks located in the linear canonical transform domain and the three-step phase-shifting digital holography, the primary color image that is hidden into a grey-scale carrier image can be encrypted into three noise-like holograms. In addition, the parameters of the chaotic Lozi map and linear canonical transform can also provide additional security to the proposed encryption scheme. Other examples of cryptography-based chaos can be found in (El Assad, S. *et al.* 2022).

### Economy

Since twenty years, one can find application of chaotic dynamical systems in economy. For example Tang et al. (Tang, T. W. *et al.* 2004) carried out an analysis of Parrondo's games with different chaotic switching strategies. The performance of Parrondo's games was compared with random and periodic switching strategies. The main idea of Parrondo's paradox, exposed in 1996, is that two individually losing games can be combined to win via deterministic or non-deterministic mixing of games (Harmer, G. P. *et al.* 2001). In (Tang, T. W. *et al.* 2004) a fair way to compare random and chaotic Parrondo's games was generalized. The logistic, tent, sinusoidal and Gaussian 1-D maps were considered together with Hénon and Lozi maps.

To play chaotic Parrondo's games, one of these chaotic generator being chosen, we consider a sequence that it generates from an initial value. Then every $n$-th iterate of such sequence determines whether Game A or B is played. Of course the outcomes of Parrondo's game are affected by the different switching strategies applied and the initial value chosen. The proportion of Game A and B played is equal for all switching strategies for a fair comparison. In conclusion, the authors found that chaotic Parondo's games can give a higher rate of winning compared to random switching strategies. This result recalls the remark made by Caponetto et al. (Caponetto, R. *et al.* 2003) that chaotic sequences can improve the performance of evolutionary algorithms versus random sequences.

Another examples can be found in (Commendatore, P. *et al.* 2015) in which Commendatore et al. proposed a new economic geography model which describes spatial distribution of industrial activity in the long run across three identical regions depending on the balancing of agglomeration and dispersion forces. It is defined by a two-dimensional piecewise smooth map depending on four parameters. They discussed the emergence of the Wada basins of coexisting attractors leading to the so-called final state sensitivity (see Figure 9). And also, in (Sushko, I. *et al.* 2023, in progress) in which Sushko et al. studied the dynamics of a financial market model with trend-followers and contrarians proposed a 2D-piecewise linear discontinuous map $F$ given by (8) (see Figure 10).

$$\begin{cases} x_{n+1} = (1 - k_1 - b)\, x_n + k_1 x_{n-1} \quad if\, |x_n - x_{n-1}| < k, \\ x_{n+1} = (1 - k_2 - b)\, x_n + k_2 x_{n-1} + m \quad if\, |x_n - x_{n-1}| > k. \end{cases} \tag{8}$$

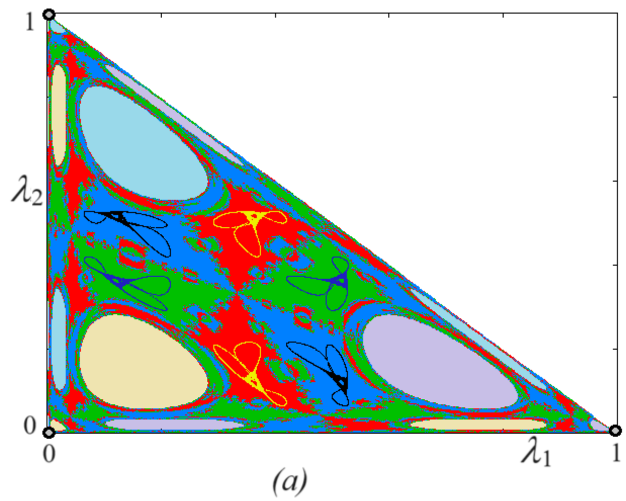



**Figure 9** 2D piecewise smooth map $G$ governing dynamics of a three region New Economic Geography model. Basins of attraction of the fixed points $(0,0)$, $(1,0)$, $(0,1)$ (attracting in Milnor sense) and of the three 2-piece chaotic attractors.

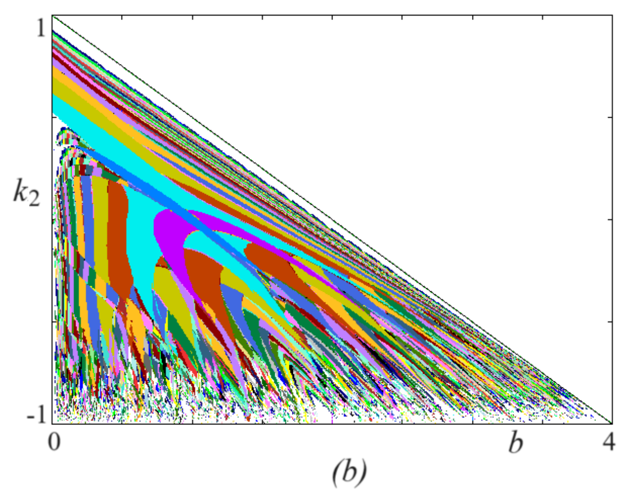



**Figure 10** Periodicity regions (where different colors are related to attracting cycles of different periods) in the $(b; k2)$-parameter plane for $k1 = -1$, $m = 1.9$, $k = 0.1$.

## THEORETICAL RESULTS

We have shown that chaotic attractors have been used for more than thirty years for applications in different fields. This does not mean that they did not advance pure mathematics.

It is difficult to list all the improvements in chaotic dynamical systems theory and bifurcation theory, so many have been made over the last half century. We can only name a few, such as the concepts of Smale'Axiom A and horseshoe, homoclinic bifurcation and Shilnikov attractors, border-collision bifurcation, ergodicity, hyperbolicity, symbolic dynamics and kneading sequences, Sinai-Bowen-Ruelle measures, fractal dimensions, general usage of fractional derivatives, fractional maps, topological entropy, etc.

I think the best example of a theory-practice-theory approach is that of chimeras. Following the discovery of the synchronized chaotic attractors (theory), research focused on network of attractors with several topologies for multiple purposes like the creation of Pseudo Random Number Generation for cryptography (Garasym, O. *et al.* 2017) (practice).

Describing the dynamical properties of synchronization of such networks, special solutions called "chimeras" and "solitary states" were highlighted (theory).

Rybalova et al. (Rybalova, E. *et al.* 2018) considered a complex system consisting of three coupled rings of nonlocally coupled chaotic maps. This multilayer network is described by the following equations:

$$\begin{cases} x_{n+1}^i = f(x_n^i, y_n^i) + \frac{\sigma_1}{2P}\sum_{j=i-P}^{j=i+P}\left[f(x_n^j, y_n^j) - f(x_n^i, y_n^i)\right] + \gamma_1 F_n^i, \\ y_{n+1}^i = b x_n^i, \\ u_{n+1}^i = f(u_n^i, v_n^i) + \frac{\sigma_2}{2R}\sum_{j=i-R}^{j=i+R}\left[f(u_n^j, v_n^j) - f(u_n^i, v_n^i)\right] + \gamma_2 G_n^i, \\ v_{n+1}^i = b x_n^i, \\ z_{n+1}^i = f(z_n^i, s_n^i) + \frac{\sigma_3}{2T}\sum_{j=i-T}^{j=i+T}\left[f(z_n^j, s_n^j) - f(z_n^i, s_n^i)\right] + \gamma_3 H_n^i, \\ s_{n+1}^i = b x_n^i, \end{cases} \tag{9}$$

The first system of equations in (9) specifies a ring network of nonlocally coupled Hénon maps with $f$ defined by (10)

$$f(x_n, y_n) = 1 - a x_n^2 + y_n, \tag{10}$$

with $a = 1.4$, $b = 0.3$, $\sigma_1 = 0.72$ and and $P = 320$. The second pair of equations corresponds to the ring of nonlocally coupled Lozi maps with $f$ defined by (11)

$$f(x_n, y_n) = 1 - a\,|x_n| + y_n, \tag{11}$$

and is analyzed for $a = 1.4$, $b = 0.3$, $\sigma_2 = 0.206$ and $R = 180$. The third pair of equations also determines the ring of nonlocally coupled Hénon maps with $a = 1.4$, $b = 0.3$, $\sigma_1 = 0.295$ and $T = 320$.

The first two rings are coupled inertially via the coupling functions $F_n^i = -G_n^i = u_n^i - x_n^i$ with the coupling coefficients $\gamma_1$ and $\gamma_2$. The third ring nodes is connected unidirectionally with the first ring units by the coupling term $\gamma_3 H_n^i$ where

$$H_n^i = f(x_n^i, y_n^i) - f(z_n^i, s_n^i), \tag{12}$$

defines the diffusive coupling with the coupling coefficient $\gamma_3$. $N$ is the number of elements in the ensemble of coupled equations in each ring. The coupling parameters $\sigma_{1,2,3}$ characterize the coupling strength, and $2P, 2R, 2T$ are the number of neighbors on each ring ($P$ (resp. $R$, $T$) neighbors on the either side of the $i$th element). The initial conditions are chosen to be randomly distributed in the interval $[-0.5, 0]$ for all the variables of the network (9).

Using numerical simulation they have demonstrated that the network of two symmetrically coupled ensembles of Hénon and Lozi maps can show a novel type of chimera state, a solitary state chimera (SSC), when the coupling between them is weak. This special structure emerges in the case if the Lozi ensemble exhibits a developed regime of solitary states. The SSC is fairly stable and is observed within a finite range of parameter variation. If the two layer network of nonlocally coupled Hénon and Lozi maps in the solitary state chimera is unidirectionally coupled to the third ring of nonlocally coupled Hénon maps, then the effect of external synchronization can be observed in a finite range of the coupling coefficient $\gamma_3$.

## CONCLUSION

The first research on chaotic dynamic systems marked the mind of the public by the beauty of the images that these attractors made it possible to draw. Nowadays applications of chaotic attractors in several domains (see (Lozi, R. 2023) for a survey) is a flourishing domain of research since three decades and can nevertheless produce wonderful images (Figures 9, 10). In the mean time, theoretical research is still very much alive and offers new mathematical tools such as chimeras, fractional differential equations and fractional mappings which in turn will allow the development of new applications.

Chaotic attractors are definitely not a mathematical curiosity.

### Acknowledgments

The author thanks Laura Gardini from University of Urbino (Italy) and Iryna Sushko from National Academy of Sciences of Ukraine, Kyiv (Ukraine) for drawing two special figures for this editorial.

### Availability of data and material

Not applicable.

### Conflicts of interest

The author declares that there is no conflict of interest regarding the publication of this paper.

## LITERATURE CITED

Abdelouahab, M.- S., Lozi, R., and Chua, L. O., 2008 Memfractance: A mathematical paradigm for circuit elements with memory. Int. J. Bifurc. Chaos **24**: 1430023.

Araujo, E. and L. S. Coelho, 2008 Particle swarm approaches using lozi map chaotic sequences to fuzzy modelling of an experimental thermal-vacuum system. Applied Soft Computing **8**: 1354–1364.

Belykh, V. N., Barabash, N. V., and Grechko, D. A. , 2023 Existence proofs for strange attractors in piecewise-smooth nonlinear lozi-hénon and belykh maps. Journal of Difference Equations and Applications pp. 1–21.

Caponetto, R., Fortuna, L., Fazzino, S., and Xibilia, M. G., 2003 Chaotic sequences to improve the performance of evolutionary algorithms. IEEE Trans. Evol. Comput. **7**: 289–304.

Chua, L. O., 1971 Particle swarm approaches using Lozi map chaotic sequences to fuzzy modelling of an experimental thermal-vacuum system. IEEE Trans. Circuit Th. **18**: 507–519.

Commendatore, P., Kubin, I., and Sushko, I., 2015 Typical bifurcation scenario in a three region symmetric new economic geography model. Mathematics and Computers in Simulation **108**: 63–80.

Davendra, D., Zelinka, I., and Senkerik, R., 2010 Chaos driven evolutionary algorithms for the task of pid control. Computers and Mathematics with Applications **60**: 1088–1104.

Dmitriev, A .S., Panas, A. I., and Starkov, S. O., 2000 Multiple access communication based on control of special chaotic trajectories. Proceedings of 2nd International Conference. Control of Oscillations and Chaos(COC-2000), St. Petersburg, Russia **3**: 518–522.

El Assad, S., Lozi, R., and Puech, W., 2022 *Special Issue "Cryptography and Its Applications in Information Security"*. Applied Science, MDPI,.

Garasym, O., Lozi, J. P., and Lozi, R., 2017 How useful randomness for cryptography can emerge from multicore-implemented complex networks of chaotic maps. Journal of difference equations and applications **23**: 821–859.

Harmer, G. P., Abbott, D., Taylor, P. G., and Parrondo J. M. R., 2001 Brownian ratchets and parrondo's games. Chaos **11**: 705A–714.

Heath, T. L., 1931 *A Manual of Greek Mathematics*. Oxford University Press, Oxford.

Høyrup J., 2011 Mesopotamian calculation: Background and contrast to greek mathematics. Contribution to IX Congresso della Società Italiana di Storia della Matematica, Genova, 17–19 novembre .

Li, T. Y. and Yorke, J. A. , 1975 Period three implies chaos. The American Mathematical Monthly **82**: 985–992.

Lorenz, E. N., 1963 Deterministic nonperiodic flow. Journal of the Atmospheric Sciences **20**: 130–141.

Lozi, R., 1995 Secure communications via chaotic synchronization in chua's circuit and bonhoeffer-van der pol equation: numerical analysis of the errors of the recovered signal. IEEE international Symposium on circuits and systems **1**: 684–687.

Lozi, R., 2023 Algorithms using the chaotic lozi map for real applications or for applications exploring some new mathematical fields: a survey. Algorithms, To appear pp. 1319–1325.

Lozi, R. and Chua, L. O., 1993 Secure communications via chaotic synchronization ii: noise reduction by cascading two identical receivers. International Journal of Bifurcation and Chaos **3**: 1319–1325.

Mandelbrot, B., 1967 How long is the coast of britain? statistical self-similarity and fractional dimension. Science, New Series **156**: 1088–1104.

Marinke, R., Araujo, J. E., Coelho, L. S., and Matko, I., 2005 Particle swarm optimization (pso) applied to fuzzy modeling in a thermal-vacuum system. Proceedings of the 5th International Conference on Hybrid Intelligent Systems, Rio de Janeiro, Brazil pp. 67–72.

Pecora, L. M. and Carroll, T. L., 1990 Synchronization in chaotic systems. Phys. Rev. Lett. **64**: 821.

Peitgen, H.-O. and Richter, P. H., 2011 *The Beauty of Fractals: Images of Complex Dynamical Systems*. Springer Verlag, Berlin.

Pluhacek, M., Senkerik, R., Davendra, D., and Zelinka, I., 2012 Designing pid controller for dc motor system by means of enhaced pso algorithm with discrete lozi map. Proceedings 26th European Conference on Modelling and Simulation,ECMS 2012, Troitzsch, K. G.; Möhring, M.; Lotzmann, U. (Eds) pp. 405–409.

Rodriguez-Vazquez, A., Huertas, J. L., Perez-Verdu, B., and Chua, L. O. , 1987 Chaos from switched-capacitor circuits: Discrete maps. Proceedings of the IEEE **75**: 1090–1106.

Rössler, O. E., 1976 Chaotic behavior in simple reaction system. Zeitschrift für Naturforschung A **31**: 259–264.

Rössler, O. E., 2020 On the Rossler Attractor. Chaos Theory and Applications **2**: 1–2.

Rybalova, E., Semenova, N., and Anishchenko, V., 2018 Solitary state chimera: Appearance, structure, and synchronization. International Symposium on Nonlinear Theory and Its Applications NOLTA 2018, Tarragona, Spain pp. 601–604.

Senkerik, R., Davendra, D, Zelinka, I., Pluhacek, M., and Kominkova Oplatkova, Z., 2013 Chaos driven differential evolution with lozi map in the task of chemical reactor optimization. Lecture Notes in Computer Science **7895**.

Strukov, D. B., Snider, G. S., Stewart, D. R., and Williams, R. S., 2008 The missing memristor found. Nature **453**: 80–83.

Su, Y. , Xu, T. , Li, T. , Zhao, J. , and Liu, S. , 2021 Optical color image encryption based on fingerprint key and phase-shifting digital holography. Optics and Lasers in Engineering **140**: 106550.

Sushko, I. , Tramontana, F. , and Gardini, L. , 2023, in progress Optical color image encryption based on fingerprint key and phase-shifting digital holography. Working Papers Series in Economics, Mathematics and Statistics, University of Urbino .

Tang, T. W. , Allison, A., and Abbott, D. , 2004 Investigation of chaotic switching strategies in parrondo's games. Fluctuation and Noise Letters **4**: L585–L596.

Wang, J., Gu, Y., Rong, K., and Xu, Q.and Zhang, X., 2022 Memristor-based lozi map with hidden hyperchaos. Mathematics **10**: 3426.

Zelinka, I., 2004 Self-organizing migrating algorithm. Studies in Fuzziness and Soft Computing **141**: 167–217.

Zelinka, I., Kojecky, L., Lampart, M., Nowakova, M. J., and Plucar, J., 2023 isoma swarm intelligence algorithm in synthesis of quantum computing circuits. Applied Soft Computing **142**: 110350.

Zhang, L. P., Wei, Z. C., Jiang, H. B., Lyu, W. P., and Bi, Q. S., 2022 Extremely hidden multistability in a class of a two dimensional maps with a cosine memristor. Chin. Phys. **B 31**: 100503.

***How to cite this article:*** Lozi, R. Are Chaotic Attractors just a Mathematical Curiosity or Do They Contribute to the Advancement of Science? *Chaos Theory and Applications*, 5(3), 133-140, 2023.