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# Apollonius Problem and Caustics of an Ellipsoid

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**ABSTRACT**

**In the paper we discuss the Apollonius Problem on the number of normals of an ellipse passing through a given point. It is known that the number is dependent on the position of the given point with respect to a certain astroida. The intersection points of the astroida and the ellipse are used to study the case when the given point is on the ellipse. The problem is then generalized for 3-dimensional space, namely for ellipsoids. The number of concurrent normals in this case is known to be dependent on the position of the given point with respect to the caustics of the ellipsoid. If the given point is on the ellipsoid then the number of normals dependends on the position of the point with respect to the intersections of the ellipsoid with its caustics. The main motivation of this paper is to find parametrizations and classify all possible cases of these intersections.**

***Keywords:*** *Ellipsoid, Apollonius, caustics, centro-surface, astroida, ellipse.*
***AMS Subject Classification (2020):*** *Primary: 53A05; Secondary: 53A04, 51M16.*

## 1. Introduction

How many normals can one draw from a point to an ellipse? In the current paper we will try to solve this problem and its generalization to 3 dimensions, using the methods of differential and integral calculus and differential geometry, which were not around when Apollonius of Perga (c. III-II centuries BC) first asked and answered this question in his famous work *Conics* [3]. Their number is not the only interesting question about these normals. For example, theorem proved by Joachimstal in 1843 states that if $AB_1$, $AB_2$, $AB_3$, and $AB_4$ are these normals, then points $B_1,\ B_2,\ B_3$, and the point diametrically opposite to $B_4$, with respect to the center $O$, of the ellipse, are concyclic [37] (see also Sect. 17.2 in [13], [34], [16], [21], [44]). There are more results related to this fact in [11], Sect. 17.7.3.

The problem about the number of normals, which Apollonius called as *the shortest* and sometimes *the longest line segments*, appeared in the fifth book of Apollonius, which survived only in Arabic translation [45]. For the outline of the solution of Apollonius, one can check [59], Chapter VII, p. 260-261. There is also a lively discussion of this problem in pages 131-135 of [55], [35]. The problem was also mentioned by V.I. Arnold in his paper [4], Chapter IV and related popular lecture [5] is available online both as a brochure and as a YouTube video. The main objective of this paper is to study in detail the cases when point $A$ is on the ellipse, and generalize these results to three dimensions for ellipsoids. Some of the results in the current paper were presented at the Maple Conference 2022 [2], again available as a YouTube video.

## 2. Apollonius problem for plane

Let the ellipse be defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{1}$$

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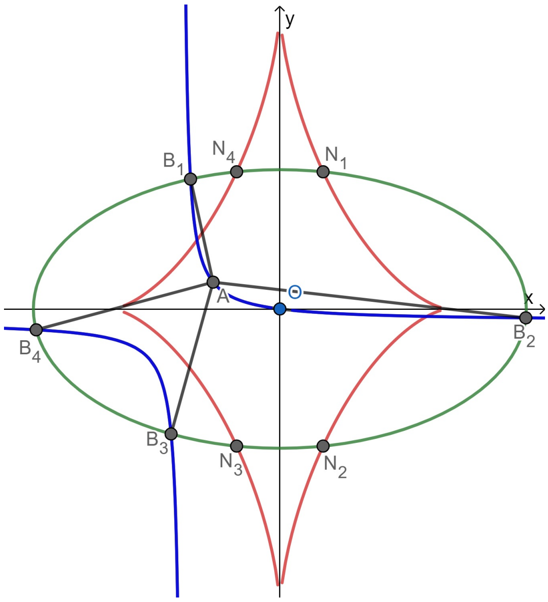


Figure 1: Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (green), its 4 normals (black), the Apollonius hyperbola $y=\frac{xY}{\epsilon X-(\epsilon-1)x}$ (blue), and astroida $\sqrt[3]{a^2X^2}+\sqrt[3]{b^2Y^2}=\sqrt[3]{(a^2-b^2)^2}$ (red). Created using GeoGebra. For more details see `https://www.geogebra.org/calculator/upkya8nx`.

where we assume that $a>b>0$. Let us take an arbitrary point $A(X,Y)$ on the plane of the ellipse. We want to find point $B(x,y)$ on the ellipse such that $AB$ is perpendicular to the tangent of the ellipse at $B$. The slope of this tangent line is $y'=-\frac{b^2x}{a^2y}$, and therefore $\frac{y-Y}{x-X}=\frac{a^2y}{b^2x}$. From this we obtain the equation of the rectangular hyperbola $y=\frac{xY}{\epsilon X-(\epsilon-1)x}$, where $\epsilon=\frac{a^2}{b^2}$. The intersection points $B_1$, $B_2$, $B_3$, and $B_4$ of this hyperbola with the ellipse give us the required normals $AB_1$, $AB_2$, $AB_3$, and $AB_4$. In his solution, Apollonius also used this hyperbola, which is now known as the Apollonius hyperbola [11], Sect. 17.5.5.6. The asymptotes of the hyperbola are $x=\frac{a^2X}{a^2-b^2}$ and $y=\frac{b^2Y}{b^2-a^2}$. One of the branches of this hyperbola passes through the center of the ellipse and therefore, there are at least 2 intersection points with the ellipse. The other branch may or may not intersect the ellipse. In the cases when $X=0$ and $Y=0$, the hyperbola degenerates to a pair of perpendicular lines $x=0, y=\frac{b^2Y}{b^2-a^2}$ and $x=\frac{a^2X}{a^2-b^2}, y=0$, respectively. Let us denote by $n(A)$ the total number of intersections of the hyperbola with the ellipse. Since the intersection points are the solutions of a fourth order equation, $n(A)$ can not exceed 4. Let us find points $A$, where $n(A)$ jumps from 4 to 2. This happens when the Apollonius hyperbola is tangent to the ellipse i.e. the slopes are equal at the intersection point: $-\frac{x}{\epsilon y}=\frac{\epsilon XY}{(\epsilon X-(\epsilon-1)x)^2}$. Using this and the equation of the ellipse, we obtain

$$\frac{x}{a}=\frac{a}{a^2-b^2}\left(\sqrt[3]{\frac{b^2Y^2X}{a^2}}+X\right),\ \frac{y}{b}=\frac{b}{b^2-a^2}\left(\sqrt[3]{\frac{a^2X^2Y}{b^2}}+Y\right),$$

which when used back in the equation of the ellipse, after some simplifications gives

$$\sqrt[3]{a^2X^2}+\sqrt[3]{b^2Y^2}=\sqrt[3]{(a^2-b^2)^2}. \qquad (2)$$

It is the equation of *astroida* in $X,Y$ coordinates. This curve is of 6th order and its parametric equations can be written as $(x,y)=\left(\frac{a^2-b^2}{a}\cos^3 t, \frac{a^2-b^2}{b}\sin^3 t\right)$. In the interior region of this astroida $n(A)=4$. Outside of the astroida $n(A)=2$. On the astroida itself $n(A)=3$, except for vertices $\left(\pm\frac{a^2-b^2}{a},0\right)$ and $\left(0,\pm\frac{a^2-b^2}{b}\right)$ of the astroida, where again $n(A)=2$. This is essentially what was done by Apollonius, which is a remarkable achievement, taking into account the mathematical tools available at the time. In [11], Sect. 17.7.4 (see also p. 204, [50]) it was mentioned that this astroida is the evolute of the ellipse and therefore drawing normals to the ellipse can be done by drawing tangent lines of the astroida.

Let us now suppose that point $A(X,Y)$ is on the ellipse: $X=x,\ Y=y$. Since the Apollonius hyperbola passes through $A(X,Y)$, one of points $B_1$, $B_2$, $B_3$, and $B_4$, coincide with $A$. For the points of ellipse (1) in astroida (2),

$n(A) = 4$. For the points of ellipse (1) outside astroida (2), $n(A) = 2$. For the intersection points $N_1,\ N_2,\ N_3$, and $N_4$ of ellipse (1) and astroida (2), $n(A) = 3$. The coordinates of these points can be easily determined: $(\pm x_0, \pm y_0)$ and $(\pm x_0, \mp y_0)$, where

$$x_0 = \sqrt{\frac{a^4(a^2-2b^2)^3}{(a^2-b^2)(a^2+b^2)^3}},\ y_0 = \sqrt{\frac{b^4(2a^2-b^2)^3}{(a^2-b^2)(a^2+b^2)^3}}.$$

Thus we proved

**Theorem 2.1.** *For ellipse (1) and astroida (2), the following cases are possible:*

1. *If $a^2 > 2b^2$ then the points $(\pm x_0, \pm y_0)$ and $(\pm x_0, \mp y_0)$ separate the ellipse into 4 regions where $n(A) = 4$ and $n(A) = 2$.*
2. *If $a^2 \le 2b^2$, then for all the points of ellipse (1), $n(A) = 2$.*

Noting this, we can say that the Apollonius problem for the number of concurrent normals of an ellipse is completely solved. There is also a three dimensional variant of this problem, where one takes point $A(X,Y,Z)$ outside of the plane of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ z = 0$ and counts the number of lines $AB$, such that $B(x,y,0)$ is on the ellipse, and $AB$ is perpendicular to the tangent of the ellipse at point $B$ (see Fig. 2). But this variant is easily reduced to the planar case. Consider the projection $A'(X,Y,0)$ of $A$ onto plane $z = 0$. If $A'B$ is a normal of the ellipse then by The Theorem of the Three Perpendiculars, $AB$ is also perpendicular to the tangent of the ellipse at point $B$. Therefore, $n(A)$ is 2, 3, or 4 depending on the position of point $A$ with respect cylindrical surface defined by the same equation for astroida (2).

Apollonius did not mention any practical uses for his results, except that these normals corresponding to minimal and maximal distances, are worth investigating for their own sake and that, in contrast to the tangents (See Appendix), the normals were not studied much by the earlier mathematicians. Because of this connection with the extremal distances, there can be applications in optics, wavefronts, mathematical billiards, etc. One of the applications of these results in astronomy can be a possible explanation for the presence of 4 images of a distant quasar, whose light is being bent around an approximately elliptical Einstein Ring formed by two galaxies 3.4 billion light-years away [27] (see also Figure 1 in [53]).

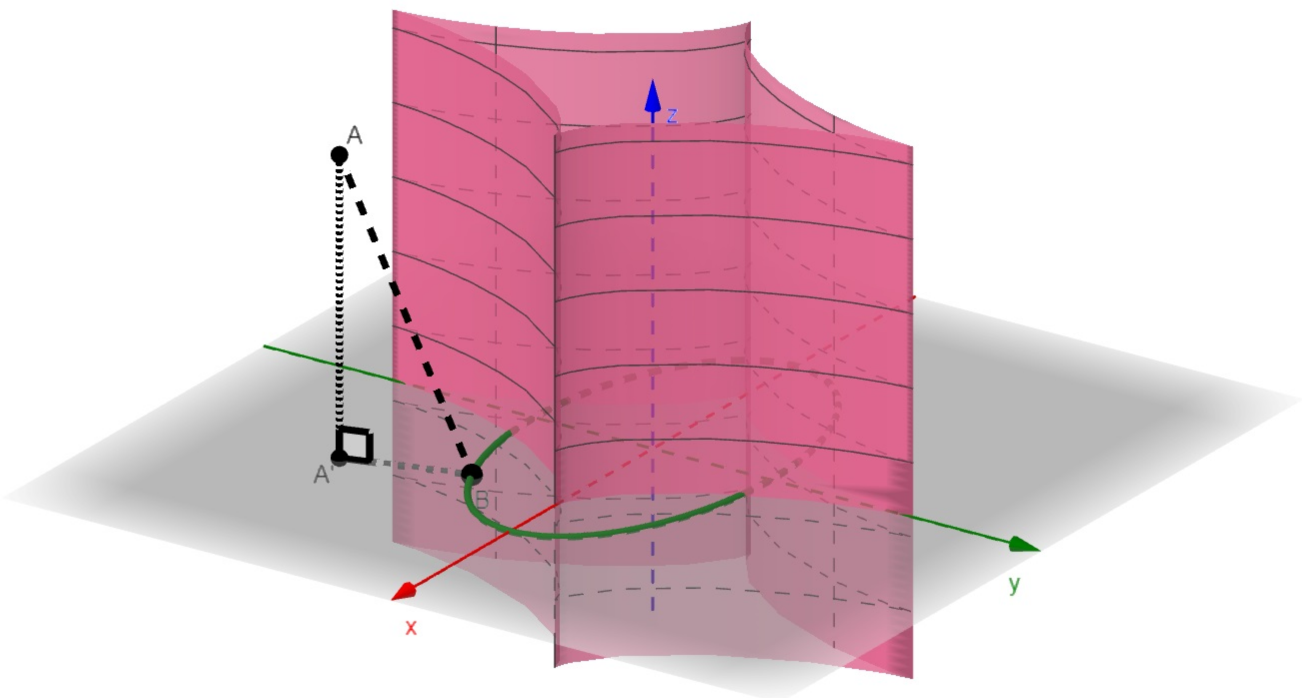


Figure 2: Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ \ z = 0$ (green), line $AB$ (black), and astroidal cylinder $\sqrt[3]{a^2X^2} + \sqrt[3]{b^2Y^2} = \sqrt[3]{(a^2-b^2)^2}$ (red). Created using GeoGebra.

## 3. Apollonius problem for space

Let us now consider three dimensional generalization of this problem. How many concurrent normals of an ellipsoid are there? In this form the problem was studied through analytic methods in [38], [20] (see also [32], [49] for geometric considerations) and generalized for higher dimensions in [47]. The answer to this question will be given in the next section. The literature about the problem of normals to surfaces of second order is vast and we refer the reader to Chapter III, Sections E2 and E3 of [48], which contains a detailed discussion of the history and many references for this 3 dimensional case and the previous planar case.

Let an ellipsoid be defined by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \tag{3}$$

where we assume that $a>b>c>0$. Let us take an arbitrary point $A(X,Y,Z)$ and find the number $n(A)$ of points $B(x,y,z)$ on the ellipsoid such that $AB$ is the normal line of the plane tangent to the ellipsoid at $B$. Since the outer normal vector of the plane tangent to the ellipsoid at $B(x,y,z)$ is $\mathbf{N}=\left(\frac{x}{a^2},\frac{y}{b^2},\frac{z}{c^2}\right)$,

$$\frac{x-X}{\frac{x}{a^2}}=\frac{y-Y}{\frac{y}{b^2}}=\frac{z-Z}{\frac{z}{c^2}}=-t,$$

where $t$ is a parameter. From this we find a parametric representation of *the cubic hyperbola* (see p. 204, [48])

$$\mathbf{r}(t)=\left(\frac{a^2X}{a^2+t},\frac{b^2Y}{b^2+t},\frac{c^2Z}{c^2+t}\right),$$

whose intersections with the ellipsoid give the base points of the normals through $A$. The asymptotes of this curve are lines

$$\mathbf{r}_1(t)=\left(t,\frac{b^2Y}{b^2-a^2},\frac{c^2Z}{c^2-a^2}\right),$$

$$\mathbf{r}_2(t)=\left(\frac{a^2X}{a^2-b^2},t,\frac{c^2Z}{c^2-b^2}\right),$$

$$\mathbf{r}_3(t)=\left(\frac{a^2X}{a^2-c^2},\frac{b^2Y}{b^2-c^2},t\right).$$

If $X=0$, $Y=0$, and $Z=0$ then the cubic hyperbola splits into a line, which served earlier as an asymptote of the cubic hyperbola, and a hyperbola:

$$\mathbf{r}(t)=\mathbf{r}_1(t),\ \mathbf{r}(t)=\left(0,\frac{b^2Y}{b^2+t},\frac{c^2Z}{c^2+t}\right);$$

$$\mathbf{r}(t)=\mathbf{r}_2(t),\ \mathbf{r}(t)=\left(\frac{a^2X}{a^2+t},0,\frac{c^2Z}{c^2+t}\right);$$

$$\mathbf{r}(t)=\mathbf{r}_3(t),\ \mathbf{r}(t)=\left(\frac{a^2X}{a^2+t},\frac{b^2Y}{b^2+t},0\right),$$

respectively. The cubic hyperbola passes through the center of the ellipsoid when $t=\pm\infty$, and goes to infinity when $t=-a^2,-b^2,-c^2$. Therefore, there are at least 2 intersections with the ellipsoid. For example, one can take the points of the ellipsoid with maximal and minimal distances from $A$. On the other hand, these intersections are determined by

$$\left(\frac{aX}{a^2+t}\right)^2+\left(\frac{bY}{b^2+t}\right)^2+\left(\frac{cZ}{c^2+t}\right)^2=1, \tag{4}$$

which is a sixth order equation with respect to $t$, and therefore can not have more than 6 real solutions. As before, let us denote the number of normals through $A$ by $n(A)$. We want to find points $A$, where $n(A)$ jumps from 2 to 4, or from 4 to 6. This happens when cubic hyperbola is tangent to the ellipsoid i.e. $\mathbf{r}'(t)=\left(-\frac{a^2X}{(a^2+t)^2},-\frac{b^2Y}{(b^2+t)^2},-\frac{c^2Z}{(c^2+t)^2}\right)$, is orthogonal to $\mathbf{N}=\left(\frac{X}{a^2+t},\frac{Y}{b^2+t},\frac{Z}{c^2+t}\right)$. This can be expressed as $\mathbf{r}'(t)\cdot\mathbf{N}=0$, or as

$$\frac{a^2X^2}{(a^2+t)^3}+\frac{b^2Y^2}{(b^2+t)^3}+\frac{c^2Z^2}{(c^2+t)^3}=0. \tag{5}$$

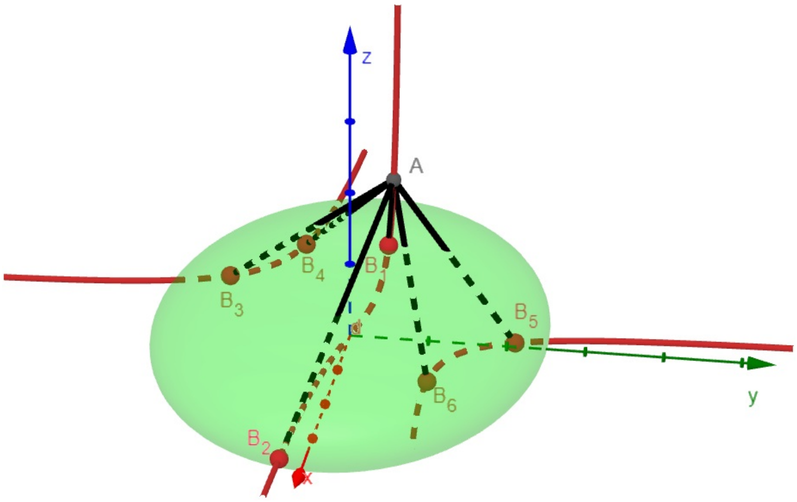


Figure 3: Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ (green), its 6 normals $AB_1, AB_2, \ldots, AB_6$ (black), cubic hyperbola $\mathbf{r}(t) = \left(\frac{a^2X}{a^2+t}, \frac{b^2Y}{b^2+t}, \frac{c^2Z}{c^2+t}\right)$ (red). Created using GeoGebra.

Equations (4) and (5) define the surface known as *Caustics of an Ellipsoid* also known as *focal surface, surface of centers, evolute of an ellipsoid*, or just *Cayley's astroida* [17] (see also p. 165, [60]). Cayley used the name *Centro-surface of an Ellipsoid*, and equations (4) and (5), which appear in p. 358 of [17], were obtained using the fact that the points of this surface are the centers of principal curvatures of ellipsoid (3) (see also p. 218 in [52]). The centers and principal radii of curvature for more general surfaces were studied by G. Monge in Sect. XXV of [46]. For a modern definition of the principal curvatures, see, for example, p. 158, [29]. A. Cayley's graph of the surface appears in p. 330 of [17] (also shown in p.116, [30]). One can also find many other images depicting this surface in various papers, dissertations, and books. See for example pp. 49-53 in [8], p. 154 in Ch. 7 of [9], [22], [36], p. 218 in [7], p. 257 in [61], p. 49 in [39] (also shown in p. 356, [12]), [62], p. 61 in [40], p. 10 in [18]. The part of the surface where the two surfaces corresponding to minimal and maximal curvatures intersect (named as "the purse"), was shown and mentioned in pages 37 and 109, respectively, of [6] (see also p. 218 in [7]). S.K. Lando gave two popular lectures about the caustics, available online, one with a demonstration of the surface at the end [43]. Another representation of the surface together with some applications of it in astronomy and physics appeared in [58] (See also [57]). According to [58], the idea of using more general caustics in cosmology is due to Ya. B. Zel'dovich (see [64], [63] and the references therein).

There are many visualizations of this surface as a physical model. Before the dawn of computer graphics and 3D printers, handmade models and sculptures represented the best medium for such mathematical objects [29]. In [42], there is a description of a model made out of gypsum by student H.A. Schwarz in the Arts Faculty (later Prof. in Univ. Berlin), which is also mentioned in Sect. 197 (p. 282), [24] (see also p. 198, [48]). Stereographic photo of one such model by unknown artist/maker from the same time period is shown in Figure 4, [19]. Two more models of this surface together with models of centro-surfaces of paraboloids and hyperboloids can be found in The Collection of Mathematical Models and Instruments at The University of Göttingen [33] (models 239 and 242). Similar models for the centers of curvature of paraboloids and hyperboloids were described in [54] (see also [15]) and p. 283 in [24] (see also [25]), respectively (see also p. 264 and p. 34, respectively, in [40]). See also the website of *The TouchGeometry Project* [23] for models of caustics of an elliptical paraboloid and a hyperboloid of one sheet in Geometry Department of Karazin University in Kharkiv, Ukraine. Another such model is in The National Museum of American History [14]. In Fig. 5, some models created using 3d printers of laboratories at ADA University are shown.

Note that in general, it is not easy to exclude the parameter $t$ from equations (4) and (5), to get an explicit equation for the caustics (see [52], p. 113 in [30]). This surface is of 12th order and its equation can be written as a $5 \times 5$ determinant (see p. 114 in [30]). But if, for example, $c = 0$, then equations (4) and (5) are transformed to

$$\left(\frac{aX}{a^2+t}\right)^2 + \left(\frac{bY}{b^2+t}\right)^2 = 1, \; \frac{a^2X^2}{(a^2+t)^3} + \frac{b^2Y^2}{(b^2+t)^3} = 0,$$

from which one can easily eliminate parameter $t$, and obtain equation (2) for the astroida. This gives us another solution for the planar case considered in the previous section. Similarly, if $b = c$ then one can introduce a new

Figure 4: Stereograph Card, Unknown artist/maker, Centro-Surface. Ellipsoid. about 1860. Gift of Weston J. and Mary M. Naef, Getty Museum Collection. Open Content program. No copyright. Used with permission.

Figure 5: 3d models of the caustics created in the laboratories of ADA University. Photo credit: Dr. Araz Yusubov (Assistant Professor, ADA). Acknowledgement: Nariman Vahabli (Lab Coordinator, ADA)

variable $Y'$, such that $(Y')^2 = Y^2 + Z^2$ and then equations (4) and (5) can be written as

$$\left(\frac{aX}{a^2+t}\right)^2 + \left(\frac{bY'}{b^2+t}\right)^2 = 1, \quad \frac{a^2X^2}{(a^2+t)^3} + \frac{b^2(Y')^2}{(b^2+t)^3} = 0,$$

from which again the parameter $t$ is easily eliminated to get

$$\sqrt[3]{a^2X^2} + \sqrt[3]{b^2(Y')^2} = \sqrt[3]{(a^2-b^2)^2},$$

or

$$\sqrt[3]{a^2X^2} + \sqrt[3]{b^2(Y^2+Z^2)} = \sqrt[3]{(a^2-b^2)^2},$$

which is a surface of revolution generated by rotating astroida (2) around $x$ axis (see Fig. 6).

## 4. Caustics of Ellipsoid in GeoGebra and Maple

In this section a method of generating the surface, based on the cartesian coordinates, will be described. The formulas for Gaussian curvature and mean curvature of an ellipsoid are given in [51], Example 5.2, and

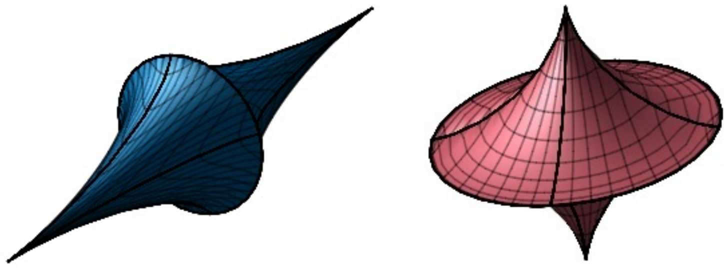

Figure 6: The surface of revolution generated by rotating the astroida around one of its axes. Created using Maple 2022 for the cases $(a, b, c) = (4, 4, 3)$ (left) and $(4.1, 3, 3)$ (right).

Corollary 13.41, p. 413 in [1] (see also Chapter 4, [10] for some applications in geodesy):

$$K(x, y) = \frac{1}{\left(abc\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)\right)^2}, \; H(x, y) = \frac{|x^2 + y^2 + z^2 - a^2 - b^2 - c^2|}{2(abc)^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{\frac{3}{2}}}.$$

The principal curvatures $k_1$ and $k_2$ are the roots of the quadratic equation $x^2 - 2Hx + K = 0$ (Corollary 13.26, p. 400, in [1]):

$$k_1 = H - \sqrt{H^2 - K}, \; k_2 = H + \sqrt{H^2 - K}.$$

The corresponding radii of the curvature are $R_1 = \frac{1}{k_1}$ and $R_2 = \frac{1}{k_2}$, and the respective centers of the curvature $C_1(x_1, y_1, z_1)$ and $C_2(x_1, y_1, z_1)$ can be determined using the formula (see p. 226, [26])

$$C_1(x_1, y_1, z_1) = (x, y, z) - R_1 \cdot \frac{\mathbf{N}}{|\mathbf{N}|}, \; C_2(x_2, y_2, z_2) = (x, y, z) - R_2 \cdot \frac{\mathbf{N}}{|\mathbf{N}|},$$

where as before outer normal is $\mathbf{N} = \left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right)$. The GeoGebra Activity demonstrating the surface but based on curvilinear coordinates (see the end of the current paper), can be found in https://www.geogebra.org. The Maple Learn document can be found in https://learn.maplesoft.com. The images created using GeoGebra and Maple 2022 are shown in Figure 7 and Figure 8, respectively.

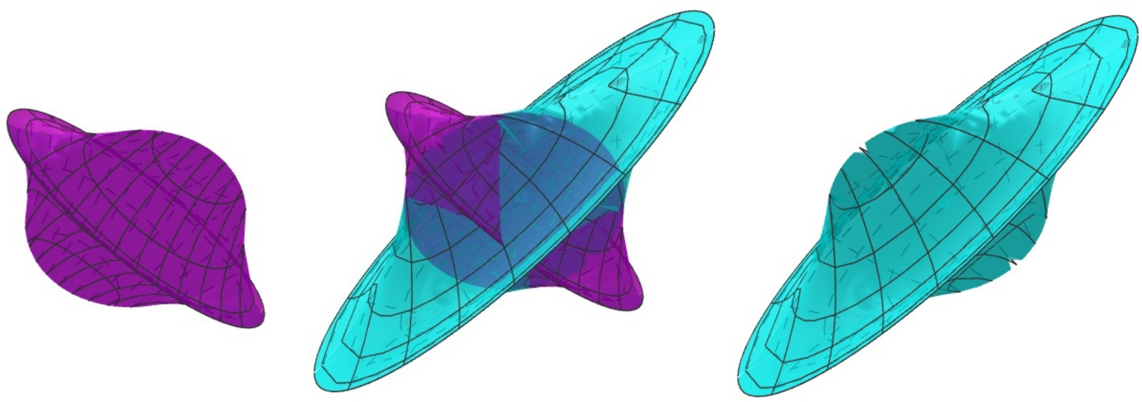

Figure 7: The centers corresponding to smaller (left, $R_2$) and greater (right, $R_1$) principal radii of curvature. The surfaces intersect (center). Created using GeoGebra.

We can now answer the question asked at the beginning of Sect. 2. The number of normals outside of the two caustics is 2 ($n(A) = 2$). For the points of the space inside of only one and both of the caustics, $n(A) = 4$ and $n(A) = 6$, respectively (see [38], p. 123-124). On the caustics, $n(A) = 3$ or $n(A) = 5$, with some exceptions on planes $X = 0$, $Y = 0$, $Z = 0$, and on the intersections of the two caustics, where again $n(A) = 2$ or $n(A) = 4$ (see Fig. 9).

In the case of an ellipsoid of revolution, for example, when $a = b$ or $b = c$, one of the caustics becomes a surface of revolution, shown in Figure 6, the other caustic degenerates to a line segment on the axis of symmetry

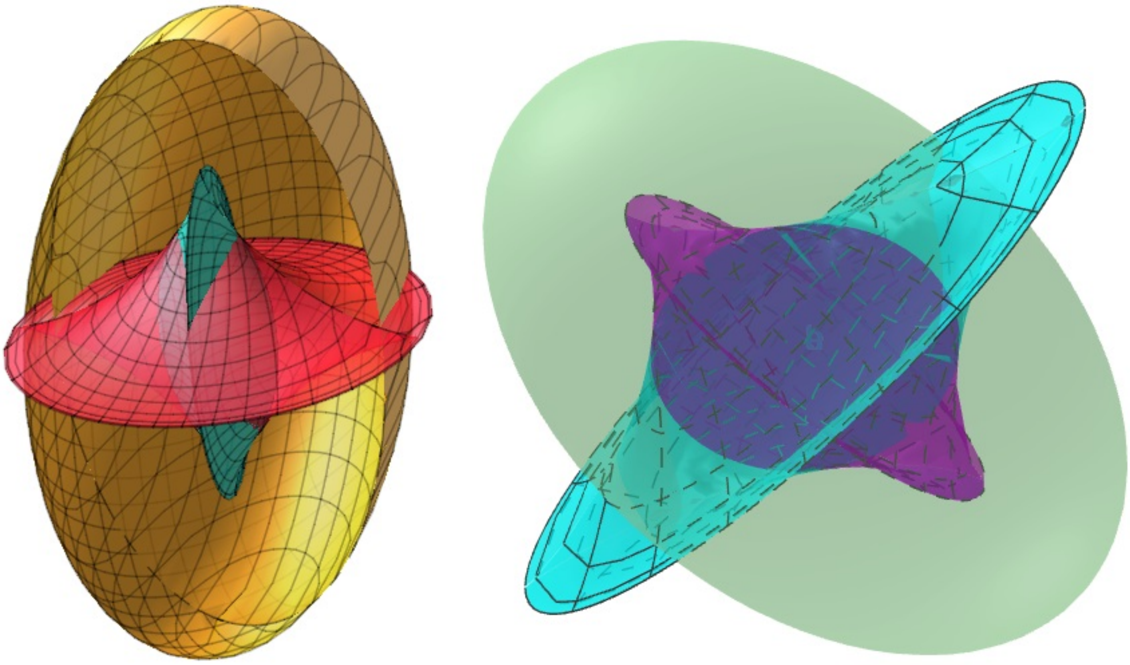

Figure 8: The ellipsoid and its caustics with transparency applied. Created using Maple 2022 (left) and GeoGebra (right).

of the surfaces shown in Figure 6, between the vertices. The number of normals of ellipsoid (3) for point $A$ on this line segment is infinite ($n(A) = \infty$), except the endpoints of this line segment where $n(A) = 2$. For the other points of the space, the situation is identical to the planar case considered in Sect. 2.

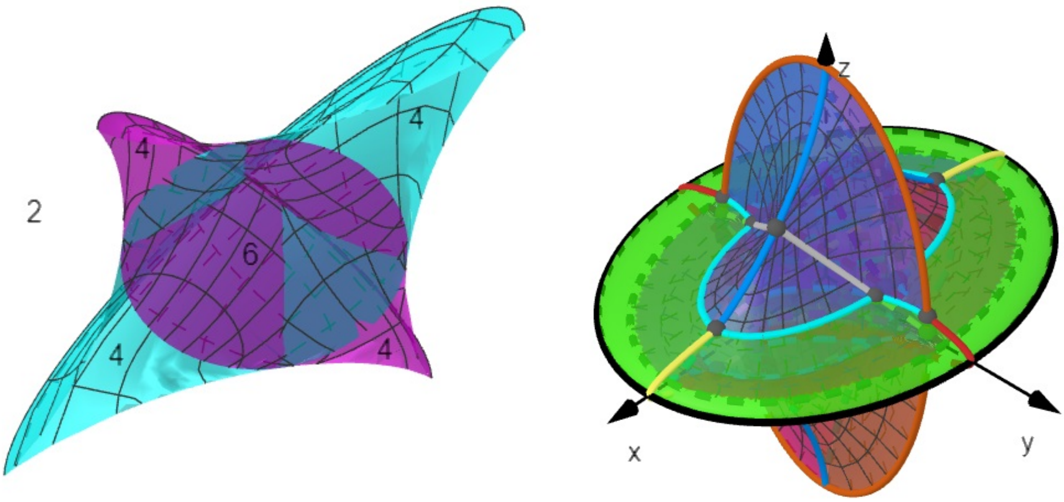


Figure 9: Left image: The number of normals in the regions of space separated by the caustics of the ellipsoid. Half of the caustics is hidden to make the inner regions visible. Right image: The caustics intersect each other and the ellipsoid along some curves separating the regions where $n(A)$ is different.

## 5. The intersections of an ellipsoid and its caustics with the coordinate planes

The intersection curves of ellipsoid (3) and its caustics (see Figure 10) with the coordinate planes $x = 0,\ y = 0,\ z = 0$ are found in p. 325, [17] (see also p. 115, [30]).

**Lemma 5.1.** *The intersections of ellipsoid (3) and its caustics with the coordinate planes are the following curves:*

1. *Ellipse* $(a\cos t, b\sin t, 0)$ *(black),*
2. *Ellipse* $(a\cos t, 0, c\sin t)$ *(yellow),*
3. *Ellipse* $(0, b\cos t, c\sin t)$ *(red),*
4. *Astroida* $\left(\frac{a^2-b^2}{a}\cos^3 t, \frac{a^2-b^2}{b}\sin^3 t, 0\right)$ *(pink),*
5. *Astroida* $\left(\frac{a^2-c^2}{a}\cos^3 t, 0, \frac{a^2-c^2}{c}\sin^3 t\right)$ *(light blue),*
6. *Astroida* $\left(0, \frac{b^2-c^2}{b}\cos^3 t, \frac{b^2-c^2}{c}\sin^3 t\right)$ *(purple),*
7. *Ellipse* $\left(\frac{a^2-c^2}{a}\cos t, \frac{b^2-c^2}{b}\sin t, 0\right)$ *(green),*
8. *Ellipse* $\left(\frac{a^2-b^2}{a}\cos t, 0, \frac{b^2-c^2}{c}\sin t\right)$ *(dark blue),*
9. *Ellipse* $\left(0, \frac{a^2-b^2}{b}\cos t, \frac{a^2-c^2}{c}\sin t\right)$ *(orange).*

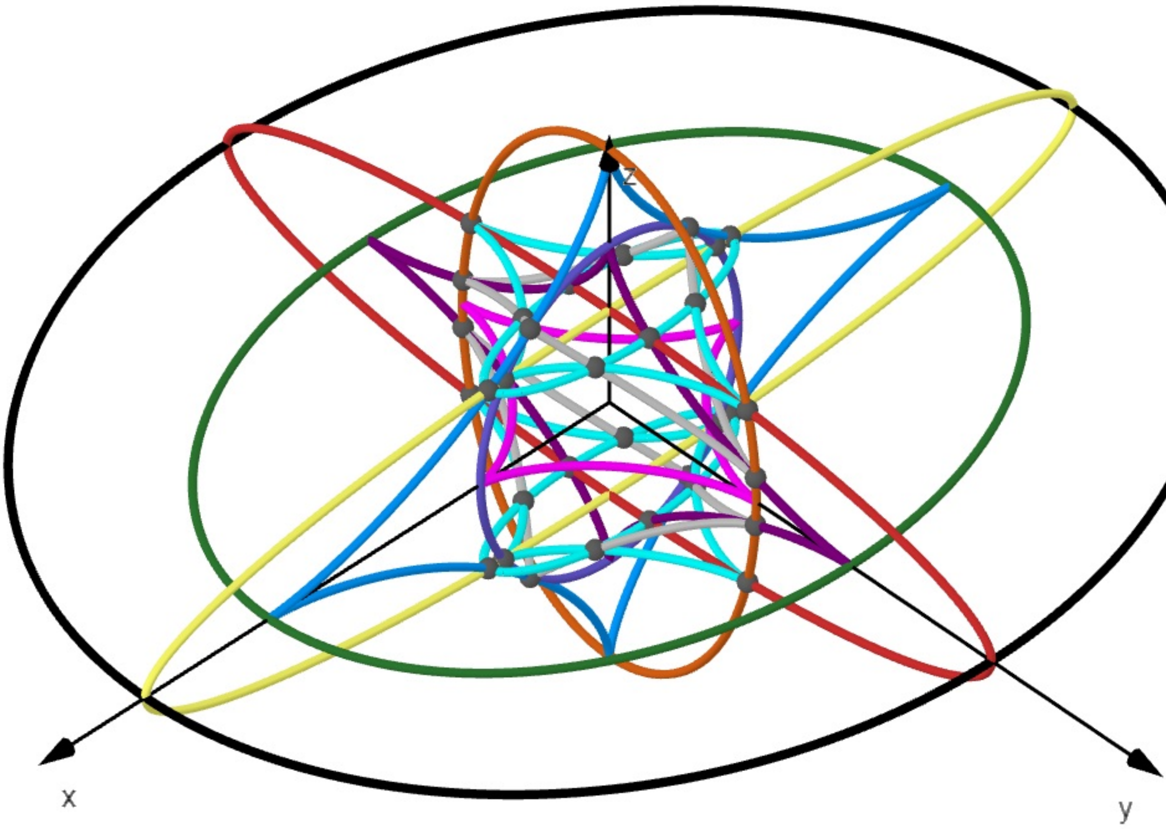


Figure 10: The intersections of a triaxial ellipsoid and its caustics with the coordinate planes, the nodal curve (grey), and the intersections of the ellipsoid with its caustics (cyan blue). The other colors are explained in Lemma 5.1. The tangency points of the two caustics and some of the intersection points are also shown. See `https://www.geogebra.org/3d/tqjwgxwg` for more details.

Let us now find intersections of Ellipses 1,2,3, Astroidas 4,5,6 and Ellipses 7,8,9, respectively.

**Lemma 5.2.** *1. If* $a^2 \geq 2b^2$*, then Ellipse 1 and Astroida 4 intersect at* $(\pm x_0, \pm y_0, 0)$ *and* $(\pm x_0, \mp y_0, 0)$*, where*

$$x_0 = \sqrt{\frac{a^4(a^2-2b^2)^3}{(a^2-b^2)(a^2+b^2)^3}},\quad y_0 = \sqrt{\frac{b^4(2a^2-b^2)^3}{(a^2-b^2)(a^2+b^2)^3}}.$$

2. *If $a^2 \geq 2c^2$, then Ellipse 2 and Astroida 5 intersect at $(\pm x_1, 0, \pm z_1)$ and $(\pm x_1, 0, \mp z_1)$, where*

$$x_1 = \sqrt{\frac{a^4(a^2-2c^2)^3}{(a^2-c^2)(a^2+c^2)^3}}, \; z_1 = \sqrt{\frac{c^4(2a^2-c^2)^3}{(a^2-c^2)(a^2+c^2)^3}}.$$

3. *If $b^2 \geq 2c^2$, then Ellipse 3 and Astroida 6 intersect at $(0, \pm y_2, \pm z_2)$ and $(0, \pm y_2, \mp z_2)$, where*

$$y_2 = \sqrt{\frac{b^4(b^2-2c^2)^3}{(b^2-c^2)(b^2+c^2)^3}}, \; z_2 = \sqrt{\frac{c^4(2b^2-c^2)^3}{(b^2-c^2)(b^2+c^2)^3}}.$$

4. *Ellipse 1 and Ellipse 7 do not have real intersection points and the coordinates of the non-real intersection points are $(\pm x^*, \pm y^*, 0)$ and $(\pm x^*, \mp y^*, 0)$, where*

$$x^* = \sqrt{\frac{a^2(a^2-c^2)^2(2b^2-c^2)}{(a^2-b^2)(2a^2b^2-a^2c^2-b^2c^2)}}, \; y^* = \sqrt{\frac{b^2(b^2-c^2)^2(2a^2-c^2)}{(b^2-a^2)(2a^2b^2-b^2c^2-a^2c^2)}}.$$

5. *If $b^2 \geq 2c^2$, then Ellipse 2 and Ellipse 8 intersect at $(\pm x_3, 0, \pm z_3)$ and $(\pm x_3, 0, \mp z_3)$, where*

$$x_3 = \sqrt{\frac{a^2(a^2-b^2)^2(2c^2-b^2)}{(a^2-c^2)(2a^2c^2-a^2b^2-b^2c^2)}}, \; z_3 = \sqrt{\frac{c^2(c^2-b^2)^2(2a^2-b^2)}{(c^2-a^2)(2a^2c^2-a^2b^2-b^2c^2)}},$$

*and $x_1 \geq x_3$, $z_1 \leq z_3$ with equality cases when $\frac{1}{a^2} + \frac{1}{c^2} = \frac{3}{b^2}$.*

6. *If $2b^2 \geq a^2 \geq 2c^2$, then Ellipse 3 and Ellipse 9 intersect at $(0, \pm y_4, \pm z_4)$ and $(0, \pm y_4, \mp z_4)$, where*

$$y_4 = \sqrt{\frac{b^2(b^2-a^2)^2(2c^2-a^2)}{(b^2-c^2)(2b^2c^2-a^2b^2-a^2c^2)}}, \; z_4 = \sqrt{\frac{c^2(c^2-a^2)^2(2b^2-a^2)}{(c^2-b^2)(2b^2c^2-a^2b^2-a^2c^2)}}.$$

*In particular, $y_2 \geq y_4$ and $z_2 \leq z_4$ if and only if $2b^4 + 2c^4 - a^2b^2 - a^2c^2 - 2b^2c^2 \geq 0$.*

7. *If $a^2 + c^2 \geq 2b^2$, then Astroida 4 and Ellipse 7 intersect at $(\pm x_5, \pm y_5, 0)$ and $(\pm x_5, \mp y_5, 0)$, where*

$$x_5 = \sqrt{\frac{(a^2-c^2)^3(2b^2-a^2-c^2)^3}{a^2(b^2-a^2)(a^2+b^2-2c^2)^3}}, \; y_5 = \sqrt{\frac{(b^2-c^2)^3(2a^2-b^2-c^2)^3}{b^2(a^2-b^2)(a^2+b^2-2c^2)}}.$$

8. *Astroida 5 and Ellipse 8 are tangent to each other at the points $(\pm x_6, 0, \pm z_6)$ and $(\pm x_6, 0, \mp z_6)$, where*

$$x_6 = \sqrt{\frac{(b^2-c^2)^3}{c^2(a^2-c^2)}}, \; z_6 = \sqrt{\frac{(a^2-b^2)^3}{a^2(a^2-c^2)}},$$

*These points also divide Astroida 5 and Ellipse 8 into parts which belong to different caustics. These points are on, in and outside ellipsoid (3) if $\frac{1}{a^2} + \frac{1}{c^2} = \frac{3}{b^2}$, $< \frac{3}{b^2}$, and $> \frac{3}{b^2}$, respectively.*

9. *If $a^2 + c^2 \leq 2b^2$, then Astroida 6 and Ellipse 9 intersect at $(0, \pm y_7, \pm z_7)$ and $(0, \pm y_7, \mp z_7)$, where*

$$y_7 = \sqrt{\frac{(b^2-a^2)^3(2c^2-b^2-a^2)^3}{b^2(c^2-b^2)(b^2+c^2-2a^2)^3}}, \; z_7 = \sqrt{\frac{(c^2-a^2)^3(2b^2-c^2-a^2)^3}{c^2(b^2-c^2)(b^2+c^2-2a^2)^3}}.$$

*These points are on, in and outside ellipsoid (3) if $2b^4 + 2c^4 - a^2b^2 - a^2c^2 - 2b^2c^2 = 0$, $< 0$, and $> 0$, respectively.*

*Proof.* By direct substitution of the given coordinates in the equations of the curves, we can prove the claim about the intersections. The inequalities involving coordinates are also proved by direct substitutions. For tangency of the curves in part 8, additionally, the slopes of the curves are compared. In parts 8 and 9, the last claim about the position with respect to ellipsoid (3), follows directly from the substitution in (3) of the coordinates $x_6, z_6$ and $y_7, z_7$, respectively. □

One can experiment with these intersection points and the curves by moving the sliders in GeoGebra Activity `https://www.geogebra.org/3d/tqjwgxwg`.

## 6. The number of normals for the points of an ellipsoid

As for an ellipse, if point $A$ is on ellipsoid (3), then one of points $B_1, B_2, \ldots, B_6$ coincides with $A$. The description all possible cases for the regions on an ellipsoid, where $n(A)$ jumps from 2 to 4, or from 4 to 6, is not as trivial as in the planar case. It seems that the problem in this setting did not attract much attention and remains unstudied. In the remaining part of the paper we will highlight the 3 cases of intersections of the ellipsoid and its caustics (see Figure 8). In general, these intersections are some curves on ellipsoid (3), and a simple parametrization for these curves is given at the end of this paper. Using the intersections of these curves with the coordinate planes, which we found in the previous section, one can categorize 3 possible cases: (i) none of the caustics intersect the ellipsoid, (ii) only one of the caustics intersects the ellipsoid, (iii) both of the caustics intersect the ellipsoid. These intersections have many different shapes and positions, and the complete categorization of all general cases is shown in Figure 11.

**Theorem 6.1.** *For ellipsoid (3) and its caustics defined by (4) and (5), the following cases are possible:*

1. *If $a^2 < 2c^2$ then there are no intersections of the caustics with ellipsoid (3),*
2. *If $b^2 < 2c^2 \leq a^2$ then only one of the caustics intersects ellipsoid (3),*
3. *If $b^2 \geq 2c^2$, then both of the caustics intersects ellipsoid (3).*

*In all cases, for the points of ellipsoid (3) lying outside of the two caustics $n(A) = 2$, for the points of ellipsoid (3) lying in only one of these caustics $n(A) = 4$, for the points of ellipsoid (3) lying in both of these caustics $n(A) = 6$, and for the intersection points of ellipsoid (3) and these caustics $n(A) = 3$ or $5$, except some of the points of ellipsoid (3), where the caustics intersect each other or these caustics intersect the coordinate planes.*

*Proof.* Let us first note that if ellipsoid (3) and one of its caustics intersect, then they should also intersect on at least one point of the coordinate planes. Indeed, suppose on the contrary that ellipsoid (3) and one of its caustics intersect but they do not intersect on any of the coordinate planes. Then one of the intersection curves should be situated completely in the 1st octant. Denote this curve by $\Gamma$. The part of the caustic bounded by $\Gamma$ is a smooth surface outside of ellipsoid (3), and therefore it has a tangent plane which does not intersect ellipsoid (3). This tangent plane contains also the corresponding normal of ellipsoid (3) because the principal radii are tangent to the caustics (see p. 312, [41]). This is a contradiction because the normals intersects ellipsoid (3). It follows that the problem of existence of intersections of ellipsoid (3) and its caustics can be studied just by their cross sections with the coordinate planes, which was done in Lemma 5.1 and Lemma 5.2. The remaining claims follow directly from the results in [38]. □

Depending on whether $a^2 \leq 2b^2$ or $a^2 > 2b^2$, the caustic corresponding to the greater principal radius (the red caustics in Figure 11) is encompassed by ellipsoid (3) or ellipsoid (3) is encompassed by this caustic. Similarly, depending on whether $a^2 + c^2 \geq 2b^2$ or $a^2 + c^2 < 2b^2$, the (blue) caustic corresponding to the smaller principal radius is encompassed by the (red) caustic corresponding to the greater principal radius or vice versa (cf. p. 326 and p. 363, [17]). It is obvious that if $a^2 + c^2 \leq 2b^2$ then $a^2 < 2b^2$. Similarly, if $b^2 < 2c^2 \leq a^2$ then $\frac{1}{a^2} + \frac{1}{c^2} < \frac{3}{b^2}$.

Detailed classification of the cases of intersection of the ellipsoid and its caustics is done in Figure 11 based on the sign of the expressions $\frac{1}{a^2} + \frac{1}{c^2} - \frac{3}{b^2}$ and $2b^4 + 2c^4 - a^2b^2 - a^2c^2 - 2b^2c^2$. Note that if $b^2 \geq 2c^2$ and $2b^4 + 2c^4 - a^2b^2 - a^2c^2 - 2b^2c^2 \geq 0$ then $\frac{1}{a^2} + \frac{1}{c^2} > \frac{3}{b^2}$. Indeed, since $\frac{1}{a^2} + \frac{1}{c^2} > \frac{3}{b^2}$ can be written as $b^2c^2 + a^2b^2 - 3a^2c^2 > 0$, it is sufficient to show that

$$b^2c^2 + a^2b^2 - 3a^2c^2 > 2b^4 + 2c^4 - a^2b^2 - a^2c^2 - 2b^2c^2.$$

This inequality can be written as $3b^2c^2 + 2a^2(b^2 - c^2) > 2b^4 + 2c^4$. Since $a > b > c$, it is sufficient to show that $3b^2c^2 + 2b^2(b^2 - c^2) \geq 2b^4 + 2c^4$, which simplifies to $b^2 \geq 2c^2$. Similarly, if $2b^4 + 2c^4 - a^2b^2 - a^2c^2 - 2b^2c^2 \geq 0$ then $a^2 + c^2 < 2b^2$. Indeed, by rewriting the given inequality we obtain $a^2 + c^2 \leq 2b^2 - \frac{c^2(a^2+b^2-2c^2)}{b^2} < 2b^2$.

**Theorem 6.2.** *If $2b^4 + 2c^4 - a^2b^2 - a^2c^2 - 2b^2c^2 \leq 0$ and $\frac{1}{a^2} + \frac{1}{c^2} \geq \frac{3}{b^2}$ then the two caustics and ellipse (3) intersect at a unique point of each octant.*

*Proof.* The intersection curve of the two caustics (see Figure 10) is called *the nodal curve* and its parametrization was given in p. 351, [17]:

$$(x(t))^2 = \frac{((\gamma - \alpha)t + \alpha)((\gamma - \alpha)t - 2\gamma)^2((\beta - \gamma)t + \gamma)^3}{-\beta\gamma a^2(\alpha\gamma + \Omega t)(3t - 2)^2}, (y(t))^2 = \frac{t^2(t-1)((\gamma - \alpha)^2 t + 3\alpha\gamma)^3}{-\alpha\gamma b^2(\alpha\gamma + \Omega t)(3t - 2)^2},$$

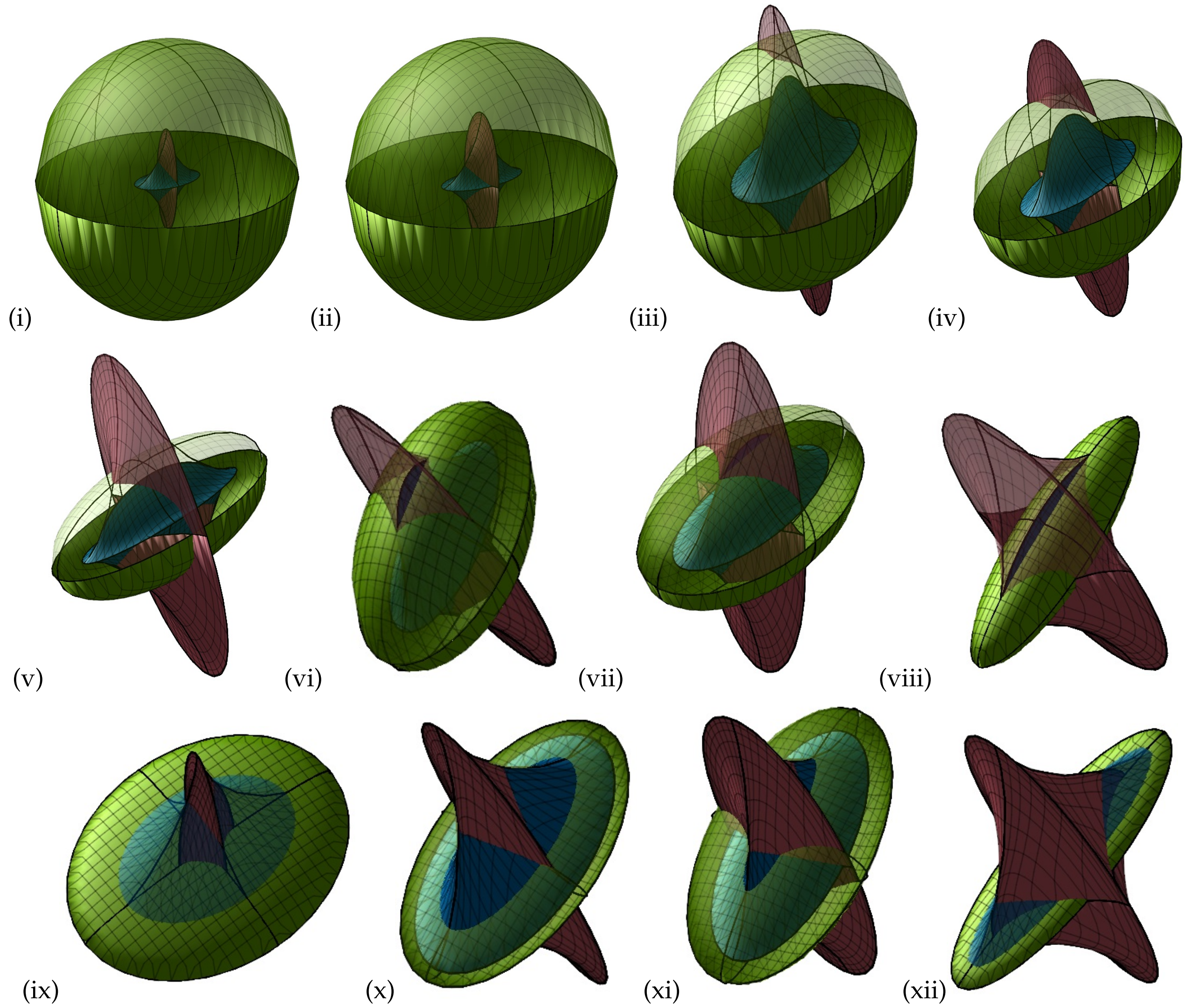

Figure 11: Classification of intersections of ellipsoid (green) and its caustics (red and blue). Created using Maple 2022 for the following values of $(a, b, c)$:
(i) $(4.7, 4.4, 4)$ ($a^2 < 2c^2$, $a^2 + c^2 < 2b^2$);
(ii) $(4.9, 4.4, 4)$ ($a^2 < 2c^2$, $a^2 + c^2 > 2b^2$);
(iii) $(4.7, 4, 3)$ ($b^2 < 2c^2 < a^2$, $a^2 + c^2 < 2b^2$);
(iv) $(5, 4, 3)$ ($b^2 < 2c^2 < a^2$, $a^2 + c^2 > 2b^2$, $a^2 < 2b^2$);
(v) $(5, 3.3, 2.5)$ ($b^2 < 2c^2 < a^2$, $a^2 > 2b^2$);
(vi) $(4, 3, 2)$ ($b^2 > 2c^2$, $a^2 + c^2 < 2b^2$, $\frac{1}{a^2} + \frac{1}{c^2} < \frac{3}{b^2}$);
(vii) $5, 3.7, 2.5$ ($b^2 > 2c^2$, $a^2 + c^2 > 2b^2$, $a^2 < 2b^2$, $\frac{1}{a^2} + \frac{1}{c^2} < \frac{3}{b^2}$);
(viii) $(5, 2.8, 1.8)$ ($b^2 > 2c^2$, $a^2 > 2b^2$, $\frac{1}{a^2} + \frac{1}{c^2} < \frac{3}{b^2}$);
(ix) $(5, 4.4, 1.6)$ ($b^2 > 2c^2$, $2b^4 + 2c^4 - a^2b^2 - a^2c^2 - 2b^2c^2 > 0$);
(x) $(4.5, 3.5, 1.4)$ ($b^2 > 2c^2$, $a^2 + c^2 < 2b^2$, $\frac{1}{a^2} + \frac{1}{c^2} > \frac{3}{b^2}$, $2b^4 + 2c^4 - a^2b^2 - a^2c^2 - 2b^2c^2 < 0$);
(xi) $(5, 3.7, 2)$ ($b^2 > 2c^2$, $a^2 + c^2 > 2b^2$, $a^2 < 2b^2$, $\frac{1}{a^2} + \frac{1}{c^2} > \frac{3}{b^2}$);
(xii) $(5, 3, 1)$ ($b^2 > 2c^2$, $a^2 > 2b^2$, $\frac{1}{a^2} + \frac{1}{c^2} > \frac{3}{b^2}$).

$$(z(t))^2 = \frac{((\gamma-\alpha)t-\gamma)((\gamma-\alpha)t+2\alpha)^2((\alpha-\beta)t-\alpha)^3}{-\alpha\beta c^2(\alpha\gamma+\Omega t)(3t-2)^2},$$

where $\alpha = b^2 - c^2$, $\beta = c^2 - a^2$, $\gamma = a^2 - b^2$, $\Omega = \alpha^2 - \beta\gamma$, and $0 \le t \le \min\left(\frac{\alpha}{\alpha-\beta}, -\frac{\gamma}{\beta-\gamma}\right)$. Note that $-\frac{\gamma}{\beta-\gamma} \ge \frac{\alpha}{\alpha-\beta}$ iff $a^2 + c^2 \ge 2b^2$. Also note that $\max\left(\frac{\alpha}{\alpha-\beta}, -\frac{\gamma}{\beta-\gamma}\right) < \frac{2}{3}$. Consider the function $f(t) = \frac{x(t)^2}{a^2} + \frac{y(t)^2}{b^2} + \frac{z(t)^2}{c^2} - 1$.

Simplifying using Maple we obtain that $f(t) = \frac{p(t)(q(t))^2}{a^4b^4c^4(3t-2)^2r(t)}$, where

$$p(t) = (a^4b^2 + a^4c^2 + a^2b^4 + c^4a^2 + b^4c^2 + c^4b^2 - 6a^2b^2c^2)(a^2 - 2b^2 + c^2)t^2$$

$$+(11a^4b^2c^2 + 11a^2b^2c^4 + 3a^2b^6 + 3b^6c^2 - b^2a^6 - c^2a^6 - a^2c^6 - b^2c^6 - a^4b^4 - 5a^4c^4 - b^4c^4 - 17a^2b^4c^2)t$$

$$+(a^2 - b^2)(b^2 - c^2)(a^2b^2 + b^2c^2 - 3a^2c^2),$$

$$q(t) = (a^2 + b^2 + c^2)(a^2 - 2b^2 + c^2)t^2 + (4b^4 + a^2b^2 + b^2c^2 - 3a^2c^2)t - 2b^4,$$

$$r(t) = (a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - a^2c^2)t + (a^2 - b^2)(b^2 - c^2).$$

Note that $r(t) > 0$ whenever $t \geq 0$. We will consider 3 cases.

Case 1. Suppose that $a^2 - 2b^2 + c^2 > 0$. Note that

$$p(0) = (a^2 - b^2)(b^2 - c^2)(a^2b^2 + b^2c^2 - 3a^2c^2) > 0,$$

$$p\left(\frac{\alpha}{\alpha - \beta}\right) = -\frac{c^2(a^2 - c^2)(b^2 - c^2)(2a^2 - b^2 - c^2)(2a^4 + 2b^4 - a^2c^2 - b^2c^2 - 2a^2b^2)}{(a^2 + b^2 - 2c^2)^2} < 0$$

$$p\,(1) = -b^4(a^2 - c^2)^2 < 0,\ p(+\infty) = +\infty.$$

Therefore, one of the zeros of $p(t)$ is in the interval $\left(0, \frac{\alpha}{\alpha-\beta}\right)$, and the other zero is in the interval $(1, +\infty)$. Similarly,

$$q(-\infty) = +\infty,\ q(0) = -2b^4 < 0,$$

$$q\left(\frac{\alpha}{\alpha - \beta}\right) = -\frac{c^2(3a^2(b^2 - c^2) + c^2(a^2 + c^2 - 2b^2))(2a^2 - b^2 - c^2)}{(a^2 + b^2 - 2c^2)^2} < 0,\ q(+\infty) = +\infty,$$

Therefore, one of the zeros of $q(t)$ is in the interval $(-\infty, 0)$, and the other zero is in the interval $\left(\frac{\alpha}{\alpha-\beta}, +\infty\right)$.

Case 2. Now suppose that $a^2 - 2b^2 + c^2 < 0$.

$$p(-\infty) = -\infty,\ p(0) > 0,$$

$$p\left(-\frac{\gamma}{\beta - \gamma}\right) = \frac{a^2(a^2 - c^2)(a^2 - b^2)(a^2 + b^2 - 2c^2)(2b^4 + 2c^4 - a^2b^2 - a^2c^2 - 2b^2c^2)}{(2a^2 - b^2 - c^2)^2} < 0.$$

Therefore, one of the zeros of $p(t)$ is in the interval $(-\infty, 0)$, and the other zero is in the interval $\left(0, -\frac{\gamma}{\beta-\gamma}\right)$. Similarly,

$$q\left(-\frac{\gamma}{\beta - \gamma}\right) = \frac{a^2(3c^2(b^2 - a^2) + a^2(a^2 + c^2 - 2b^2))(a^2 + b^2 - 2c^2)}{(2a^2 - b^2 - c^2)^2} < 0,$$

$$q(1) = a^4 - a^2c^2 + c^4 > 0,\ q(+\infty) = -\infty.$$

Therefore, one of the zeros of $q(t)$ is in the interval $\left(-\frac{\gamma}{\beta-\gamma}, 1\right)$, and the other zero is in the interval $(1, +\infty)$.

Case 3. Let us now suppose that $a^2 - 2b^2 + c^2 = 0$. In this case $p(t)$ and $q(t)$ are linear functions and $-\frac{\gamma}{\beta-\gamma} = \frac{\alpha}{\alpha-\beta}$. As in Case 1, $p(0) > 0$, $p\left(\frac{\alpha}{\alpha-\beta}\right) < 0$, and therefore, the only root of $p(t)$ is in the interval $\left(0, \frac{\alpha}{\alpha-\beta}\right)$. As in Case 2, $q\left(-\frac{\gamma}{\beta-\gamma}\right) < 0$, $q(1) > 0$, and therefore, the only root of $q(t)$ is in the interval $\left(-\frac{\gamma}{\beta-\gamma}, 1\right)$.

We proved that in all the cases there is only one zero $t_0$ of $f(t)$ in the interval $0 \leq t \leq \min\left(\frac{\alpha}{\alpha-\beta}, -\frac{\gamma}{\beta-\gamma}\right)$. $\square$

Note that $t_0$ is the solution of quadratic equation $p(t) = 0$ with minus sign before the square root of its discriminant. Coordinates $x(t_0),\ y(t_0),\ z(t_0)$ of the intersection point of the two caustics and ellipsoid (3) in the first quadrant can be obtained from substitution of $t_0$ in Cayley's parametrization of the nodal curve. The obtained expressions are not simple and we will not include them here (See Figure 10, the intersection point of the cyan blue curves). Note also that if $\frac{1}{a^2} + \frac{1}{c^2} = \frac{3}{b^2}$ then $(x(t_0),\ y(t_0),\ z(t_0)) = (x_6, 0, z_6)$, and if $2b^4 + 2c^4 - a^2b^2 - a^2c^2 - 2b^2c^2 = 0$ then $(x(t_0),\ y(t_0),\ z(t_0)) = (0, y_7, z_7)$.

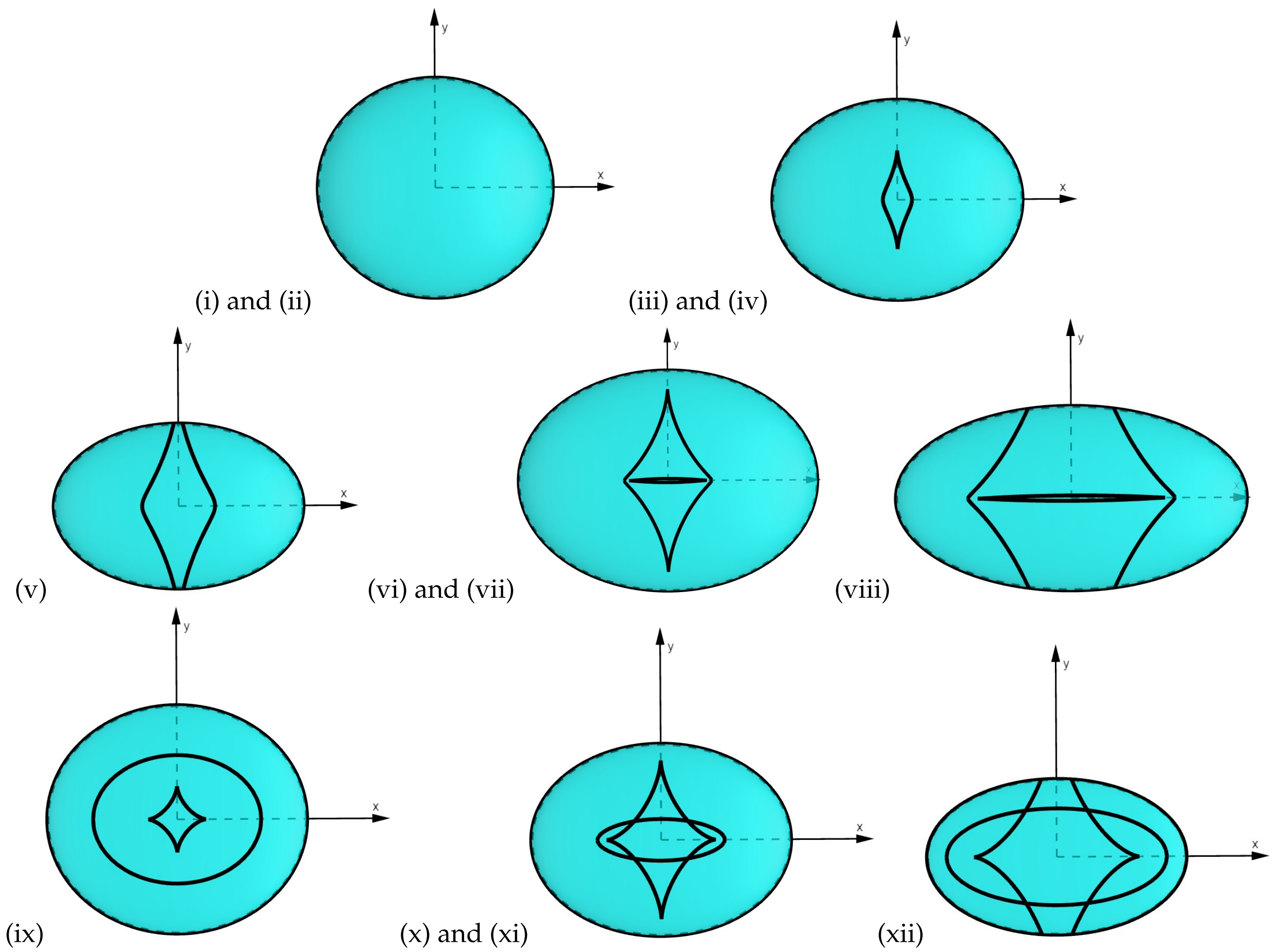


Figure 12: Classification of the cases for positions of regions divided by the intersection curves (black) of ellipsoid (cyan blue) and its caustics (not shown). View from the top of $z$-axis. Created using GeoGebra (cf. Fig. 11). See `https://www.geogebra.org/3d/uergsexn` for more details.
(i) and (ii) ($a^2 < 2c^2$);
(iii) and (iv) ($b^2 < 2c^2 < a^2$, $a^2 < 2b^2$);
(v) ($b^2 < 2c^2 < a^2$, $a^2 > 2b^2$);
(vi) and (vii) ($b^2 > 2c^2$, $a^2 < 2b^2$, $\frac{1}{a^2} + \frac{1}{c^2} < \frac{3}{b^2}$);
(viii) ($b^2 > 2c^2$, $a^2 > 2b^2$, $\frac{1}{a^2} + \frac{1}{c^2} < \frac{3}{b^2}$);
(ix) ($b^2 > 2c^2$, $2b^4 + 2c^4 - a^2b^2 - a^2c^2 - 2b^2c^2 > 0$);
(x) and (xi) ($b^2 > 2c^2$, $a^2 < 2b^2$, $\frac{1}{a^2} + \frac{1}{c^2} > \frac{3}{b^2}$, $2b^4 + 2c^4 - a^2b^2 - a^2c^2 - 2b^2c^2 < 0$);
(xii) ($b^2 > 2c^2$, $a^2 > 2b^2$, $\frac{1}{a^2} + \frac{1}{c^2} > \frac{3}{b^2}$).

In the remaining part of the paper we will find a parametrization for the intersection curves of each of the caustics with ellipsoid (3). First of all, note that ellipsoid (3) can be parametrized in curvilinear coordinates $\xi$, $\eta$ as

$$(x(\xi,\eta))^2 = -\frac{a^2(a^2+\xi)(a^2+\eta)}{\beta\gamma},\ (y(\xi,\eta))^2 = -\frac{b^2(b^2+\xi)(b^2+\eta)}{\alpha\gamma},\ (z(\xi,\eta))^2 = -\frac{c^2(c^2+\xi)(c^2+\eta)}{\alpha\beta},$$

where $-a^2 \le \xi, \eta \le -c^2$. In p. 324, [17] (see also p. 218, [52]) a similar parametrization of the caustics is given:

$$(x(\xi,\eta))^2 = -\frac{(a^2+\xi)^3(a^2+\eta)}{a^2\beta\gamma},\ (y(\xi,\eta))^2 = -\frac{(b^2+\xi)^3(b^2+\eta)}{b^2\alpha\gamma},\ (z(\xi,\eta))^2 = -\frac{(c^2+\xi)^3(c^2+\eta)}{c^2\alpha\beta},$$

where again $-a^2 \le \xi, \eta \le -c^2$. By substituting the last three equalities in (3), we obtain

$$\eta(\xi) = -\frac{1 + \frac{(a^2+\xi)^3}{\beta\gamma a^2} + \frac{(b^2+\xi)^3}{\alpha\gamma b^2} + \frac{(c^2+\xi)^3}{\alpha\beta c^2}}{\frac{(a^2+\xi)^3}{\beta\gamma a^4} + \frac{(b^2+\xi)^3}{\alpha\gamma b^4} + \frac{(c^2+\xi)^3}{\alpha\beta c^4}}.$$

Substitution of this equality in the above parametrization of the caustics and taking $\xi = t$ gives

$$(x(t))^2 = \frac{a^2(a^2+t)^3((b^2+c^2)t+3b^2c^2)}{(a^2-c^2)(a^2-b^2)((a^2b^2+a^2c^2+b^2c^2)t+3a^2b^2c^2)},$$

$$(y(t))^2 = \frac{b^2(b^2+t)^3((a^2+c^2)t+3a^2c^2)}{(b^2-a^2)(b^2-c^2)((a^2b^2+a^2c^2+b^2c^2)t+3a^2b^2c^2)},$$

$$(z(t))^2 = \frac{c^2(c^2+t)^3((a^2+b^2)t+3a^2b^2)}{(c^2-a^2)(c^2-b^2)((a^2b^2+a^2c^2+b^2c^2)t+3a^2b^2c^2)},$$

where $-a^2 \le t \le -b^2$ and $-b^2 \le t \le -c^2$ correspond to the two intersections of the caustics with ellipsoid (3) (See Figure 10, the cyan blue curves). This parametrization determines the curves on ellipsoid (3) which separate the regions of the ellipsoid with different values for $n(A)$, thus completing the solution of the Apollonius problem for triaxial ellipsoid (see Fig. 12). Several of the preceding results conducted by means of the intersections with the coordinate planes, might have been conducted more simply by means of the last parametrization.

## 7. Conclusion

In the paper the Apollonius problems for 2 dimensions (ellipse) and 3 dimensions (ellipsoid) were discussed. The number of concurrent normals of an ellipse (an ellipsoid) is dependent on the position of the point of concurrency with respect to the caustics of the ellipse (the ellipsoid). The cases when the point of concurrency is on the ellipse (the ellipsoid), required the study of several different cases of intersections of the caustics with the given ellipse (ellipsoid). It would be interesting to generalize the results to 4 (see [39]) and higher dimensions.

## 8. Appendix

The tangent lines of an ellipse, and the tangent lines and planes of an ellipsoid are much easier to study than the normals. For completeness, the problem on the number of concurrent tangent lines (planes) of an ellipse (ellipsoid) will be discussed here. Let us take point $A(X,Y)$ outside of ellipse (1) and find points $B(x_1,y_1)$ and $C(x_2,y_2)$ on ellipse (1) such that $AB$ and $AC$ are tangent lines of ellipse (1) at $B$ and $C$, respectively. Since $AB$ and $AC$ have the same slopes as ellipse (1) at points $B$ and $C$, respectively, we have $\frac{y-Y}{x-X} = -\frac{b^2x}{a^2y}$, which can be written as $\frac{x(x-X)}{a^2} + \frac{y(y-Y)}{b^2} = 0$. This is equation of an ellipse through points $O$ and $A$, with center at $\left(\frac{X}{2}, \frac{Y}{2}\right)$ and semiaxes parallel to the semiaxes of the original ellipse (see Fig. 13). Its intersections with the original ellipse (1) are on line $\frac{xX}{a^2} + \frac{yY}{b^2} = 1$, which is obtained from subtraction of the equations of the ellipses. The coordinates of points $B$ and $C$ are then determined by

$$x_1 = a \cdot \frac{\frac{X}{a} - \frac{Y}{b}\sqrt{\left(\frac{X}{a}\right)^2 + \left(\frac{Y}{b}\right)^2 - 1}}{\left(\frac{X}{a}\right)^2 + \left(\frac{Y}{b}\right)^2}, \; y_1 = b \cdot \frac{\frac{Y}{b} + \frac{X}{a}\sqrt{\left(\frac{X}{a}\right)^2 + \left(\frac{Y}{b}\right)^2 - 1}}{\left(\frac{X}{a}\right)^2 + \left(\frac{Y}{b}\right)^2},$$

$$x_2 = a \cdot \frac{\frac{X}{a} + \frac{Y}{b}\sqrt{\left(\frac{X}{a}\right)^2 + \left(\frac{Y}{b}\right)^2 - 1}}{\left(\frac{X}{a}\right)^2 + \left(\frac{Y}{b}\right)^2}, \; y_2 = b \cdot \frac{\frac{Y}{b} - \frac{X}{a}\sqrt{\left(\frac{X}{a}\right)^2 + \left(\frac{Y}{b}\right)^2 - 1}}{\left(\frac{X}{a}\right)^2 + \left(\frac{Y}{b}\right)^2}.$$

If we denote by $t(A)$ the total number of tangent lines of the ellipse passing through $A$, then $t(A) = 2$ outside of the ellipse, $t(A) = 0$ inside of the ellipse, and $t(A) = 1$ on the ellipse. The equations of the tangent lines are $\frac{y-Y}{x-X} = \frac{y_i-Y}{x_i-X}$ $(i = 1, 2)$.

Similarly, for ellipsoid (3), if $A$ is outside of ellipsoid (3), then point $B$, such that $AB$ is tangent to ellipsoid (3) can be determined by intersecting (3) with another ellipsoid $\frac{x(x-X)}{a^2} + \frac{y(y-Y)}{b^2} + \frac{z(z-Z)}{c^2} = 0$. All these intersection points are on plane $\frac{xX}{a^2} + \frac{yY}{b^2} + \frac{zZ}{c^2} = 1$ (see Fig. 14). The number of tangent lines $tl(A)$ and tangent planes $tp(A)$ of ellipsoid (3), which pass through $A$ is infinity ($tl(A) = tp(A) = \infty$) for exterior points of ellipsoid (3), $tl(A) = tp(A) = 0$ for interior points, and $tl(A) = \infty$ and $tp(A) = 1$ for the points of ellipsoid (3) itself.

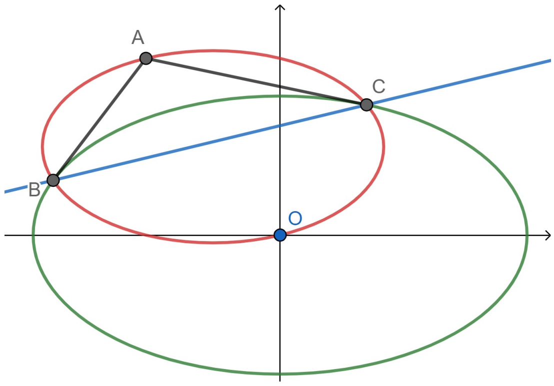


Figure 13: Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (green), its 2 tangents (black), line $\frac{xX}{a^2} + \frac{yY}{b^2} = 1$ (blue), and ellipse $\frac{x(x-X)}{a^2} + \frac{y(y-Y)}{b^2} = 0$ (red). See `https://www.geogebra.org/calculator/em993rev` for more details.

By solving $\frac{xX}{a^2} + \frac{yY}{b^2} + \frac{zZ}{c^2} = 1$ for $z$, we obtain $z = -\frac{\left(xX\,b^2 + yY\,a^2 - a^2b^2\right)c^2}{a^2b^2Z}$. By using this in (3) and taking $x = t$, we obtain parametrization for the intersection curve of ellipsoid (3) with plane $\frac{xX}{a^2} + \frac{yY}{b^2} + \frac{zZ}{c^2} = 1$ (see Fig. 14) as $B\,(t, y(t), z(t))$, where $z(t) = -\frac{\left(tX\,b^2 + y(t)Y\,a^2 - a^2b^2\right)c^2}{a^2b^2Z}$, and
$y(t) = \frac{b^2\left(-XY\,c^2t + c^2a^2Y \pm \sqrt{-X^2Z^2b^2c^2t^2 + 2X\,Z^2a^2b^2c^2t + Y^2Z^2a^4c^2 - Y^2Z^2a^2c^2t^2 + Z^4a^4b^2 - Z^4a^2b^2t^2 - Z^2a^4b^2c^2}\right)}{a^2(Y^2c^2 + Z^2b^2)}$.
Here $t_1 \le t \le t_2$, where $t_1, t_2$ ($t_1 \le t_2$) are the solutions of quadratic equation

$$\left(\frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2}\right)t^2 - 2Xt - a^2\left(\frac{Y^2}{b^2} + \frac{Z^2}{c^2} - 1\right) = 0.$$

There are other ways to parametrize the intersection of a quadric surface with a plane (see [56] and its references). One of them can be implemented using GeoGebra command IntersectConic (IntersectPath) (see https://www.geogebra.org/).

It is now straightforward to find the tangent lines and tangent planes passing through $A(X, Y, Z)$ and $B\,(t_0, y(t_0), z(t_0))$ for $t_1 \le t_0 \le t_2$. The equation of the tangent line can be written as

$$\frac{x - X}{t_0 - X} = \frac{y - Y}{y(t_0) - Y} = \frac{z - Z}{z(t_0) - Z}.$$

Using the components of normal vector $\mathbf{N} = \left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right)$ of ellipsoid (3) at point $B\,(t_0, y(t_0), z(t_0))$, the equation of the tangent plane can be written as

$$\frac{t_0(x - X)}{a^2} + \frac{y(t_0)(y - Y)}{b^2} + \frac{z(t_0)(z - Z)}{c^2} = 0.$$

The intersections of a triaxial ellipsoid with planes have many applications outside of mathematics [28], [31].

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## Availability of data and materials

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## Competing interests

The authors declare that they have no competing interests.

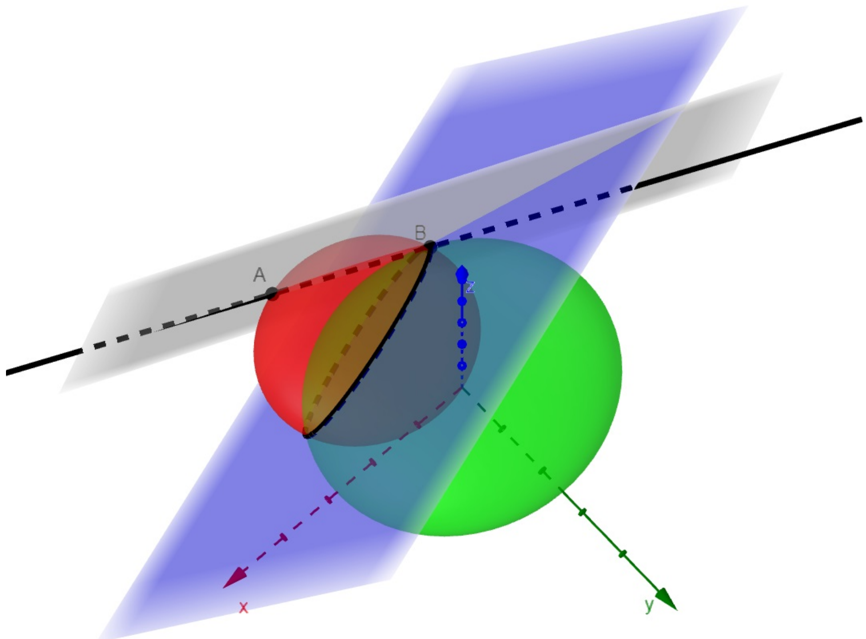


Figure 14: Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ (green), plane $\frac{xX}{a^2}+\frac{yY}{b^2}+\frac{zZ}{c^2}=1$ (blue), ellipsoid $\frac{x(x-X)}{a^2}+\frac{y(y-Y)}{b^2}+\frac{z(z-Z)}{c^2}=0$ (red), their intersection curve (ellipse, black), tangent plane (grey), and tangent line (black). See `https://www.geogebra.org/3d/uvh3uqdj` for more details.

### Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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