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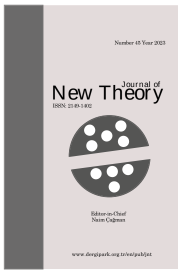

Altered Numbers of Fibonacci Number Squared

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Abstract − We investigate two types of altered Fibonacci numbers obtained by adding or subtracting a specific value $\{a\}$ from the square of the n^{th} Fibonacci numbers $G_{F_{(n)}}^{(2)}(a)$ and $H_{F_{(n)}}^{(2)}(a)$. These numbers are significant as they are related to the consecutive products of the Fibonacci numbers. As a result, we establish consecutive sum-subtraction relations of altered Fibonacci numbers and their Binet-like formulas. Moreover, we explore greatest common divisor (GCD) sequences of r-successive terms of altered Fibonacci numbers represented by $\left\{G_{F(n),r}^{(2)}(a)\right\}$ and $\left\{H_{F(n),r}^{(2)}(a)\right\}$ such that $r \in \{1,2,3\}$ and $a \in \{1,4\}$. The sequences are based on the GCD properties of consecutive terms of the Fibonacci numbers and structured as periodic or Fibonacci sequences.

Keywords *Altered Fibonacci number, greatest common divisor (GCD) sequence, Fibonacci sequence*

Mathematics Subject Classification (2020) 11B39, 11B50

1. Introduction

It is known [1] that the Fibonacci sequence is defined recursively as

$$F_n = F_{n-1} + F_{n-2}$$

for $n \geq 2$ with initial values $F_0 = 0$ and $F_1 = 1$ (A000045 in OEIS). As a similar, L_n is the n^{th} term in the Lucas sequence (A000032) and defined by

$$L_n = L_{n-1} + L_{n-2}, \qquad L_0 = 2 \quad \text{and} \quad L_1 = 1$$

Their characteristic equation is $x^2 = x + 1$ and its roots are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Hence, the Binet formulas for the Fibonacci F_n and Lucas L_n numbers are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n$$

Binet formulas can be used to prove certain properties of the Fibonacci and Lucas numbers. For instance, for negative subscripts the n^{th} Fibonacci number can be established as $F_{-n} = (-1)^{n+1}F_n$, for all $n \geq 1$, or two useful identities can be confirmed the Cassini identity and the d'Ocagne identity [1-3], respectively,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

and

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$$F_mF_{n+1} = F_nF_{m+1} + (-1)^nF_{m-n}, \quad m > n \geq 1$$

Additionally, the formulas sum and subtraction for the Fibonacci numbers squared are

$$F_{m+n+1}^2 + F_{m-n}^2 = F_{2m+1}F_{2n+1} \tag{1}$$

and

$$F_{m+n}^2 - F_{m-n}^2 = F_{2m}F_{2n} \tag{2}$$

Many sum properties [1-3] can be provided as examples of sequences derived from the Fibonacci numbers. The sum of the Fibonacci numbers is $\sum_{i=1}^{n} F_i = F_{n+2} - 1$ (A000071 in OEIS [2]), and the sum of even-indices Fibonacci numbers is $\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1$ (A027941 in [2]). These findings have been scrutinized as the altered Fibonacci sequences [4]. The sum of odd-indices Fibonacci numbers is $\sum_{i=1}^{n} F_{2i-1} = F_nL_n$ (A001906 in [2]). The sum of the Fibonacci numbers squared between F_1 and F_n is $\sum_{i=1}^{n} F_i^2 = F_nF_{n+1}$ (A180662, The golden rectangle numbers in [2]).

In the literature, numerous researchers [4-8] have developed novel sequences utilizing Fibonacci numbers and analyzed some of their basic properties. Dudley et al. [4] studied two altered Fibonacci sequences $\{G_n\} = \{F_n + (-1)^n\}$ and $\{H_n\} = \{F_n - (-1)^n\}$, concerned with number sequences A000071 and A027941, using the equations given by Theorem 1 in [4]

$$F_{4k} + 1 = F_{2k-1}L_{2k+1} \qquad F_{4k} - 1 = F_{2k+1}L_{2k-1} \tag{3}$$

$$F_{4k+1} + 1 = F_{2k+1}L_{2k} \qquad F_{4k+1} - 1 = F_{2k}L_{2k+1} \tag{4}$$

$$F_{4k+2} + 1 = F_{2k+2}L_{2k} \qquad F_{4k+2} - 1 = F_{2k}L_{2k+2} \tag{5}$$

$$F_{4k+3} + 1 = F_{2k+1}L_{2k+2} \qquad F_{4k+3} - 1 = F_{2k+2}L_{2k+1} \tag{6}$$

Some of those are easily obtained according to whether n is odd or even in the Cassini identity. Moreover, $\{(G_n, G_{n+1})\}_{n\geq 0}$ and $\{(H_n, H_{n+1})\}_{n\geq 0}$ sequences are defined by using the greatest common divisor (GCD) of the numbers G_n and H_n considering Equations 3-6 are multiplication cases. These sequences produce Fibonacci subsequences, such as $(G_{4k}, G_{4k+1}) = L_{2k+1}$, $(G_{4k+2}, G_{4k+3}) = F_{2k+2}$, $(H_{4k}, H_{4k+1}) = F_{2k+1}$, and $(H_{4k+2}, H_{4k+3}) = L_{2k+2}$ [4]. Hernandez and Luca [5] proved the existence of an integer c in the form of an infinite number $(F_n + a, F_m + b) > e^{(cm)}$ of any positive integer $n < m$, according to various n and m for the positive integers a and b. Chen [6] defined a sequence $\{F_n + a\}_{n\geq 0}$ such that $a \in \mathbb{Z}$, called a shifted Fibonacci sequence, and established a sequence $\{f_n(a)\}_{n\geq 0} = \{(F_n + a, F_{n+1} + a)\}_{n\geq 0}$, referred to as a GCD sequence of the shifted Fibonacci sequence. He showed that some successive terms of the altered and shifted sequences have different behavior, such as $f_{4n-1}(1) = F_{2n-1}$, $f_{4n+1}(1) = L_{2n}$, $f_{4n-1}(-1) = L_{2n-1}$, and $f_{4n+1}(-1) = F_{2n}$. The author showed that $\{f_n(a)\}$ is bounded from above if $a \neq \pm 1$. In [7], in addition to the properties of $\{f_n(a)\}$, Spilker showed that for two integers a and n if $m = a^4 - 1$ is not 0 and $f_n(a)$ divides $a^2 + (-1)^n$, then $f_n(a)$ is simply periodic such that a period p is defined by $F_p \equiv 0 \pmod m$. Koken [8] defined the altered sequences $\{L_n^+\}_{n>0}$ and $\{L_n^-\}_{n>0}$ such that $L_{4k}^+ = 5F_{2k+1}F_{2k-1}$, $L_{4k+1}^+ = 5F_{2k+1}F_{2k}$, $L_{4k+2}^+ = L_{2k+2}L_{2k}$, and $L_{4k+3}^+ = L_{2k+2}L_{2k+1}$ and $L_{4k}^- = L_{2k+1}L_{2k-1}$, $L_{4k+1}^- = L_{2k+1}L_{2k}$, $L_{4k+2}^- = 5F_{2k+2}F_{2k}$, and $L_{4k+3}^- = 5F_{2k+2}F_{2k+1}$. Furthermore, he presented the numbers $L_{4k,1}^+ = 5F_{2k+1}, k \geq 1$, $L_{4k-2,1}^+ = L_{2k}, k \geq 1$, $L_{4k,1}^- = L_{2k+1}$ and $L_{4k+2,1}^- = 5F_{2k+2}$ where $L_{n,r}^{\pm} = \left(L_n^{\pm}, L_{n+r}^{\pm}\right)$ denotes r-successive GCD numbers. Besides, the GCD numbers $L_{n,r}^+$ and $L_{n,r}^-$ are obtained by $r \in \{2,3,4\}$. For over 50 years, many authors [9-14] have studied to determine all such numbers of the forms w^2, w^3, $w^2 \pm 1$, and $w^3 \pm 1$ in the Fibonacci sequences. Marques

[15] has considered the Fibonacci variant of the Brocard-Ramanujan equation and claimed that the Diophantine equation

$$F_n F_{n+1} \cdots F_{n+k-2} F_{n+k-1} + 1 = F_m^2 \tag{7}$$

has no solution according to the positive integer values k, m, and n. However, according to equations $F_2 F_4 + 1 = F_1 F_4 + 1 = F_3^2$ and $F_2 F_6 + 1 = F_1 F_6 + 1 = F_4^2$, it can be observed that the Fibonacci Brocard-Ramanujan version in Equality 7 has solutions. Szalay [16] obtains the solutions of the equations by accepting a correct version of the result of Marques [15] more general than the Fibonacci Brocard-Ramanujan equation in Equality 7. Pongsriiam [17] has continued to search for the solutions of the Diophantine equations:

$$F_{n_1} F_{n_2} \cdots F_{n_{k-1}} F_{n_k} \pm 1 = F_m^2 \quad \text{and} \quad L_{n_1} L_{n_2} \cdots L_{n_{k-1}} L_{n_k} \pm 1 = F_m^2$$

such that $0 \leq n_1 < n_2 < \cdots < n_{k-1} < n_k$, $m \geq 0$, and $k \geq 1$.

Inspired by previous research on altered Fibonacci numbers and the Brocard-Ramanujan equation, this study aims to explore their applications and altered sequences of Fibonacci numbers squared. This investigation is continued by the question of whether it is possible to define altered Fibonacci sequences, specifically those of the form $\{F_n^2 \pm a\}$. Unlike [18,19], related to the sum of sequences of k-consecutive Fibonacci numbers, the paper considers the results of altered Fibonacci numbers squared through the following sums:

$$\sum_{j=1}^{2n} F_j F_{j+1} = F_{2n+1}^2 - 1 \quad \text{or} \quad \sum_{j=2}^{2n} F_j F_{j+1} = F_{2n+1}^2 - 2$$

and

$$\sum_{j=1}^{2n-1} F_j F_{j+1} = F_{2n}^2$$

Koken [20] investigate two types altered Lucas numbers $G_{L(n)}^{(2)}(a)$ and $H_{L(n)}^{(2)}(a)$. Since these numbers form as the consecutive products of the Fibonacci numbers, they give the GCD sequences of r–successive terms of altered Lucas numbers denoted $\left\{G_{L(n),r}^{(2)}(a)\right\}$ and $\left\{H_{L(n),r}^{(2)}(a)\right\}$ such that $r \in \{1,2\}$ and $a \in \{1,9\}$. We show that these sequences are periodic or Fibonacci sequences.

This present paper is organized as follows: Section 2 provides brief definitions and properties. Section 3 defines two altered sequences and investigates some of their properties. This includes analyzing the sum and difference, Binet formula, and closed forms for the numbers $G_{F(n)}^{(2)}(a) = F_n^2 + (-1)^n a$ and $H_{F(n)}^{(2)}(a) = F_n^2 - (-1)^n a$. Section 4 establishes two types of r-successive altered Fibonacci GCD sequences, referred to as $G_{F(n),r}^{(2)}(a)$ and $H_{F(n),r}^{(2)}(a)$, and investigates these sequences according to the cases $r \in \{1,2,3\}$ for the values $G_{F(n)}^{(2)}(a)$ and $H_{F(n)}^{(2)}(a)$ such that $a \in \{1,4\}$.

2. Preliminaries

This section defines two types of altered numbers derived by using a value $\{a\}$ from the n^{th} Fibonacci number squared. It works on taking values $\{\pm 1\}$ instead of $\{a\}$.

Definition 2.1. The n^{th} altered Fibonacci numbers denoted by $G_{F_{(n)}^{(2)}}(a)$ and $H_{F_{(n)}^{(2)}}(a)$ are defined as

$$G_{F(n)}^{(2)}(a) = F_n^2 + (-1)^n a \tag{8}$$

and

$$H_{F(n)}^{(2)}(a) = F_n^2 - (-1)^n a \tag{9}$$

where F_n be the n^{th} Fibonacci number and $a \in \mathbb{Z}$.

For example, particular values $G_{F_{(n)}}^{(2)}(1)$ and $H_{F_{(n)}}^{(2)}(1)$ numbers are provided in Table 1, and they follow $G_{F_{(n)}}^{(2)}(1) = H_{F_{(n)}}^{(2)}(-1)$ and $H_{F_{(n)}}^{(2)}(1) = G_{F_{(n)}}^{(2)}(-1)$.

Table 1. First few terms of $G_{F(n)}^{(2)}(1)$ and $H_{F_{(n)}}^{(2)}(1)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$G_{F(n)}^{(2)}(1)$	1	0	2	3	10	24	65	168	442	1155	3026	7920	20737
$H_{F_{(n)}}^{(2)}(1)$	-1	2	0	5	8	26	63	170	440	1157	3024	7922	20735

Table 1 shows that $G_{F_{(3n)}}^{(2)}(1)$ and $H_{F_{(3n)}}^{(2)}(1)$ are odd, and the others are even, any increasing sequences with special values except the first few values. The general terms of the sequences $\left\{G_{F_{(n)}}^{(2)}(1)\right\}$ and $\left\{H_{F_{(n)}}^{(2)}(1)\right\}$ can be given as follows:

Theorem 2.2. Let $G_{F_{(n)}}^{(2)}(1)$ and $H_{F_{(n)}}^{(2)}(1)$ denote the n^{th} altered Fibonacci numbers. Then,

$$G_{F_{(n)}}^{(2)}(1) = F_{n+1}F_{n-1} \tag{10}$$

and

$$H_{F_{(n)}}^{(2)}(1) = F_{n+2}F_{n-2} \tag{11}$$

Proof.

If $m = k + 1$ and $n = k$ in Equation 1, then $G_{F_{(2k)}}^{(2)}(1) = F_{2k-1}F_{2k+1}$, for $a = 1$ and $n = 2k$ in Equation 8. In addition, if $m = k + 1$ and $n = k$ in Equation (2), then $G_{F_{(2k+1)}}^{(2)}(1) = F_{2k+2}F_{2k}$, for $a = 1$ and $n = 2k + 1$ in Equation 8. The number $G_{F_{(n)}}^{(2)}(1)$ is observed from these equations for $n = 2k$ and $n = 2k + 1$.

If $m = k + 2$ and $n = k$ in Equation 1, then $H_{F_{(2k+1)}}^{(2)}(1) = F_{2k+3}F_{2k-1}$, for $a = 1$ and $n = 2k + 1$ in Equation 9. For $m = k + 2$ and $n = k$ in Equation 2, $H_{F_{(2k)}}^{(2)}(1) = F_{2k+2}F_{2k-2}$ when $n = 2k$ in Equation 9. The number $H_{F_{(n)}}^{(2)}(1)$ is observed from these equations for $n = 2k$ and $n = 2k + 1$. □

We have conducted research on several addition and subtraction identities of numbers $G_{F_{(n)}}^{(2)}(1)$ and $H_{F_{(n)}}^{(2)}(1)$.

Theorem 2.3. Let $G_{F_{(n)}}^{(2)}(1)$ and $H_{F_{(n)}}^{(2)}(1)$ be the n^{th} altered Fibonacci numbers. Then,

$$G_{F_{(n)}}^{(2)}(1) + G_{F_{(n+1)}}^{(2)}(1) = H_{F_{(n)}}^{(2)}(1) + H_{F_{(n+1)}}^{(2)}(1) = F_{2n+1} \tag{12}$$

$$G_{F_{(n+1)}}^{(2)}(1) - G_{F_{(n-1)}}^{(2)}(1) = H_{F_{(n+1)}}^{(2)}(1) - H_{F_{(n-1)}}^{(2)}(1) = F_{2n} \tag{13}$$

$$2G_{F_{(n+1)}}^{(2)}(1) + G_{F_{(n)}}^{(2)}(1) - G_{F_{(n-1)}}^{(2)}(1) = F_{2n+2} \tag{14}$$

$$2H_{F_{(n+1)}^{(2)}}(1) + H_{F_{(n)}^{(2)}}(1) - H_{F_{(n-1)}^{(2)}}(1) = F_{n+1}L_{n+1} \tag{15}$$

PROOF.

From Equations 10 and 11 and the identities $F_n^2 + F_{n+1}^2 = F_{2n+1}$ and $F_nL_n = F_{2n}$,

$$H_{F_{(n)}^{(2)}}(1) + H_{F_{(n+1)}^{(2)}}(1) = F_{n+2}F_{n-2} + (F_{n+2} + F_{n+1})F_{n-1} = F_{2n+1}$$

and

$$G_{F(n+1)}^{(2)}(1) - G_{F(n-1)}^{(2)}(1) = F_n(F_{n+1} + F_n - F_{n-2}) = F_nL_n$$

The others in Equations 12 and 13 are obtained similarly. If Equations 12 and 13 are summed side-to-side collection, then Equations 14 and 15 are obtained. □

2.1. Altered Fibonacci Sequences $G_{F_{(n)}^{(2)}}(F_t^2)$ and $H_{F_{(n)}^{(2)}}(F_t^2)$

This subsection generalizes the value $\{a\}$ in Equations 8 and 9 as the square of t^{th} Fibonacci numbers such that $t \in \mathbb{Z}$.

Theorem 2.4. Let $G_{F_{(n)}^{(2)}}(F_t^2)$ and $H_{F_{(n)}^{(2)}}(F_t^2)$ denote the n^{th} altered Fibonacci numbers. Then,

$$G_{F_{(n)}^{(2)}}(F_t^2) = F_{n+t}F_{n-t}, \qquad t \text{ is odd} \tag{16}$$

and

$$H_{F_{(n)}^{(2)}}(F_t^2) = F_{n+t}F_{n-t}, \qquad t \text{ is even} \tag{17}$$

where F_t^2 is the square of the t^{th} Fibonacci numbers.

PROOF.

Let t is odd. If $m = k + (t+1)/2$ and $n = k - (t-1)/2$ are taken in Equation 1, for $a = F_t^2$ and $n = 2k$ in Equation 8, then $G_{F_{(2k)}^{(2)}}(F_t^2) = F_{(2k)+t}F_{(2k)-t}$. Moreover, if values of $m = k + (t+1)/2$ and $n = k - (t-1)/2$ are considered in Equation 2, according to $a = F_t^2$ and $n = 2k+1$ in Equation 8, then $G_{F_{(2k+1)}^{(2)}}(F_t^2) = F_{(2k+1)+t}F_{(2k+1)-t}$.

Similarly, let t is even. If $m = k + t/2$ and $n = k - t/2$ in Equations 1 and 2, then the desired result is obtained. □

As a result, the sum of two successive altered Fibonacci numbers equals the Fibonacci number, and no alike Fibonacci recurrence relation is provided. However, a Binet-like formula for the numbers $G_{F_{(n)}^{(2)}}(F_t^2)$ and $H_{F_{(n)}^{(2)}}(F_t^2)$ can be obtained by using the Fibonacci Binet formula.

Theorem 2.5. Let $G_{F_{(n)}^{(2)}}(F_t^2)$ and $H_{F_{(n)}^{(2)}}(F_t^2)$ be the n^{th} altered Fibonacci numbers. Then,

$$G_{F_{(n)}^{(2)}}(F_t^2) = \frac{(\alpha^{2n} + \beta^{2n}) + (-1)^n(\alpha^{2t} + \beta^{2t})}{5}, \qquad t \text{ is odd} \tag{18}$$

and

$$H_{F_{(n)}^{(2)}}(F_t^2) = \frac{(\alpha^{2n} + \beta^{2n}) - (-1)^n(\alpha^{2t} + \beta^{2t})}{5}, \qquad t \text{ is even} \tag{19}$$

Proof.

Let t is odd. If we substitute the Fibonacci Binet formula in Equation 16. Then,

$$G_{F_{(n)}^{(2)}}(F_t^2) = \frac{(\alpha^{n+t} - \beta^{n+t})(\alpha^{n-t} - \beta^{n-t})}{(\alpha - \beta)^2}$$

By using $\alpha - \beta = \sqrt{5}$ and $\alpha\beta = -1$, the desired expression is obtained. The other appeared as an application of the Fibonacci Binet formula in Equation 17. □

As a result of Equations 18 and 19, Binet-like formulas for the numbers $G_{F_{(n)}^{(2)}}(1)$ and $H_{F_{(n)}^{(2)}}(1)$ are

$$G_{F_{(n)}^{(2)}}(1) = \frac{(\alpha^{2n} + \beta^{2n}) + (-1)^n 3}{5} = \frac{L_{2n} + (-1)^n 3}{5}$$

and

$$H_{F_{(n)}^{(2)}}(1) = \frac{(\alpha^{2n} + \beta^{2n}) - (-1)^n 7}{5} = \frac{L_{2n} - (-1)^n 7}{5}$$

More details about the sequences $a(n) = F_n F_{n+2}$ and $b(n) = F_n F_{n+4}$ can be found in (A059929) and (A192883). We study the special terms of the altered Fibonacci numbers $G_{F_{(n)}^{(2)}}(F_t^2) = H_{F_{(n)}^{(2)}}(-F_t^2)$ and $H_{F_{(n)}^{(2)}}(F_t^2) = G_{F_{(n)}^{(2)}}(-F_t^2)$. The altered number $G_{F_{(n)}^{(2)}}(4) = F_{n+3}F_{n-3}$ is the case $t = 3$ in Equation 16. Furthermore, the sequence $x(n) = F_{n+3}F_{n-3}$ has been studied in the literature (A292612) with its different applications. The altered number $H_{F_{(n)}^{(2)}}(9) = F_{n+4}F_{n-4}$ is the case $t = 4$ in Equation 17. In addition, the sequence $b(n) = F_{n+4}F_{n-4}$ has been studied in the literature (A292612) with its different applications. However, $H_{F_{(n)}^{(2)}}(4)$ and $G_{F_{(n)}^{(2)}}(9)$ could not be generalized as the product of Fibonacci or Lucas numbers.

3. Altered Fibonacci GCD Sequences $G_{F_{(n),r}^{(2)}}(a)$ and $H_{F_{(n),r}^{(2)}}(a)$

A GCD of two Fibonacci numbers is a Fibonacci number, such as $(F_m, F_n) = F_{(m,n)}$ and $(F_m, F_n) = (F_n, F_r)$, for all $m = qn + r$ such that $m, n, r, q \in \mathbb{N}$. Thus, two successive Fibonacci numbers are relatively prime, i.e., $(F_n, F_{n+1}) = 1$ and $\left(F_{qn-1}, F_n\right) = (F_n, F_{n+2}) = 1$ [1-3]. This section investigates properties related to GCD of two numbers whose indices differ r from the altered sequences $\left\{G_{F_{(n)}^{(2)}}(a)\right\}$ and $\left\{H_{F_{(n)}^{(2)}}(a)\right\}$.

Definition 3.1. Let $G_{F_{(n)}^{(2)}}(a)$ and $H_{F_{(n)}^{(2)}}(a)$ be the n^{th} altered Fibonacci numbers. Then,

$$G_{F_{(n),r}^{(2)}}(a) = \left(G_{F_{(n)}^{(2)}}(a), G_{F_{(n+r)}^{(2)}}(a)\right)$$

and

$$H_{F_{(n),r}^{(2)}}(a) = \left(H_{F_{(n)}^{(2)}}(a), H_{F_{(n+r)}^{(2)}}(a)\right)$$

The sequences $\left\{G_{F_{(n),r}^{(2)}}(a)\right\}$ and $\left\{H_{F_{(n),r}^{(2)}}(a)\right\}$ formed by these numbers are called the r-successive altered Fibonacci GCD sequences.

Table 2 shows $\left\{G_{F_{(n),1}^{(2)}}(1)\right\}$ and $\left\{H_{F_{(n),1}^{(2)}}(1)\right\}$ are not increasing or decreasing but can be periodic sequences.

Table 2. 1-successive altered Fibonacci GCD numbers $G_{F(n),1}^{(2)}(1)$ and $H_{F(n),1}^{(2)}(1)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$G_{F(n),1}^{(2)}(1)$	1	2	1	1	2	1	1	2	1	1	2	1	1	2	1	1
$H_{F(n),1}^{(2)}(1)$	1	2	5	1	2	1	1	10	1	1	2	1	5	2	1	1

The following theorem investigates whether 1-successive altered Fibonacci GCD sequences take special values in certain periods.

Theorem 3.2. Let $G_{F(n),1}^{(2)}(1)$ and $H_{F(n),1}^{(2)}(1)$ be the n^{th} 1-successive altered Fibonacci GCD numbers. Then,

$$G_{F(n),1}^{(2)}(1) = \begin{cases} 2, & n \equiv 1 \pmod 3 \\ 1, & \text{otherwise} \end{cases}$$

and

$$H_{F(n),1}^{(2)}(1) = \begin{cases} 10, & n \equiv 7 \pmod{15} \\ 5, & n \equiv 2,12 \pmod{15} \\ 2, & n \equiv 1,4,10,13 \pmod{15} \\ 1, & \text{otherwise} \end{cases}$$

PROOF.

According to Equation 10, $G_{F(n),1}^{(2)}(1) = (F_{n+1}F_{n-1}, F_nF_{n+2})$. Since $(F_{n+1}, F_n) = (F_{n+1}, F_{n+2}) = (F_{n-1}, F_n) = 1$, then $G_{F(n),1}^{(2)}(1) = (F_{n-1}, F_{n+2})$. Therefore, let $(F_{n-1}, F_{n+2}) = d$. By using $(F_x, F_y) = F_{(x,y-x)}$, $(F_{n-1}, F_{n+2}) = F_{(n-1,3)} = F_3, n \equiv 1 \pmod 3$. Otherwise, $(F_{n-1}, F_{n+2}) = F_1$.

According to Equation 11, $H_{F(n),1}^{(2)}(1) = (F_{n+2}F_{n-2}, F_{n-1}F_{n+3})$. Since $(F_{n+2}, F_{n+3}) = (F_{n-2}, F_{n-1}) = 1$, then $H_{F(n),1}^{(2)}(1) = (F_{n-2}, F_{n+3})(F_{n+2}, F_{n-1})$. Thus, if $(F_{n-2}, F_{n+3}) = F_{(n-2,5)} = F_5$, $n \equiv 2 \pmod 5$ and $(F_{n+2}, F_{n-1}) = F_{(n-1,3)} = F_3$, $n \equiv 1 \pmod 3$, then we can obtain desired results by using the Chinese remainder theorem. □

Table 3 manifests that the 2-successive altered Fibonacci GCD sequence $\left\{G_{F(n),2}^{(2)}(1)\right\}$, for $n \geq 2$, takes values according to a specific increasing sequence, and the sequence $\left\{H_{F(n),2}^{(2)}(1)\right\}$ is seen periodic.

Table 3. 2-successive altered Fibonacci GCD numbers $G_{F(n),2}^{(2)}(1)$ and $H_{F(n),2}^{(2)}(1)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$G_{F(n),2}^{(2)}(1)$	1	3	2	3	5	24	13	21	34	165	89	144	233	1131	610	987
$H_{F(n),2}^{(2)}(1)$	1	1	8	1	1	2	1	1	8	1	1	2	1	1	8	1

Some properties of the aforesaid sequences are as follows:

Theorem 3.3. Let $G_{F(n),2}^{(2)}(1)$ and $H_{F(n),2}^{(2)}(1)$ be the n^{th} 2-successive altered Fibonacci GCD numbers. Then,

$$G_{F(n),2}^{(2)}(1) = \begin{cases} 3F_{n+1}, & n \equiv 1 \pmod 4 \\ F_{n+1}, & \text{otherwise} \end{cases}$$

and

$$H_{F(n),2}^{(2)}(1) = \begin{cases} 8, & n \equiv 2 \pmod 6 \\ 2, & n \equiv 5 \pmod 6 \\ 1, & \text{otherwise} \end{cases}$$

PROOF.

According to Equation 10, $G_{F_{(n),2}}^{(2)}(1) = \left(G_{F_{(n)}}^{(2)}(1), G_{F_{(n+2)}}^{(2)}(1)\right) = F_{n+1}(F_{n-1}, F_{n+3})$. Therefore, $(F_{n-1}, F_{n+3}) = F_{(n-1,4)} = F_4$, $n \equiv 1 \pmod 4$ by using $\left(F_x, F_y\right) = F_{(x,y-x)}$. Otherwise, it is seen that $(F_{n-1}, F_{n+3}) = F_{(n-1,4)} = F_2$ or F_1.

According to Equation 11, $H_{F(n),2}^{(2)}(1) = (F_{n+2}F_{n-2}, F_nF_{n+4})$. Because of $(F_{n+2}, F_n) = (F_{n+2}, F_{n+4}) = (F_{n-2}, F_n) = 1$, we study on $H_{F(n),2}^{(2)}(1) = (F_{n-2}, F_{n+4})$. Thus, $H_{F(n),2}^{(2)}(1) = F_{(n-2,6)} = F_6$, $n \equiv 2 \pmod 6$. Otherwise, the others are $H_{F(n),2}^{(2)}(1) = F_{(n-2,6)} = F_3$, $n \equiv 5 \pmod 6$; $H_{F(n),2}^{(2)}(1) = F_2, n \equiv 0,4 \pmod 6$; or $H_{F(n),2}^{(2)}(1) = F_1, n \equiv 1,3 \pmod 6$. □

Theorem 3.4. Let $G_{F_{(n),2}}^{(2)}(1)$ be the n^{th} 2-successive altered Fibonacci GCD number. Then,

$$G_{F_{(n),2}}^{(2)}(1) + G_{F_{(n+1),2}}^{(2)}(1) = \begin{cases} F_{n+1} + L_{n+2}, & n \equiv 1 \pmod 4 \\ L_{n+3}, & n \equiv 0 \pmod 4 \\ F_{n+3}, & \text{otherwise} \end{cases}$$

PROOF.

According to $G_{F_{(n),2}}^{(2)}(1)$ in Theorem 3.3,

$$G_{F_{(n),2}}^{(2)}(1) + G_{F_{(n+1),2}}^{(2)}(1) = \begin{cases} F_{n+2} + 3F_{n+1}, & n \equiv 1 \pmod 4 \\ 3F_{n+2} + F_{n+1}, & n \equiv 0 \pmod 4 \\ F_{n+1} + F_{n+2}, & \text{otherwise} \end{cases}$$

$$= \begin{cases} F_{n+3} + 2F_{n+1}, & n \equiv 1 \pmod 4 \\ F_{n+2} + F_{n+4}, & n \equiv 0 \pmod 4 \\ F_{n+3}, & \text{otherwise} \end{cases}$$

by the identity $F_{n+1} + F_{n-1} = L_n$. □

This study continues according to the particular values of the numbers $G_{F_{(n)}}^{(2)}(4) = F_{n+3}F_{n-3}$ and $H_{F_{(n)}}^{(2)}(9) = F_{n+4}F_{n-4}$ provided in Table 4.

Table 4. Altered Lucas numbers $G_{F(n)}^{(2)}(4)$ and $H_{F_{(n)}}^{(2)}(9)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$G_{F(n)}^{(2)}(4)$	4	-3	5	0	13	21	68	165	445	1152	3029	7917	20740
$H_{F_{(n)}}^{(2)}(9)$	-9	10	-8	13	0	34	55	178	432	1165	3016	7930	20727

By utilizing properties divisibility and GCD of Fibonacci numbers, GCD sequences $G_{F_{(n),r}}^{(2)}(4)$, $r \in \{1,2,3\}$, of the sequences $G_{F(n)}^{(2)}(4)$ presented in Table 4 are observed periodic.

$$G_{F_{(n),1}}^{(2)}(4) = (F_{n+3}F_{n-3}, F_{n+4}F_{n-2}) = \begin{cases} F_5F_7, & n \equiv 17 \pmod{35} \\ F_7, & n \equiv 3,10,24,31 \pmod{35} \\ F_5, & n \equiv 2,7,12,22,27,32 \pmod{35} \\ 1, & \text{otherwise} \end{cases}$$

$$G_{F_{(n),2}}^{(2)}(4) = (F_{n+3}F_{n-3}, F_{n+5}F_{n-1}) = \begin{cases} F_8, & n \equiv 3 \pmod 8 \\ F_4, & n \equiv 1,5,7 \pmod 8 \\ 1, & \text{otherwise} \end{cases}$$

and

$$G_{F_{(n),3}^{(2)}}(4) = (F_{n+3}F_{n-3}, F_{n+6}F_n) = \begin{cases} F_3F_9, & n \equiv 3 \pmod 8 \\ F_3^2, & n \equiv 1,5,7 \pmod 8 \\ 1, & \text{otherwise} \end{cases}$$

We haven't got a closed-form expression for the numbers $G_{F_{(n)}^{(2)}}(9) = H_{F_{(n)}^{(2)}}(-9)$ and $G_{F_{(n)}^{(2)}}(-4) = H_{F_{(n)}^{(2)}}(4)$. Thus, the properties of the GCD sequences $G_{F_{(n),r}^{(2)}}(9)$ and $H_{F_{(n),r}^{(2)}}(4)$, $r \in \{1,2,3\}$, have been investigated by using MAPLE up to $n < 100$. It is seen that all sequences are bounded and periodic sequences.

4. Conclusion

In this study, we derived two types of altered numbers of the Fibonacci numbers squared, defined as $G_{F_{(n)}^{(2)}}(a) = F_n^2 + (-1)^n a$ and $H_{F_{(n)}^{(2)}}(a) = F_n^2 - (-1)^n a$, for $a \in \mathbb{Z}$. We observed that the numbers $G_{F_{(n)}^{(2)}}(1)$ and $H_{F_{(n)}^{(2)}}(1)$ correspond to an extraordinary multiplication of the Fibonacci numbers. Furthermore, their generalizations $G_{F_{(n)}^{(2)}}(F_t^2)$ and $H_{F_{(n)}^{(2)}}(F_t^2)$ exhibit the same unique Fibonacci multiplication as follows:

$$G_{F_{(n)}^{(2)}}(F_t^2) = F_{n+t}F_{n-t}, \qquad t \text{ is odd}$$

and

$$H_{F_{(n)}^{(2)}}(F_t^2) = F_{n+t}F_{n-t}, \qquad t \text{ is even}$$

Therefore, we researched r-successive altered Fibonacci GCD sequences $\left\{G_{F_{(n),r}^{(2)}}(a)\right\}$ and $\left\{H_{F_{(n),r}^{(2)}}(a)\right\}$, where $a \in \{-1,1\}$ and $r \in \{1,2\}$. We could refer that the sequences $\left\{G_{F_{(n),2}^{(2)}}(1)\right\}$ and $\left\{H_{F_{(n),4}^{(2)}}(1)\right\}$ are Fibonacci subsequences. The other GCD sequences are periodic and bounded. In future studies, other properties of the sequences $\left\{G_{F_{(n),r}^{(2)}}(F_t^2)\right\}$ and $\left\{H_{F_{(n),r}^{(2)}}(F_t^2)\right\}$ and their r-successive GCD sequences are worth studying. Besides, matrix and graph applications may be handled.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the second author's master's thesis supervised by the first author. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

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