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Bonferroni arithmetic mean operator of trapezoidal fuzzy multi numbers and its decision-making application to crafting the ideal student dormitory

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Fuzzy multi set, Trapezoidal fuzzy numbers, Trapezoidal fuzzy multi numbers, Bonferroni arithmetic mean, Multiple attribute decision making

Abstract — Trapezoidal fuzzy multi-numbers (TFM-numbers) are widely used in the decision-making process when choosing among various potential values for alternatives. In this context, we present a methodology for multiple attribute decision-making problems in terms of TFM-numbers. This is why we have developed an aggregation technique known as the TFM-Bonferroni arithmetic mean operator. This operator is utilized to aggregate information represented by TFM-numbers. We then gave an examination of its properties and discussed its special cases. Furthermore, we introduce an approach designed to tackle multiple attribute decision-making as part of TFM environments. We subsequently apply this approach to solve multi-attribute decision-making problems. To illustrate its practicality, we provide an example in daily life. Finally, we offer an analysis table that facilitates a comparative evaluation of our proposed approach against existing methods.

Subject Classification (2020): 94D05, 47S40

1. Introduction

In this century, among the various paradigm shifts observed in mathematics and science, uncertainty is perhaps the most striking. There has been a gradual shift from the traditional understanding that views uncertainty as an undesirable situation and believes it should be avoided in all possible cases to an alternative perspective that deals with uncertainty and claims that it is impossible to avoid in science. Mathematicians, logicians, and philosophers have been grappling with problems of uncertainty for a long time. Recently, such problems have become very important for scientists and researchers in the fields of computer science and artificial intelligence. Researchers are continually proposing new theories due to the importance of being able to mathematically express uncertain concepts that classical logic cannot define. One of the most well-known theories in this regard is the fuzzy set theory, suggested by Zadeh [1]. Fuzzy sets, an extension of the classical sets, have been implemented in many

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areas to overcome uncertainties, including situations that are not strictly categorized as true or false and where clear boundaries are absent. For example, Sarkar et al. [2] built an article of fuzzy in the planning of transportation and regulation of traffic. Şahin et al. [3, 4] proposed two articles to show the usage of fuzzy logic to conduct a study on education. They built an application of artificial intelligence and aimed to see the effect of national human rights in the context of the protection and promotion of human rights.

In time, fuzzy sets have been expanded and diversified by scientists substantially. For instance, Dijkman et al. [5] introduced some types of fuzzy numbers. Then, they proposed operations and worked on relationships between these operations. Additionally, a median method was introduced by Srinivasan et al. [6]. They aimed to access optimum solutions for a decision-making problem related to transportation. Dubois et al. [7] proposed new properties of transformation on probability-possibility and gave a paper on symmetric triangular fuzzy numbers. Then, a new method was developed by Roseline and Amirtharaj [8] for the ranking of generalized trapezoidal fuzzy numbers. Afterward, they suggested a generalized fuzzy Hungarian method to get an initial solution for a problem about transportation. Readers can find further studies on trapezoidal and triangular fuzzy numbers. For more details, see [9–13]. Over time, the theory was expanded by many authors, and new types of fuzzy numbers were suggested and studied.

Fuzzy sets assign membership values within the range $[0, 1]$ to elements of the universe. However, this membership value may be inadequate to provide comprehensive information for certain problems, particularly when each element has different membership values. To cope with this limitation, a different generalization of fuzzy sets known as multi-fuzzy sets (also referred to as fuzzy bags) was introduced by Yager [17]. They are extended of both fuzzy sets and multi-sets. Then, Ramakrishnan and Sebastian [18] and Sebastian and John [19] expanded Yager's concepts to aggregate vague information and uncertainty. At the same time, extensive research have been conducted on multi-fuzzy sets [20–24]. Due to the possibility of multiple occurrences with different membership functions, trapezoidal fuzzy multi-numbers on the real number set $\mathbb{R}$ introduced by Uluçay et al. [25]. This extension combines elements of both multi-fuzzy sets and fuzzy numbers by permitting the repeated occurrence of any element. Further enhancing this concept, Keleş [26] defined the notions of value and ambiguity using α-cut sets. Keleş also developed similarity and distance measures and applied them to solve multi-attribute decision-making problems in the context of TFM-numbers. In addition, Şahin et al. [27] proposed a novel approach to multi-criteria decision-making by introducing the concepts of dice vector similarity and weighted dice vector similarity measures. Readers can find more studies of TFM-numbers in [28–30]. Thus far, many generalizations have been conducted, such as linear diophantine fuzzy (LDF) sets [14–16], containing reference parameters.

Lately, multi-criteria decision-making methods, closely related to fuzzy logic, have become widely studied for decision-making problems such as selecting the best alternative or ranking alternatives. Despite the relatively recent adoption of fuzzy logic in fields such as finance, education, agriculture, automotive, and others, the number of studies conducted in these areas is increasing every day. These studies mostly focus on decision-making and have led to the development of various decision-making operators. Some of these operators are the Bonferroni mean operators, which were developed by Bonferroni [31] in 1950. They can mainly find the interrelationships among arguments, which has an important role in the multi-criteria decision-making process. After the operators were introduced, many scientists studied Bonferroni operators. For instance, Yager [32] proposed a paper forming a frame on Bonferroni mean operators. Zhu et al. [33] conducted a study on Bonferroni geometric means of hesitant fuzzy sets. Xu [34] proposed comprehensive studies on Bonferroni mean operators extended into hesitant fuzzy elements. Wan and Zhu [35] proposed triple Bonferroni harmonic mean operators.

Further, they proposed an application to multi-attribute group decision-making problems given with a triangular intuitionistic fuzzy environment. Wang et al. [36] introduced a study on Archimedean Bonferroni mean operators. Perez et al. [37] presented a novel operator to combine the heavy-induced prioritized Bonferroni. Garg et al. [38] extended the Archimedean Bonferroni mean operators to complex Pythagorean fuzzy information and developed a decision-making strategy. Yahya et al. [39] used Dombi Bonferroni mean operator to analyze of medical diagnosis. Kesen and Deli [40] extended the Bonferroni harmonic mean operator to TFM-numbers. Then, they applied the operator to a decision-making problem. Hait et al. [41] conducted a study on Bonferroni mean-type pre-aggregation operators to emphasize the systematized introduction of the Bonferroni mean-type pre-aggregation operators. Radenovic et al. [42] introduced a paper on Bonferroni mean operators given with a square root fuzzy set environment.

As far as we know, no article on Bonferroni aggregation operators on TFM-numbers has been introduced. To fill this gap, this article has been proposed. The method given in the paper provides flexibility to decision-makers due to its parameter-containing structure. This feature of the operator provides a serious space of action for decision-makers. Moreover, the operator is a strong tool to find the interrelationship among aggregated arguments.

The paper consists of seven sections. Section 2 provides definitions for fuzzy sets, fuzzy multi-sets, and TFM-numbers, including some of their properties and operations. Section 3 introduces an aggregation method known as the TFM-Bonferroni arithmetic mean operator, which is designed to aggregate TFM information. This section also investigates its special cases and properties. Section 4 presents an algorithm for multiple attribute decision-making problems. Section 5 applies the proposed TFM-Bonferroni arithmetic mean operator to multi-attribute decision-making problems, providing an example to illustrate the obtained outputs. Section 6 offers an analytical perspective on the proposed approach, including a brief comparative analysis with existing methodologies. To conclude, Section 7 presents our findings and conclusions.

2. Preliminaries

This section provides some essential notions about fuzzy numbers, fuzzy sets, fuzzy-multi sets, and TFM-numbers used in the following sections.

Definition 2.1. [1] Let X be a non-empty set and $\mu_F : X \to [0,1]$. Then, $F = \{\langle x, \mu_F(x)\rangle : x \in X\}$ is called a fuzzy set over X.

Definition 2.2. [18] Let X be a non-empty set. A multi-fuzzy set G on X is defined as:

$$G = \left\{\left\langle x, \mu_G^1(x), \mu_G^2(x), \cdots, \mu_G^i(x), \cdots \right\rangle : x \in X\right\}$$

where $\mu_G^i : X \to [0,1]$, for all $i \in \{1, 2, \cdots, p\}$ and $x \in X$.

Definition 2.3. [43] Let $w_N \in [0,1]$, $x_i, y_i, z_i, t_i \in \mathbb{R}$, and $x_i \le y_i \le z_i \le t_i$. A generalized trapezoidal fuzzy number (GTF-number) $N = \langle (x_i, y_i, z_i, t_i); w_N\rangle$ is a special fuzzy set on the real number set $\mathbb{R}$. Its membership function is given as follows:

$$\mu_N(x) = \begin{cases} (x - x_i)w_N/(y_i - x_i), & x_i \le x < y_i \\ w_N, & y_i \le x \le z_i \\ (t_i - x)w_N/(t_i - z_i), & z_i < x \le t_i \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.4. [25] Let $\eta_N^s \in [0,1]$, $s \in \{1, 2, \cdots, p\}$, and $x_i, y_i, z_i, t_i \in \mathbb{R}$ such that $x_i \le y_i \le z_i \le t_i$. Then, trapezoidal fuzzy multi-number (TFM-number) is shown by $N = \left\langle (x_i, y_i, z_i, t_i); \eta_N^1, \eta_N^2, \cdots, \eta_N^P \right\rangle$

is a special fuzzy multi-set on the real numbers set $\mathbb{R}$ and its membership functions are defined as follows:

$$\mu_N^s(x) = \begin{cases} (x - x_i)\eta_N^s/(y_i - x_i), & x_i \le x < y_i \\ \eta_N^s, & y_i \le x \le z_i \\ (t_i - x)\eta_N^s/(t_i - z_i), & z_i < x \le t_i \\ 0, & \text{otherwise} \end{cases}$$

Throughout this paper, let $\mho(\mathbb{R}^+)$ represent the set of all the TFM-number on $\mathbb{R}^+$, $I_n := \{1, 2, \cdots, n\}$, and $I_m := \{1, 2, \cdots, m\}$.

Definition 2.5. [25] Let $N_1 = \left\langle (x_1, y_1, z_1, t_1); \eta_{N_1}^1, \eta_{N_1}^2, \cdots, \eta_{N_1}^P \right\rangle, N_2 = \left\langle (x_2, y_2, z_2, t_2); \eta_{N_2}^1, \eta_{N_2}^2, \cdots, \eta_{N_2}^P \right\rangle \in \mho(\mathbb{R}^+)$, $\gamma \neq 0$, and $\gamma \in \mathbb{R}$. Then,

i. $N_1 + N_2 = \left\langle (x_1 + x_2, y_1 + y_2, z_1 + z_2, t_1 + t_2); \eta_{N_1}^1 + \eta_{N_2}^1 - \eta_{N_1}^1.\eta_{N_2}^1, \eta_{N_1}^2 + \eta_{N_2}^2 - \eta_{N_1}^2.\eta_{N_2}^2, \cdots, \eta_{N_1}^P + \eta_{N_2}^P - \eta_{N_1}^P.\eta_{N_2}^P \right\rangle$

ii. $$N_1 \times N_2 = \begin{cases} \left\langle (x_1x_2, y_1y_2, z_1z_2, t_1t_2); \eta_{N_1}^1.\eta_{N_2}^1, \eta_{N_1}^2.\eta_{N_2}^2, \cdots, \eta_{N_1}^P.\eta_{N_2}^P \right\rangle & (t_1 > 0, t_2 > 0) \\ \left\langle (x_1t_2, y_1z_2, z_1y_2, t_1x_2); \eta_{N_1}^1.\eta_{N_2}^1, \eta_{N_1}^2.\eta_{N_2}^2, \cdots, \eta_{N_1}^P.\eta_{N_2}^P \right\rangle & (t_1 < 0, t_2 > 0) \\ \left\langle (t_1t_2, z_1z_2, y_1y_2, x_1x_2); \eta_{N_1}^1.\eta_{N_2}^1, \eta_{N_1}^2.\eta_{N_2}^2, \cdots, \eta_{N_1}^P.\eta_{N_2}^P \right\rangle & (t_1 < 0, t_2 < 0) \end{cases}$$

iii. $\gamma N_1 = \left\langle (\gamma x_1, \gamma y_1, \gamma z_1, \gamma t_1); 1 - \left(1 - \eta_{N_1}^1\right)^\gamma, 1 - \left(1 - \eta_{\bar{N}_1}^2\right)^\gamma, \cdots, 1 - \left(1 - \eta_{N_1}^p\right)^\gamma \right\rangle$ $(\gamma > 0)$

iv. $N_1^\gamma = \left\langle (x_1^\gamma, y_1^\gamma, z_1^\gamma, t_1^\gamma); \left(\eta_{N_1}^1\right)^\gamma, \left(\eta_{N_1}^2\right)^\gamma, \cdots, \left(\eta_{N_1}^P\right)^\gamma \right\rangle$ $(\gamma \ge 0)$

Definition 2.6. [40] Let $N_1 = \left\langle (x_1, y_1, z_1, t_1); \eta_{N_1}^1, \eta_{N_1}^2, \cdots, \eta_{N_2}^P \right\rangle, N_2 = \left\langle (x_2, y_2, z_2, t_2); \eta_{N_2}^1, \eta_{N_2}^2, \cdots, \eta_{N_2}^P \right\rangle \in \mho(\mathbb{R}^+)$.

i. If $x_1 < x_2$, $y_1 < y_2$, $z_1 < z_2$, $t_1 < t_2$, and $\eta_{N_1}^1 < \eta_{N_2}^1, \eta_{N_1}^2 < \eta_{N_2}^2, \cdots, \eta_{N_1}^P < \eta_{N_2}^P$, then $N_1 < N_2$.

ii. If $x_1 > x_2$, $y_1 > y_2$, $z_1 > z_2$, $t_1 > t_2$, and $\eta_{N_1}^1 > \eta_{N_2}^1, \eta_{N_1}^2 > \eta_{N_2}^2, \cdots, \eta_{N_1}^P > \eta_{N_2}^P$, then $N_1 > N_2$.

iii. If $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$, $t_1 = t_2$, and $\eta_{N_1}^1 = \eta_{N_2}^1, \eta_{N_1}^2 = \eta_{N_2}^2, \cdots, \eta_{N_1}^P = \eta_{N_2}^P$, then $N_1 = N_2$.

Definition 2.7. [44] Let $N = \left\langle (x_1, y_1, z_1, t_1); \eta_N^1, \eta_N^2, \cdots, \eta_N^P \right\rangle$ be a TFM-number. Value of N denoted by $Val(N)$ based on centroid point denoted by $deff(N_i)$ is computed as follows:

$$Val(N) = \frac{\sum\limits_{i=1}^{P} deff(N_i)}{P}$$

where

$$deff(N_i) = \frac{\int\limits_{x_1}^{y_1} x\frac{(x-x_1)\eta_N^i}{(y_1-x_1)}dx + \int\limits_{y_1}^{z_1} x\eta_N^i dx + \int\limits_{z_1}^{t_1} x\frac{(t_1-x)\eta_N^i}{(t_1-z_1)}dx}{\int\limits_{x_1}^{y_1} \frac{(x-x_1)\eta_N^i}{(y_1-x_1)}dx + \int\limits_{y_1}^{z_1} \eta_N^i dx + \int\limits_{z_1}^{t_1} \frac{(t_1-x)\eta_N^i}{(t_1-z_1)}dx}, i \in \{1, 2, \cdots, P\}$$

Definition 2.8. [40] Let $N = \left\langle (x, y, z, t); \eta_N^1, \eta_N^2, \cdots, \eta_N^P \right\rangle$ be a TFM-number and P show number of η_N^s. Then, score of N denoted $S(N)$ is defined as follows:

$$S(N) = \frac{t^2 + z^2 - x^2 - y^2}{2.P} \sum_{s=1}^{P} \eta_N^s$$

2.1. Critic Method for Determining of Weight of Criteria

CRITIC (Criteria Importance Through Intercriteria Correlation) Method, developed by Diakoulaki et al. [45], is used to determine the relative importance of criteria in a multi-criteria decision-making process. It takes into consideration the correlations between criteria to assign weights to each criterion and helps decision-makers to determine the weight of each criterion by means of values in the decision matrix. The steps of the application of the method are given as follows:

Step 1. Construct the decision matrix according to decision makers' preferences:

$$(D_{ij})_{m\times n} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1x} \\ x_{21} & x_{22} & \cdots & x_{2x} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}$$

Step 2. Find the normalized decision matrix as follows:

$$(\bar{D}_{ij})_{m\times n} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1r} \\ r_{21} & r_{22} & \cdots & r_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{pmatrix}$$

where

$$r_{ij} = \begin{cases} \frac{x_{ij} - \min\limits_{k\in I_n}\{x_{ik}\}}{\max\limits_{k\in I_n}\{x_{ik}\} - \min\limits_{k\in I_n}\{x_{ik}\}}, & \text{for benefit attribute} \\ \frac{\max\limits_{k\in I_n}\{x_{ik}\} - x_{ij}}{\max\limits_{k\in I_n}\{x_{ik}\} - \min\limits_{k\in I_n}\{x_{ik}\}}, & \text{for cost attribute} \end{cases}$$

such that $i \in I_m$ and $j \in I_n$.

Step 3. Construct the relation-coefficient matrix as follows:

$$(RCM)_{n\times n} = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn} \end{pmatrix}$$

where

$$\rho_{jk} = \frac{\sum\limits_{i=1}^{m}(r_{ij} - \bar{r_j}).(r_{ik} - \bar{r_k})}{\sqrt{\sum\limits_{i=1}^{m}(r_{ij} - \bar{r_j})^2 . \sum\limits_{i=1}^{m}(r_{ik} - \bar{r_k})^2}}$$

such that $j, k \in I_n$. Here, $\bar{r_j}$ and $\bar{r_k}$ are arithmetic means of r_{ij} and r_{ik}, respectively.

Step 4. The Critic method aims to get information from contrast and conflicts in the criteria. In this context, combining two concepts and expressing aggregated information in jth criterion, c_j is computed as follows:

$$c_j = \sigma_j \sum_{k=1}^{n}(1 - \rho_{jk})$$

where

$$\sigma_j = \sqrt{\frac{\sum\limits_{i=1}^{m}(r_{ij} - \bar{r_j})^2}{m-1}}$$

such that $j \in I_n$.

Step 5. Compute weights of criteria as follows:

$$w_j = \frac{c_j}{\sum\limits_{k=1}^{n} c_j}$$

Example 2.9. Assume that a committee wants to choose among five alternatives according to four criteria. The committee will give scores ranging between 0 and 1 to each alternative according to the criteria. Since the weights of the criteria are unknown, the final decision cannot be made. To surpass this obstacle, the committee decided to use the CRITIC method as follows:

Step 1. Construct the decision matrix according to decision makers' preferences:

$$(D_{ij})_{5\times 4} = \begin{pmatrix} 0.35 & 0.43 & 0.21 & 0.56 \\ 0.16 & 0.23 & 0.67 & 0.28 \\ 0.65 & 0.68 & 0.91 & 0.56 \\ 0.32 & 0.12 & 0.65 & 0.81 \\ 0.23 & 0.11 & 0.71 & 0.38 \end{pmatrix}$$

Step 2. Find the normalized decision matrix as follows:

$$(\bar{D}_{ij})_{5\times 4} = \begin{pmatrix} 0.612 & 0.561 & 0.000 & 0.471 \\ 1.000 & 0.210 & 0.657 & 1.000 \\ 0.000 & 1.000 & 1.000 & 0.471 \\ 0.673 & 0.017 & 0.628 & 0.000 \\ 0.857 & 0.000 & 0.714 & 0.811 \end{pmatrix}$$

Step 3. Construct the relation-coefficient matrix as follows:

$$(RCM)_{4\times 4} = \begin{pmatrix} 1.000 & -0.859 & -0.343 & 0.430 \\ -0.859 & 1.000 & 0.121 & -0.059 \\ -0.343 & 0.121 & 1.000 & 0.099 \\ 0.430 & -0.059 & 0.099 & 1.000 \end{pmatrix}$$

Step 4. Compute c_j $(j \in I_4)$ as follows:

$$c_j = (1.445, 1.610, 1.144, 0.968)$$

Step 5. Compute weights of criteria w_j $(j \in I_4)$ as follows:

$$w_j = (0.279, 0.311, 0.221, 0.187)$$

3. Bonferroni Arithmetic Mean Operator on TFM-Numbers

This section develops an aggregation method called the TFM-Bonferroni arithmetic mean operator. It is useful for aggregating the TFM-information. This section inspires from [11–13, 34, 46].

Definition 3.1. Let $N_i = \left\langle (x_i, y_i, z_i, t_i); \eta^1_{N_i}, \eta^2_{N_i}, \cdots, \eta^P_{N_i} \right\rangle$ $(i \in I_n)$ be a TFM-numbers' collection and $p, q > 0$. Then, TFM Bonferroni arithmetic mean operator characterized by $TFMBAM^{(p,q)}$ is defined as follows:

$$TFMBAM^{(p,q)}(N_1, N_2, \cdots, N_n) = \left(\frac{1}{n.(n-1)} \bigoplus_{i,j=1, i\neq j}^{n} \left(N_i^p \otimes N_j^q \right) \right)^{\frac{1}{p+q}} \tag{3.1}$$

Theorem 3.2. Let $N_i = \left\langle (x_i, y_i, z_i, t_i); \eta^1_{N_i}, \eta^2_{N_i}, \cdots, \eta^P_{N_i} \right\rangle$ $(i \in I_n)$ be a TFM-numbers' collection and $p, q > 0$. Then, aggregated value computed by $TFMBAM^{(p,q)}$ operator is also a TFM-number and computed as follows:

$$TFMBAM^{(p,q)}(N_1, N_2, \cdots, N_n) = \Bigg\langle \Bigg(\left(\frac{1}{n.(n-1)} \sum_{i,j=1, i\neq j}^{n} \left(x_i^p . x_j^q \right) \right)^{\frac{1}{p+q}}, \left(\frac{1}{n.(n-1)} \sum_{i,j=1, i\neq j}^{n} \left(y_i^p . y_j^q \right) \right)^{\frac{1}{p+q}},$$

$$\left(\frac{1}{n.(n-1)} \sum_{i,j=1, i\neq j}^{n} \left(z_i^p . z_j^q \right) \right)^{\frac{1}{p+q}}, \left(\frac{1}{n.(n-1)} \sum_{i,j=1, i\neq j}^{n} \left(t_i^p . t_j^q \right) \right)^{\frac{1}{p+q}} \Bigg);$$

$$\left(1 - \prod_{i,j=1, i\neq j}^{n} \left(1 - \left(\eta_{N_i}^1\right)^p . \left(\eta_{N_j}^1\right)^q \right)^{\frac{1}{n.(n-1)}} \right)^{\frac{1}{p+q}}, \tag{3.2}$$

$$\left(1 - \prod_{i,j=1, i\neq j}^{n} \left(1 - \left(\eta_{N_i}^2\right)^p . \left(\eta_{N_j}^2\right)^q \right)^{\frac{1}{n.(n-1)}} \right)^{\frac{1}{p+q}}, \cdots,$$

$$\left(1 - \prod_{i,j=1, i\neq j}^{n} \left(1 - \left(\eta_{N_i}^P\right)^p . \left(\eta_{N_j}^P\right)^q \right)^{\frac{1}{n.(n-1)}} \right)^{\frac{1}{p+q}} \Bigg\rangle$$

such that $i, j \in I_n$ and $i \neq j$.

Proof.

Let $N_i = \left\langle (x_i, y_i, z_i, t_i); \eta_{N_i}^1, \eta_{N_i}^2, \cdots, \eta_{N_i}^P \right\rangle$ $(i \in I_n)$ be a TFM-numbers' collection and $p, q > 0$. Firstly, we need to show that:

$$\bigoplus_{i,j=1, i\neq j}^{n} \left(N_i^p \otimes N_j^q \right) = \Bigg\langle \Bigg(\sum_{i,j=1, i\neq j}^{n} \left(x_i^p . x_j^q \right), \sum_{i,j=1, i\neq j}^{n} \left(y_i^p . y_j^q \right),$$

$$\sum_{i,j=1, i\neq j}^{n} \left(z_i^p . z_j^q \right), \sum_{i,j=1, i\neq j}^{n} \left(t_i^p . t_j^q \right) \Bigg);$$

$$\prod_{i,j=1, i\neq j}^{n} \left(\eta_{N_i}^1\right)^p . \left(\eta_{N_j}^1\right)^q, \tag{3.3}$$

$$\prod_{i,j=1, i\neq j}^{n} \left(\eta_{N_i}^2\right)^p . \left(\eta_{N_j}^2\right)^q, \cdots,$$

$$\prod_{i,j=1, i\neq j}^{n} \left(\eta_{N_i}^P\right)^p . \left(\eta_{N_j}^P\right)^q \Bigg\rangle$$

By the operational rules given in Definition 2.5

$$N_i^p \otimes N_j^q = \left\langle \left(x_i^p . x_j^q, y_i^p . y_j^q, z_i^p . z_j^q, t_i^p . t_j^q \right); \left(\eta_{N_i}^1\right)^p . \left(\eta_{N_j}^1\right)^q, \left(\eta_{N_i}^2\right)^p . \left(\eta_{N_j}^2\right)^q, \cdots, \left(\eta_{N_i}^P\right)^p . \left(\eta_{N_j}^P\right)^q \right\rangle$$

If we use mathematical induction on n;

i. When $n = 2$, we obtain:

$$\bigoplus_{i,j=1,i\neq j}^{2}\left(N_i^p \otimes N_j^q\right) = (N_1^p \otimes N_2^q) \oplus (N_2^p \otimes N_1^q)$$

$$= \langle (x_1^p.x_2^q + x_2^p.x_1^q, y_1^p.y_2^q + y_{2.}^p y_1^q, z_1^p.z_2^q + z_2^p.z_1^q, t_1^p.t_2^q + t_2^p.t_1^q);$$

$$\left(\eta_{N_i}^1\right)^p.\left(\eta_{N_j}^1\right)^q \oplus \left(\eta_{N_j}^1\right)^p.\left(\eta_{N_i}^1\right)^q$$

$$\left(\eta_{N_i}^2\right)^p.\left(\eta_{N_j}^2\right)^q \oplus \left(\eta_{N_j}^2\right)^p.\left(\eta_{N_i}^2\right)^q, \cdots,$$

$$\left(\eta_{N_i}^P\right)^p.\left(\eta_{N_j}^P\right)^q \oplus \left(\eta_{N_j}^P\right)^p.\left(\eta_{N_i}^P\right)^q\rangle$$

$$= \left\langle \left(\sum_{i,j=1,i\neq j}^{2}\left(x_i^p.x_j^q\right), \sum_{i,j=1,i\neq j}^{2}\left(y_i^p.y_j^q\right), \sum_{i,j=1,i\neq j}^{2}\left(z_i^p.z_j^q\right), \sum_{i,j=1,i\neq j}^{2}\left(t_i^p.t_j^q\right)\right); 1-\prod_{i,j=1,i\neq j}^{2}\left(1-\left(\eta_{N_i}^1\right)^p.\left(\eta_{N_j}^1\right)^q\right), 1-\prod_{i,j=1,i\neq j}^{2}\left(1-\left(\eta_{N_i}^2\right)^p.\left(\eta_{N_j}^2\right)^q\right), \cdots, 1-\prod_{i,j=1,i\neq j}^{2}\left(1-\left(\eta_{N_i}^P\right)^p.\left(\eta_{N_j}^P\right)^q\right)\right\rangle \tag{3.4}$$

Therefore, when $n = 2$, (3.3) is right.

ii. Suppose when $n = k$, (3.3) is right, i.e

$$\bigoplus_{i,j=1,i\neq j}^{k}\left(N_i^p \otimes N_j^q\right) = \left\langle \left(\sum_{i,j=1,i\neq j}^{k}\left(x_i^p.x_j^q\right), \sum_{i,j=1,i\neq j}^{k}\left(y_i^p.y_j^q\right), \sum_{i,j=1,i\neq j}^{k}\left(z_i^p.z_j^q\right), \sum_{i,j=1,i\neq j}^{k}\left(t_i^p.t_j^q\right)\right); 1-\prod_{i,j=1,i\neq j}^{k}\left(1-\left(\eta_{N_i}^1\right)^p.\left(\eta_{N_j}^1\right)^q\right), 1-\prod_{i,j=1,i\neq j}^{k}\left(1-\left(\eta_{N_i}^2\right)^p.\left(\eta_{N_j}^2\right)^q\right), \cdots, 1-\prod_{i,j=1,i\neq j}^{k}\left(1-\left(\eta_{N_i}^P\right)^p.\left(\eta_{N_j}^P\right)^q\right)\right\rangle \tag{3.5}$$

We need to prove it is true for $n = k + 1$ as well. From (3.1), for $n = k + 1$, the following equality is obtained:

$$\bigoplus_{i,j=1,i\neq j}^{k+1}\left(N_i^p \otimes N_j^q\right) = \bigoplus_{i,j=1,i\neq j}^{k}\left(N_i^p \otimes N_j^q\right) \oplus \bigoplus_{j=1}^{k}\left(N_{k+1}^p \otimes N_j^q\right) \oplus \bigoplus_{i=1}^{k}\left(N_i^p \otimes N_{k+1}^q\right) \tag{3.6}$$

By (3.1), we obtain

$$\bigoplus_{j=1}^{k}\left(N_{k+1}^{p}\otimes N_{j}^{q}\right)=\Bigg\langle\left(\sum_{j=1}^{k}x_{k+1}^{p}.x_{j}^{q},\sum_{j=1}^{k}y_{k+1}^{p}.y_{j}^{q},\sum_{j=1}^{k}z_{k+1}^{p}.z_{j}^{q},\sum_{j=1}^{k}t_{k+1}^{p}.t_{j}^{q}\right); 1-\prod_{j=1}^{k}\left(1-\left(\eta_{N_{k+1}}^{1}\right)^{p}.\left(\eta_{N_{j}}^{1}\right)^{q}\right), 1-\prod_{j=1}^{k}\left(1-\left(\eta_{N_{k+1}}^{2}\right)^{p}.\left(\eta_{N_{j}}^{k}\right)^{q}\right),\cdots, 1-\prod_{j=1}^{k}\left(1-\left(\eta_{N_{k+1}}^{P}\right)^{p}.\left(\eta_{N_{j}}^{P}\right)^{q}\right)\Bigg\rangle \tag{3.7}$$

and

$$\bigoplus_{i=1}^{k}\left(N_{i}^{p}\otimes N_{k+1}^{q}\right)=\Bigg\langle\left(\sum_{i=1}^{k}x_{i.}^{p}x_{k+1}^{q},\sum_{i=1}^{k}y_{i}^{p}.y_{k+1}^{q},\sum_{i=1}^{k}z_{i}^{p}.z_{k+1}^{q},\sum_{i=1}^{k}t_{i}^{p}.t_{k+1}^{q}\right); 1-\prod_{i=1}^{k}\left(1-\left(\eta_{N_{i}}^{1}\right)^{p}.\left(\eta_{N_{k+1}}^{1}\right)^{q}\right), 1-\prod_{i=1}^{k}\left(1-\left(\eta_{N_{i}}^{2}\right)^{p}.\left(\eta_{N_{k+1}}^{k}\right)^{q}\right),\cdots, 1-\prod_{i=1}^{k}\left(1-\left(\eta_{N_{i}}^{P}\right)^{p}.\left(\eta_{N_{k+1}}^{P}\right)^{q}\right)\Bigg\rangle \tag{3.8}$$

Therefore, from (3.5)-(3.8), we obtain:

$$\begin{aligned}\bigoplus_{i,j=1,i\neq j}^{k+1}\left(N_{i}^{p}\otimes N_{j}^{q}\right)=&\Bigg\langle\left(\sum_{i,j=1,i\neq j}^{k}\left(x_{i}^{p}.x_{j}^{q}\right),\sum_{i,j=1,i\neq j}^{k}\left(y_{i}^{p}.y_{j}^{q}\right),\sum_{i,j=1,i\neq j}^{k}\left(z_{i}^{p}.z_{j}^{q}\right),\sum_{i,j=1,i\neq j}^{k}\left(t_{i}^{p}.t_{j}^{q}\right)\right);\\ &1-\prod_{i,j=1,i\neq j}^{k}\left(1-(\eta_{N_{i}}^{1})^{p}.\left(\eta_{N_{j}}^{1}\right)^{q}\right),\\ &1-\prod_{i,j=1,i\neq j}^{k}\left(1-(\eta_{N_{i}}^{2})^{p}.\left(\eta_{N_{j}}^{2}\right)^{q}\right),\cdots,\\ &1-\prod_{i,j=1,i\neq j}^{k}\left(1-\left(\eta_{N_{i}}^{P}\right)^{p}.\left(\eta_{N_{j}}^{P}\right)^{q}\right)\Bigg\rangle\\ &\oplus\Bigg\langle\left(\sum_{j=1}^{k}x_{k+1}^{p}.x_{j}^{q},\sum_{j=1}^{k}y_{k+1}^{p}.y_{j}^{q},\sum_{j=1}^{k}z_{k+1}^{p}.z_{j}^{q},\sum_{j=1}^{k}t_{k+1}^{p}.t_{j}^{q}\right);\\ &1-\prod_{j=1}^{k}\left(1-\left(\eta_{N_{k+1}}^{1}\right)^{p}.\left(\eta_{N_{j}}^{1}\right)^{q}\right),\\ &1-\prod_{j=1}^{k}\left(1-\left(\eta_{N_{k+1}}^{2}\right)^{p}.\left(\eta_{N_{j}}^{k}\right)^{q}\right),\cdots,\\ &1-\prod_{j=1}^{k}\left(1-\left(\eta_{N_{k+1}}^{P}\right)^{p}.\left(\eta_{N_{j}}^{P}\right)^{q}\right)\Bigg\rangle\end{aligned} \tag{3.9}$$

$$\oplus\left\langle\left(\sum_{i=1}^{k} x_i^p.x_{k+1}^q, \sum_{i=1}^{k} y_i^p.y_{k+1}^q, \sum_{i=1}^{k} z_i^p.z_{k+1}^q, \sum_{i=1}^{k} t_i^p.t_{k+1}^q\right);\right.$$

$$1-\prod_{i=1}^{k}\left(1-\left(\eta_{N_i}^1\right)^p.\left(\eta_{N_{k+1}}^1\right)^q\right),$$

$$1-\prod_{i=1}^{k}\left(1-\left(\eta_{N_i}^2\right)^p.\left(\eta_{N_{k+1}}^k\right)^q\right),\cdots,$$

$$\left.1-\prod_{i=1}^{k}\left(1-\left(\eta_{N_i}^P\right)^p.\left(\eta_{N_{k+1}}^P\right)^q\right)\right\rangle$$

$$=\left\langle\left(\sum_{i,j=1,i\neq j}^{k+1} x_i^p.x_j^q, \sum_{i,j=1,i\neq j}^{k+1} y_i^p.y_j^q, \sum_{i,j=1,i\neq j}^{k+1} z_i^p.z_j^q, \sum_{i,j=1,i\neq j}^{k+1} t_i^p.t_j^q\right);\right.$$

$$1-\prod_{i,j=1,i\neq j}^{k+1}\left(1-\left(\eta_{N_i}^1\right)^p.\left(\eta_{N_j}^1\right)^q\right),$$

$$1-\prod_{i,j=1,i\neq j}^{k+1}\left(1-\left(\eta_{N_i}^2\right)^p.\left(\eta_{N_j}^2\right)^q\right),\cdots,$$

$$\left.1-\prod_{i,j=1,i\neq j}^{k+1}\left(1-\left(\eta_{N_i}^P\right)^p.\left(\eta_{N_j}^P\right)^q\right)\right\rangle$$

Thus, when $n = k + 1$, (3.9) is right. Therefore, (3.3) is right, for all n, and the proof is done. □

Theorem 3.3. Let $N_i = \left\langle(x_i, y_i, z_i, t_i); \eta_{N_i}^1, \eta_{N_i}^2, \cdots, \eta_{N_i}^P\right\rangle$ $(i \in I_n)$ be a TFM-numbers' collection and $p, q > 0$. Then,

$$TFMBAM^{(p,q)}(N_1, N_2, \cdots, N_n) = \left(\frac{1}{n.(n-1)}\bigoplus_{i,j=1,i<j}^{n}\left(N_i^p\otimes N_j^q\right)\oplus\left(N_j^p\otimes N_i^q\right)\right)^{\frac{1}{p+q}}$$

$$=\left\langle\left(\frac{1}{n.(n-1)}\sum_{i,j=1,i<j}^{n}(x_i)^p.(x_j)^q+(x_j)^p.(x_i)^q\right)^{\frac{1}{p+q}},\right.$$

$$\left(\frac{1}{n.(n-1)}\sum_{i,j=1,i<j}^{n}(y_i)^p.(y_j)^q+(y_j)^p.(y_i)^q\right)^{\frac{1}{p+q}},$$

$$\left(\frac{1}{n.(n-1)}\sum_{i,j=1,i<j}^{n}(z_i)^p.(z_j)^q+(z_j)^p.(z_i)^q\right)^{\frac{1}{p+q}},$$

$$\left(\frac{1}{n.(n-1)}\sum_{i,j=1,i<j}^{n}(t_i)^p.(t_j)^q+(t_j)^p.(t_i)^q\right)^{\frac{1}{p+q}};$$

$$\left(1-\prod_{i,j=1,i<j}^{n}\left(\left(1-\left(\eta_{N_i}^1\right)^p.\left(\eta_{N_j}^1\right)^q\right).\left(1-\left(\eta_{N_j}^1\right)^p.\left(\eta_{N_i}^1\right)^q\right)\right)^{\frac{1}{n.(n-1)}}\right)^{\frac{1}{p+q}},$$

$$\left(1-\prod_{i,j=1,i<j}^{n}\left(\left(1-\left(\eta_{N_i}^2\right)^p.\left(\eta_{N_j}^2\right)^q\right).\left(1-\left(\eta_{N_j}^2\right)^p.\left(\eta_{N_i}^2\right)^q\right)\right)^{\frac{1}{n.(n-1)}}\right)^{\frac{1}{p+q}},\cdots$$

$$\left.\left(1-\prod_{i,j=1,i<j}^{n}\left(\left(1-\left(\eta_{N_i}^P\right)^p.\left(\eta_{N_j}^P\right)^q\right).\left(1-\left(\eta_{N_j}^P\right)^p.\left(\eta_{N_i}^P\right)^q\right)\right)^{\frac{1}{n.(n-1)}}\right)^{\frac{1}{p+q}}\right\rangle$$

Lemma 3.4. Let $N_i = \left\langle (x_i, y_i, z_i, t_i); \eta^1_{N_i}, \eta^2_{N_i}, \cdots, \eta^P_{N_i} \right\rangle$ $(i \in I_n)$ be a TFM-numbers' collection and $p, q > 0$. If we interchange parameters p and q, we obtain another property called idempotent commutativity. It is given as follows:

Because $\left(N_i^p \otimes N_j^q\right) \oplus \left(N_j^p \otimes N_i^q\right) = \left(N_i^q \otimes N_j^p\right) \oplus \left(N_j^q \otimes N_i^p\right)$ $(i, j \in I_n$ and $i < j)$, by interchanging parameters p and q, we get:

$$
\begin{aligned}
TFMBAM^{(p,q)}(N_1, N_2, \cdots, N_n) &= \left(\frac{1}{n.(n-1)} \bigoplus_{i,j=1, i<j}^{n} \left(N_i^p \otimes N_j^q\right) \oplus \left(N_j^p \otimes N_i^q\right) \right)^{\frac{1}{p+q}} \\
&= \Bigg\langle \left(\frac{1}{n.(n-1)} \sum_{i,j=1,i<j}^{n} (x_i)^p.(x_j)^q \oplus (x_j)^p.(x_i)^q \right)^{\frac{1}{p+q}}, \\
&\quad \left(\frac{1}{n.(n-1)} \sum_{i,j=1,i<j}^{n} (y_i)^p.(y_j)^q \oplus (y_j)^p.(y_i)^q \right)^{\frac{1}{p+q}}, \\
&\quad \left(\frac{1}{n.(n-1)} \sum_{i,j=1,i<j}^{n} (z_i)^p.(z_j)^q \oplus (z_j)^p.(z_i)^q \right)^{\frac{1}{p+q}}, \\
&\quad \left(\frac{1}{n.(n-1)} \sum_{i,j=1,i<j}^{n} (t_i)^p.(t_j)^q \oplus (t_j)^p.(t_i)^q \right)^{\frac{1}{p+q}}; \\
&\quad \left(1 - \prod_{i,j=1,i<j}^{n} \left(\left(1 - \left(\eta^1_{N_i}\right)^p . \left(\eta^1_{N_j}\right)^q\right) . \left(1 - \left(\eta^1_{N_j}\right)^p . \left(\eta^1_{N_i}\right)^q\right) \right)^{\frac{1}{n.(n-1)}} \right)^{\frac{1}{p+q}}, \\
&\quad \left(1 - \prod_{i,j=1,i<j}^{n} \left(\left(1 - \left(\eta^2_{N_i}\right)^p . \left(\eta^2_{N_j}\right)^q\right) . \left(1 - \left(\eta^2_{N_j}\right)^p . \left(\eta^2_{N_i}\right)^q\right) \right)^{\frac{1}{n.(n-1)}} \right)^{\frac{1}{p+q}}, \cdots \\
&\quad \left(1 - \prod_{i,j=1,i<j}^{n} \left(\left(1 - \left(\eta^P_{N_i}\right)^p . \left(\eta^P_{N_j}\right)^q\right) . \left(1 - \left(\eta^P_{N_j}\right)^p . \left(\eta^P_{N_i}\right)^q\right) \right)^{\frac{1}{n.(n-1)}} \right)^{\frac{1}{p+q}} \Bigg\rangle \\
&= \Bigg\langle \left(\frac{1}{n.(n-1)} \sum_{i,j=1,i<j}^{n} (x_i)^q.(x_j)^p + (x_j)^q.(x_i)^p \right)^{\frac{1}{q+p}}, \\
&\quad \left(\frac{1}{n.(n-1)} \sum_{i,j=1,i<j}^{n} (y_i)^q.(y_j)^p + (y_j)^q.(y_i)^p \right)^{\frac{1}{q+p}}, \\
&\quad \left(\frac{1}{n.(n-1)} \sum_{i,j=1,i<j}^{n} (z_i)^q.(z_j)^p + (z_j)^q.(z_i)^p \right)^{\frac{1}{q+p}}, \\
&\quad \left(\frac{1}{n.(n-1)} \sum_{i,j=1,i<j}^{n} (t_i)^q.(t_j)^p + (t_j)^q.(t_i)^p \right)^{\frac{1}{q+p}}; \\
&\quad \left(1 - \prod_{i,j=1,i<j}^{n} \left(\left(1 - \left(\eta^1_{N_i}\right)^q . \left(\eta^1_{N_j}\right)^p\right) . \left(1 - \left(\eta^1_{N_j}\right)^q . \left(\eta^1_{N_i}\right)^p\right) \right)^{\frac{1}{n.(n-1)}} \right)^{\frac{1}{q+p}}, \\
&\quad \left(1 - \prod_{i,j=1,i<j}^{n} \left(\left(1 - \left(\eta^2_{N_i}\right)^q . \left(\eta^2_{N_j}\right)^p\right) . \left(1 - \left(\eta^2_{N_j}\right)^q . \left(\eta^2_{N_i}\right)^p\right) \right)^{\frac{1}{n.(n-1)}} \right)^{\frac{1}{q+p}}, \cdots \\
&\quad \left(1 - \prod_{i,j=1,i<j}^{n} \left(\left(1 - \left(\eta^P_{N_i}\right)^q . \left(\eta^P_{N_j}\right)^p\right) . \left(1 - \left(\eta^P_{N_j}\right)^q . \left(\eta^P_{N_i}\right)^p\right) \right)^{\frac{1}{n.(n-1)}} \right)^{\frac{1}{q+p}} \Bigg\rangle \\
&= TFMBAM^{(q,p)}(N_1, N_2, \cdots, N_n)
\end{aligned}
$$

Example 3.5. Suppose we have three TFM-numbers as follows:

$$N_1 = \langle(0.1, 0.4, 0.5, 0.6); 0.5, 0.3, 0.4, 0.2\rangle$$
$$N_2 = \langle(0.1, 0.2, 0.5, 0.8); 0.9, 0.6, 0.3, 0.5\rangle$$
$$N_3 = \langle(0.2, 0.3, 0.3, 0.4); 0.7, 0.8, 0.3, 0.4\rangle$$

Then, considering the operations of TFM-numbers given in the Definition 2.5 and (3.2), for $p, q = 1$, we have:

$$N_1^1 \otimes N_2^1 = \langle(0.01, 0.08, 0.25, 0.48); 0.45, 0.18, 0.12, 0.1\rangle$$
$$N_2^1 \otimes N_1^1 = \langle(0.01, 0.08, 0.25, 0.48); 0.45, 0.18, 0.12, 0.1\rangle$$
$$N_1^1 \otimes N_3^1 = \langle(0.02, 0.12, 0.15, 0.24); 0.35, 0.24, 0.12, 0.08\rangle$$
$$N_3^1 \otimes N_1^1 = \langle(0.02, 0.12, 0.15, 0.24); 0.35, 0.24, 0.12, 0.08\rangle$$
$$N_2^1 \otimes N_3^1 = \langle(0.02, 0.06, 0.15, 0.32); 0.63, 0.48, 0.09, 0.20\rangle$$
$$N_3^1 \otimes N_2^1 = \langle(0.02, 0.06, 0.15, 0.32); 0.63, 0.48, 0.09, 0.20\rangle$$

and then we obtain:

$$TFMBAM^{(1,1)}(N_1, N_2, N_3) = \langle(0.129, 0.294, 0.428, 0.588); 0.700, 0.559, 0.331, 0.358\rangle$$
$$TFMBAM^{(2,2)}(N_1, N_2, N_3) = \langle(0.131, 0.300, 0.435, 0.600); 0.707, 0.577, 0.333, 0.370\rangle$$
$$TFMBAM^{(1,3)}(N_1, N_2, N_3) = \langle(0.138, 0.307, 0.441, 0.615); 0.7324, 0.599, 0.335, 0.382\rangle$$
$$TFMBAM^{(3,1)}(N_1, N_2, N_3) = \langle(0.138, 0.307, 0.441, 0.615); 0.7324, 0.599, 0.335, 0.382\rangle$$
$$TFMBAM^{(10,2)}(N_1, N_2, N_3) = \langle(0.162, 0.341, 0.468, 0.682); 0.779, 0.674, 0.352, 0.427\rangle$$

Proposition 3.6. Let $N_i = \left\langle(x_i, y_i, z_i, t_i); \eta^1_{N_i}, \eta^2_{N_i}, \cdots, \eta^P_{N_i}\right\rangle$ and $M_i = \left\langle(k_i, l_i, m_i, n_i); \eta^1_{M_i}, \eta^2_{M_i}, \cdots, \eta^P_{M_i}\right\rangle$ $(i \in I_n)$ be two collections of TFM-numbers.

i. ***(Monotonicity)*** Based on Definition 2.6, if $x_i \leq k_i, y_i \leq l_i, z_i \leq m_i, t_i \leq n_i$ $(i \in I_n)$ and $\eta^1_{N_i} \leq \eta^1_{M_i}$, $\eta^2_{N_i} \leq \eta^2_{M_i}, \cdots, \eta^p_{N_i} \leq \eta^p_{M_i}$, then

$$TFMBAM^{(p,q)}(N_1, N_2, \cdots, N_n) \leq TFMBAM^{(p,q)}(M_1, M_2, \cdots, M_n)$$

ii. ***(Commutativity)*** If $(\dot{N}_1, \dot{N}_2, \cdots, \dot{N}_n)$ any permutation of $(N_1, N_2, \cdots, N_n)$, then

$$\begin{aligned} TFMBAM^{(p,q)}(N_1, N_2, \cdots, N_n) &= \left(\frac{1}{n.(n-1)} \bigoplus_{i,j=1, i\neq j}^{n} \left(N_i^p \otimes N_j^q\right)\right)^{\frac{1}{p+q}} \\ &= \left(\frac{1}{n.(n-1)} \bigoplus_{i,j=1, i\neq j}^{n} \left(\dot{N}_i^{\,p} \otimes \dot{N}_j^{\,q}\right)\right)^{\frac{1}{p+q}} \\ &= TFMBAM^{(p,q)}(\dot{N}_1, \dot{N}_2, \cdots, \dot{N}_n) \end{aligned}$$

iii. ***(Boundedness)***

$$N^- \leq TFMBAM^{(p,q)}(N_1, N_2, \cdots, N_n) \leq N^+$$

where

$$N^+ = \left\langle\left(\max_{i\in I_n}\{x_i\}, \max_{i\in I_n}\{y_i\}, \max_{i\in I_n}\{z_i\}, \max_{i\in I_n}\{t_i\}\right); \max_{i\in I_n}\left\{\eta^1_{N_i}\right\}, \max_{i\in I_n}\left\{\eta^2_{N_i}\right\}, \cdots, \max_{i\in I_n}\left\{\eta^P_{N_i}\right\}\right\rangle$$

and

$$N^{-}=\left\langle\left(\min_{i\in I_n}\{x_i\},\min_{i\in I_n}\{y_i\},\min_{i\in I_n}\{z_i\},\min_{i\in I_n}\{t_i\}\right);\min_{i\in I_n}\left\{\eta^1_{N_i}\right\},\min_{i\in I_n}\left\{\eta^2_{N_i}\right\},\cdots,\min_{i\in I_n}\left\{\eta^P_{N_i}\right\}\right\rangle$$

iv. ***(Idempotent Commutativity)*** If we interchange parameters p and q, we have:

$$TFMBAM^{(p,q)}(N_1,N_2,\cdots,N_n)=\left(\frac{1}{n.(n-1)}\bigoplus_{i,j=1,i<j}^{n}\left(N_i^p\otimes N_j^q\right)\oplus\left(N_j^p\otimes N_i^q\right)\right)^{\frac{1}{p+q}}$$

$$=\left(\frac{1}{n.(n-1)}\bigoplus_{i,j=1,i<j}^{n}\left(N_i^q\otimes N_j^p\right)\oplus\left(N_j^q\otimes N_i^p\right)\right)^{\frac{1}{q+p}}$$

$$=TFMBAM^{(q,p)}(N_1,N_2,\cdots,N_n)$$

If we change values of p and q, special cases of the $TFMBAM^{(p,q)}$ taken as follows:

Case 1. If $q=0$, then operator $TFMBAM^{(p,q)}$ converts into a TFM mean operator:

$$TFMBAM^{(p,0)}(N_1,N_2,\cdots,N_n)=\left(\frac{1}{n}\bigoplus_{i=1}^{n}N_i^p\left(\frac{1}{n-1}\bigoplus_{j=1,j\neq i}^{n}N_j^0\right)\right)^{\frac{1}{p+0}}$$

$$=\left(\frac{1}{n}\bigoplus_{i=1}^{n}(N_i^p)\right)^{\frac{1}{p}}$$

$$=\left\langle\left(\left(\frac{1}{n}\sum_{i=1}^{n}(x_i^p)\right)^{\frac{1}{p}},\left(\frac{1}{n}\sum_{i=1}^{n}(y_i^p)\right)^{\frac{1}{p}},\left(\frac{1}{n}\sum_{i=1}^{n}(z_i^p)\right)^{\frac{1}{p}},\left(\frac{1}{n}\sum_{i=1}^{n}(t_i^p)\right)^{\frac{1}{p}}\right);\right.$$

$$\left(1-\prod_{i=1}^{n}\left(1-\left(\eta^1_{N_i}\right)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}},\left(1-\prod_{i=1}^{n}\left(1-\left(\eta^2_{N_i}\right)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}},\cdots,$$

$$\left.\left(1-\prod_{i=1}^{n}\left(1-\left(\eta^P_{N_i}\right)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}\right\rangle$$

Case 2. If $p=2$ and $q=0$, then operator $TFMBAM^{(p,q)}$ converts into a TFM square mean operator:

$$TFMBAM^{(2,0)}(N_1,N_2,\cdots,N_n)=\left(\frac{1}{n}\bigoplus_{i=1}^{n}\left(N_i^2\right)\right)^{\frac{1}{2}}$$

$$=\left\langle\left(\left(\frac{1}{n}\sum_{i=1}^{n}\left(x_i^2\right)\right)^{\frac{1}{2}},\left(\frac{1}{n}\sum_{i=1}^{n}\left(y_i^2\right)\right)^{\frac{1}{2}},\left(\frac{1}{n}\sum_{i=1}^{n}\left(z_i^2\right)\right)^{\frac{1}{2}},\left(\frac{1}{n}\sum_{i=1}^{n}\left(t_i^2\right)\right)^{\frac{1}{2}}\right);\right.$$

$$\left(1-\prod_{i=1}^{n}\left(1-\left(\eta^1_{N_i}\right)^2\right)^{\frac{1}{n}}\right)^{\frac{1}{2}},\left(1-\prod_{i=1}^{n}\left(1-\left(\eta^2_{N_i}\right)^2\right)^{\frac{1}{n}}\right)^{\frac{1}{2}},\cdots,$$

$$\left.\left(1-\prod_{i=1}^{n}\left(1-\left(\eta^P_{N_i}\right)^2\right)^{\frac{1}{n}}\right)^{\frac{1}{2}}\right\rangle$$

Case 3. If $p = 1$ and $q = 0$, then operator $TFMBAM^{(p,q)}$ converts into a TFM arithmetic operator:

$$\begin{aligned}TFMBAM^{(1,0)}(N_1, N_2, \cdots, N_n) = & \ \tfrac{1}{n}\bigoplus_{i=1}^{n}(N_i)\\ = & \ \Bigg\langle\left(\tfrac{1}{n}\sum_{i=1}^{n}x_i, \tfrac{1}{n}\sum_{i=1}^{n}y_i, \tfrac{1}{n}\sum_{i=1}^{n}z_i, \tfrac{1}{n}\sum_{i=1}^{n}t_i\right);\\ & \left(1-\prod_{i=1}^{n}\left(1-\left(\eta^1_{N_i}\right)\right)^{\frac{1}{n}}\right), \left(1-\prod_{i=1}^{n}\left(1-\left(\eta^2_{N_i}\right)\right)^{\frac{1}{n}}\right), \cdots,\\ & \left(1-\prod_{i=1}^{n}\left(1-\left(\eta^P_{N_i}\right)\right)^{\frac{1}{n}}\right)\Bigg\rangle\end{aligned}$$

Case 4. If $p = q = 1$, then operator $TFMBAM^{(p,q)}$ converts into a TFM interrelated square mean operator:

$$\begin{aligned}TFMBAM^{(1,1)}(N_1, N_2, \cdots, N_n) = & \left(\frac{1}{n.(n-1)}\bigoplus_{i,j=1, i\neq j}^{n}(N_i \otimes N_j)\right)^{\frac{1}{2}}\\ = & \Bigg\langle\Bigg(\left(\frac{1}{n.(n-1)}\sum_{i,j=1,i\neq j}^{n}(x_i.x_j)\right)^{\frac{1}{2}}, \left(\frac{1}{n.(n-1)}\sum_{i,j=1,i\neq j}^{n}(y_i.y_j)\right)^{\frac{1}{2}},\\ & \left(\frac{1}{n.(n-1)}\sum_{i,j=1,i\neq j}^{n}(z_i.z_j)\right)^{\frac{1}{2}}, \left(\frac{1}{n.(n-1)}\sum_{i,j=1,i\neq j}^{n}(t_i.t_j)\right)^{\frac{1}{2}}\Bigg);\\ & \left(1-\prod_{i,j=1,i\neq j}^{n}\left(1-\left(\eta^1_{N_i}\right).\left(\eta^1_{N_j}\right)\right)^{\frac{1}{n.(n-1)}}\right)^{\frac{1}{2}},\\ & \left(1-\prod_{i,j=1,i\neq j}^{n}\left(1-\left(\eta^2_{N_i}\right).\left(\eta^2_{N_j}\right)\right)^{\frac{1}{n.(n-1)}}\right)^{\frac{1}{2}}, \cdots\\ & \left(1-\prod_{i,j=1,i\neq j}^{n}\left(1-\left(\eta^P_{N_i}\right).\left(\eta^P_{N_j}\right)\right)^{\frac{1}{n.(n-1)}}\right)^{\frac{1}{2}}\Bigg\rangle\end{aligned}$$

Definition 3.7. Let $N_i = \left\langle(x_j, y_j, z_j, t_j); \eta^1_{N_i}, \eta^2_{N_i}, \cdots, \eta^P_{N_i}\right\rangle$ $(i \in I_n)$ be a TFM-numbers' collection, $p, q > 0$, and N_i's weight vector is $w = (w_1, w_2, \cdots, w_n)^T$. Here, w_i is N_i's importance degree, satisfying $w_i \in [0, 1]$ $(i \in I_n)$ such that $\sum\limits_{i=1}^{n} w_i = 1$. Then, the weighted TFM Bonferroni arithmetic mean operator denoted by $TFMBAM^{(p,q)}_w$ $(N_1, N_2, \cdots, N_n)$ is defined as follows:

$$TFMBAM^{(p,q)}_w(N_1, N_2, \cdots, N_n) = \left(\frac{1}{n.(n-1)}\bigoplus_{i,j=1,i\neq j}^{n}\left(w_i.N_i^p \otimes w_j.N_j^q\right)\right)^{\frac{1}{p+q}} \quad (3.10)$$

Theorem 3.8. Let $N_i = \left\langle(x_i, y_i, z_i, t_i); \eta^1_{N_i}, \eta^2_{N_i}, \cdots, \eta^P_{N_i}\right\rangle$ $(i \in I_n)$ be a TFM-numbers' collection, $p, q > 0$, and N_i's weight vector is $w = (w_1, w_2, \cdots, w_n)^T$. Here, w_i is N_i's importance degree, satisfying $w_i \in [0, 1]$, $i \in I_n$ such that $\sum\limits_{i=1}^{n} w_i = 1$. Then, aggregated value by using the $TFMBAM^{(p,q)}_w$ an operator is a TFM-number and computed as follows:

$$TFMBAM_w^{(p,q)}(N_1,N_2,\cdots,N_n)=\Bigg\langle\left(\left(\frac{1}{n.(n-1)}\sum_{i,j=1,i\neq j}^{n}\left(w_i.x_i^p w_j.x_j^q\right)\right)^{\frac{1}{p+q}},\left(\frac{1}{n.(n-1)}\sum_{i,j=1,i\neq j}^{n}\left(w_i.y_i^p w_j.y_j^q\right)\right)^{\frac{1}{p+q}},\left(\frac{1}{n.(n-1)}\sum_{i,j=1,i\neq j}^{n}\left(w_i.z_i^p w_j.z_j^q\right)\right)^{\frac{1}{p+q}},\left(\frac{1}{n.(n-1)}\sum_{i,j=1,i\neq j}^{n}\left(w_i.t_i^p w_j.t_j^q\right)\right)^{\frac{1}{p+q}}\right); \tag{3.11}$$
$$\left(1-\prod_{i,j=1,i\neq j}^{n}\left[1-\left(1-\left(1-\left(\eta_{N_i}^1\right)^p\right)^{w_i}\right).\left(1-\left(1-\left(\eta_{N_j}^1\right)^q\right)^{w_j}\right)\right]^{\frac{1}{n.(n-1)}}\right)^{\frac{1}{p+q}},$$
$$\left(1-\prod_{i,j=1,i\neq j}^{n}\left[1-\left(1-\left(1-\left(\eta_{N_i}^2\right)^p\right)^{w_i}\right).\left(1-\left(1-\left(\eta_{N_j}^2\right)^q\right)^{w_j}\right)\right]^{\frac{1}{n.(n-1)}}\right)^{\frac{1}{p+q}},\cdots,$$
$$\left(1-\prod_{i,j=1,i\neq j}^{n}\left[1-\left(1-\left(1-\left(\eta_{N_i}^P\right)^p\right)^{w_i}\right).\left(1-\left(1-\left(\eta_{N_j}^P\right)^q\right)^{w_j}\right)\right]^{\frac{1}{n.(n-1)}}\right)^{\frac{1}{p+q}}\Bigg\rangle$$

4. Proposed Multi-Criteria Decision-Making Algorithm

By considering Bonferroni arithmetic mean operator of generalized hesitant TFM-numbers proposed by Deli [12], we developed an algorithm for multi attribute making problems.

Definition 4.1. [25] Let $Z = \{z_i | i \in I_m\}$ be alternatives' set, $C = \{c_j | j \in I_n\}$ set of criteria, and $w = (w_1, w_2, \cdots, w_n)$ be weights' set. Here, w_j $(j \in I_n)$ is the weight of criteria c_j such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$. Then, the characteristic of the alternative z_i on criteria c_j is represented by the TFM-number $\overline{N}_{ij}$. All the possible values that the alternative z_i $(i \in I_m)$ satisfies the criteria c_j $(j \in I_n)$ represented in the following TFM decision matrix $(\overline{N}_{ij})_{m\times n}$;

$$(\overline{N}_{ij})_{m\times n}=\begin{pmatrix}\overline{N}_{11} & \overline{N}_{12} & \cdots & \overline{N}_{1n}\\ \overline{N}_{21} & \overline{N}_{22} & \cdots & \overline{N}_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ \overline{N}_{m1} & \overline{N}_{m2} & \cdots & \overline{N}_{mn}\end{pmatrix}$$

Note 4.2. In next example, Table 1 [40] as follows will be used as linguistic terms table.

Table 1. TFM-numbers of linguistic terms

Linguistic terms	TFM-numbers
Definitely-low(DL)	$\langle(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4\rangle$
Too-Low(TL)	$\langle(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1\rangle$
Very-Low(VL)	$\langle(0.10, 0.15, 0.15, 0.20); 0.2, 0.4, 0.5, 0.3\rangle$
Low(L)	$\langle(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1\rangle$
Fairly-low(FL)	$\langle(0.15, 0.20, 0.25, 0.30); 0.4, 0.6, 0.2, 0.5\rangle$
Medium(M)	$\langle(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8\rangle$
Fairly-high(FH)	$\langle(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4\rangle$
High(H)	$\langle(0.40, 0.45, 0.50, 0.55); 0.8, 0.9, 0.3, 0.6\rangle$
Very-High(VH)	$\langle(0.45, 0.55, 0.65, 0.75); 0.7, 0.8, 0.6, 0.3\rangle$
Too-High(TH)	$\langle(0.50, 0.60, 0.70, 0.80); 0.1, 0.7, 0.8, 0.9\rangle$
Definitely-high(DH)	$\langle(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2\rangle$

The Proposed Algorithm Steps:

Step 1. Present TFM decision matrix showing results of evaluation of the expert based on the characteristics of the alternative z_i $(i \in I_m)$ satisfying the attribute c_j $(j \in I_n)$ based on Table 1 as follows:

$$(\overline{N}_{ij})_{m\times n} = \begin{pmatrix} \overline{N}_{11} & \overline{N}_{12} & \cdots & \overline{N}_{1n} \\ \overline{N}_{21} & \overline{N}_{22} & \cdots & \overline{N}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{N}_{m1} & \overline{N}_{m2} & \cdots & \overline{N}_{mn} \end{pmatrix}$$

Step 2. Find the weights of the criteria as follows:

Substep 1. Construct a matrix consisting of real numbers by the value of TFM-numbers obtained from defuzzification of each element of the decision matrix $(\overline{N}_{ij})_{m\times n}$ by using Definition 2.7 as follows:

$$(D_{ij})_{m\times n} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1x} \\ x_{21} & x_{22} & \cdots & x_{2x} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}$$

Substep 2. Find the weights of criteria according to values in $(D_{ij})_{m\times n}$ matrix by using critic method given in Subsection 2.1:

$$w = (w_1, w_2, \cdots, w_n)$$

Step 3. For all i $(i \in I_m)$, find the aggregation values by (3.11) to get the ultimate performance value corresponding to the alternative z_i $(i \in I_m)$ as follows:

$$\overline{N}_i = TFMBAM_w^{(p,q)}(\overline{N}_{i1}, \overline{N}_{i2}, \cdots, \overline{N}_{in}), (i \in I_m)$$

Step 4. Calculate the score value whose formula is given in Definition 2.8 for each $(\overline{N}_i)$ $(i \in I_m)$ and rank all the alternatives.

5. Illustrative Example of the Proposed Algorithm for Crafting the Ideal Student Dormitory

This section presents an example to demonstrate the efficiency of the method.

Example 5.1. Assume that the board of directors of a college aims to build a dorm for students of the college. The board doesn't know how kind of dorm should be built. After examining all the student dorms in the city, they will select one from the list of the five dorms ($Z = \{z_i | i \in I_5\}$) that best matches their preferences and construct one similar to it. Furthermore, the board has the following four attributes to be regarded:

i. Visuality (c_1)

ii. Green surrounding (c_2)

iii. Earthquake resistance (c_3)

iv. Building cost (c_4)

Step 1. The board evaluated alternatives and attributes. Results are presented in the TFM decision matrix $(\overline{N}_{ij})_{5\times 4}$ as follows:

$$(\overline{N}_{ij})_{5\times4} = \begin{pmatrix} \langle(0.10,0.15,0.15,0.20);0.2,0.4,0.5,0.3\rangle & \langle(0.15,0.20,0.25,0.30);0.4,0.6,0.2,0.5\rangle & \langle(0.30,0.35,0.40,0.45);0.6,0.1,0.8,0.4\rangle & \langle(0.45,0.55,0.65,0.75);0.7,0.8,0.6,0.3\rangle \\ \langle(0.05,0.10,0.15,0.20);0.2,0.3,0.4,0.1\rangle & \langle(0.10,0.20,0.20,0.30);0.3,0.4,0.8,0.1\rangle & \langle(0.25,0.30,0.35,0.40);0.4,0.5,0.6,0.8\rangle & \langle(0.40,0.45,0.50,0.55);0.8,0.9,0.3,0.6\rangle \\ \langle(0.70,0.80,0.90,1.00);0.7,0.8,0.9,0.2\rangle & \langle(0.50,0.60,0.70,0.80);0.1,0.7,0.8,0.9\rangle & \langle(0.10,0.20,0.20,0.30);0.3,0.4,0.8,0.1\rangle & \langle(0.01,0.05,0.10,0.15);0.1,0.2,0.3,0.4\rangle \\ \langle(0.25,0.30,0.35,0.40);0.4,0.5,0.6,0.8\rangle & \langle(0.05,0.10,0.15,0.20);0.2,0.3,0.4,0.1\rangle & \langle(0.50,0.60,0.70,0.80);0.1,0.7,0.8,0.9\rangle & \langle(0.25,0.30,0.35,0.40);0.4,0.5,0.6,0.8\rangle \\ \langle(0.50,0.60,0.70,0.80);0.1,0.7,0.8,0.9\rangle & \langle(0.70,0.80,0.90,1.00);0.7,0.8,0.9,0.2\rangle & \langle(0.40,0.45,0.50,0.55);0.8,0.9,0.3,0.6\rangle & \langle(0.10,0.20,0.20,0.30);0.3,0.4,0.8,0.1\rangle \end{pmatrix}$$

Step 2.

Substep 1. Construct a matrix consisting of real numbers by defuzzification of each element of the decision matrix $(\overline{N}_{ij})_{m\times n}$ by using Definition 2.7 as follows:

$$(D_{ij})_{m\times n} = \begin{pmatrix} 0.1500 & 0.2250 & 0.3750 & 0.6000 \\ 0.1250 & 0.2000 & 0.3250 & 0.4750 \\ 0.8500 & 0.6500 & 0.2000 & 0.0779 \\ 0.3250 & 0.1250 & 0.6500 & 0.3250 \\ 0.6500 & 0.8500 & 0.4750 & 0.2000 \end{pmatrix}$$

Substep 2. Find the weights of criteria according to criteria in the decision-making problem and values in matrix $(D_{ij})_{m\times n}$ by using the critic method given in Subsection 2.1:

$$w = (0.328, 0.250, 0.197, 0.223)$$

Step 3. For all i $(i \in I_5)$, we find the aggregation values according to (3.11), for $p = 1$ and $q = 1$, to access the ultimate performance of the alternatives N_i $(i \in I_5)$ as follows:

$$\begin{aligned}
\overline{N}_1 = {} & TFMBAM_w^{(1,1)}\left(\overline{N}_{11}, \overline{N}_{12}, \overline{N}_{13}, \overline{N}_{14}\right) \\
= {} & \left(\frac{1}{4.(4-1)} \bigoplus_{s,t=1,s\neq t}^{4} \left(w_s.\overline{N}_{1s} \otimes w_t.\overline{N}_{1t}\right)\right)^{\frac{1}{1+1}} \\
= {} & \Big(\tfrac{1}{12}\Big(w_1.\overline{N}_{11} \otimes w_2.\overline{N}_{12} \oplus w_2.\overline{N}_{12} \otimes w_1.\overline{N}_{11} \oplus w_1.\overline{N}_{11} \otimes w_3.\overline{N}_{13} \oplus w_3.\overline{N}_{13} \otimes w_1.\overline{N}_{11} \\
& \oplus w_1.\overline{N}_{11} \otimes w_4.\overline{N}_{14} \oplus w_4.\overline{N}_{14} \otimes w_1.\overline{N}_{11} \oplus w_2.\overline{N}_{12} \otimes w_4.\overline{N}_{14} \oplus w_4.\overline{N}_{14} \otimes w_2.\overline{N}_{12} \\
& \oplus w_3.\overline{N}_{13} \otimes w_4.\overline{N}_{14} \oplus w_4.\overline{N}_{14} \otimes w_3.\overline{N}_{13}\Big)\Big)^{\frac{1}{2}} \\
= {} & \langle(0.0555, 0.0708, 0.0813, 0.0966); 0.1440, 0.1611, 0.1735, 0.1092\rangle
\end{aligned}$$

$$\begin{aligned}
\overline{N}_2 = {} & TFMBAM_w^{(1,1)}\left(\overline{N}_{21}, \overline{N}_{22}, \overline{N}_{23}, \overline{N}_{24}\right) \\
= {} & \langle(0.0420, 0.0590, 0.0685, 0.0848); 0.1275, 0.1770, 0.1744, 0.1152\rangle \\
\overline{N}_3 = {} & TFMBAM_w^{(1,1)}\left(\overline{N}_{31}, \overline{N}_{32}, \overline{N}_{33}, \overline{N}_{34}\right) \\
= {} & \langle(0.0781, 0.1007, 0.1168, 0.1395); 0.0841, 0.1881, 0.2914, 0.1288\rangle
\end{aligned}$$

$$\begin{aligned}
\overline{N}_4 = {} & TFMBAM_w^{(1,1)}\left(\overline{N}_{41}, \overline{N}_{42}, \overline{N}_{43}, \overline{N}_{44}\right) \\
= {} & \langle(0.0594, 0.0746, 0.0896, 0.1046); 0.0791, 0.1585, 0.2067, 0.2660\rangle
\end{aligned}$$

$$\begin{aligned}\overline{N}_5 = &\ TFMBAM_w^{(1,1)}\left(\overline{N}_{51}, \overline{N}_{52}, \overline{N}_{53}, \overline{N}_{54}\right)\\ = &\ \langle(0.1040, 0.1268, 0.1423, 0.1649); 0.1494, 0.2782, 0.2966, 0.1559\rangle\end{aligned}$$

Step 4. The scores of $\overline{N}_i$ $(i \in I_5)$ $(s(\overline{N}_i))$ are calculated as follows:

$$s(\overline{N}_1) = 0.00057, s(\overline{N}_2) = 0.00049, s(\overline{N}_3) = 0.00146, s(\overline{N}_4) = 0.00087, \text{ and } s(\overline{N}_5) = 0.00226$$

Moreover, all the alternatives are ranked as follows:

$$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$$

As a result, the board should choose the z_5 in the alternatives, which is the best option.

Table 2. Rankings for some alternatives in terms of different $TFMBAM_w^{(p,q)}$ of Example 5.1

(p,q)	i	1	2	3	4	5	Ranking
$(1.0, 1.0)$	$s(\overline{N}_i)$	0.00057	0.00049	0.00146	0.00087	0.00226	$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$
$(1.0, 2.0)$	$s(\overline{N}_i)$	0.00242	0.00186	0.00602	0.00335	0.00854	$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$
$(2.0, 1.0)$	$s(\overline{N}_i)$	0.00242	0.00186	0.00602	0.00335	0.00854	$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$
$(3.0, 1.0)$	$s(\overline{N}_i)$	0.00567	0.00392	0.01390	0.00736	0.01802	$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$
$(1.0, 3.0)$	$s(\overline{N}_i)$	0.00567	0.00392	0.01390	0.00736	0.01802	$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$
$(3.0, 2.0)$	$s(\overline{N}_i)$	0.00772	0.00543	0.01944	0.01002	0.02583	$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$
$(2.0, 3.0)$	$s(\overline{N}_i)$	0.00772	0.00543	0.01944	0.01002	0.02583	$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$
$(5.0, 5.0)$	$s(\overline{N}_i)$	0.02146	0.01378	0.05746	0.02588	0.06978	$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$
$(1.0, 0.5)$	$s(\overline{N}_i)$	0.00016	0.00014	0.00044	0.00026	0.00069	$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$
$(0.5, 1.0)$	$s(\overline{N}_i)$	0.00016	0.00014	0.00044	0.00026	0.00069	$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$
$(2.0, 0.5)$	$s(\overline{N}_i)$	0.00161	0.00120	0.00415	0.00221	0.00558	$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$
$(0.5, 2.0)$	$s(\overline{N}_i)$	0.00161	0.00120	0.00415	0.00221	0.00558	$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$
$(3.0, 0.5)$	$s(\overline{N}_i)$	0.0091	0.00322	0.01223	0.00623	0.01492	$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$
$(0.5, 3.0)$	$s(\overline{N}_i)$	0.0091	0.00322	0.01223	0.00623	0.01492	$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$
$(4.0, 1.0)$	$s(\overline{N}_i)$	0.01010	0.00643	0.02387	0.01260	0.029173	$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$
$(1.0, 4.0)$	$s(\overline{N}_i)$	0.01010	0.00643	0.02387	0.01260	0.029173	$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$
$(10, 2.0)$	$s(\overline{N}_i)$	0.04034	0.02205	0.08164	0.04696	0.09285	$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$

6. Comparison of Studies

A comparison of the proposed method with some other methods given in [13,25,27,40,47] is presented below based on Example 5.1. The developed method, known as the TFM-Bonferroni arithmetic mean operator, proves to be a useful tool for multiple attribute decision-making problems. To see its performance and compare it with existing methods studied in [13, 25, 27, 40, 47], we conduct a comprehensive comparative study. The resulting rankings of alternatives are summarized in Table 3. Upon reviewing Table 3, it becomes evident that the ranking order of alternatives is generally consistent among various methods. Furthermore, when different values of (p, q) are chosen, the ranking order remains the same generally in existing approaches given in [13, 25, 27, 40, 47]. Thus, our proposed method exhibits versatility and can be effectively used alongside existing methods to tackle multi-attribute decision-making problems given with TFM information. Additionally, our developed method offers flexibility, as demonstrated by the solutions presented in Table 2 for Example 5.1 with varying values of (p, q). The results exhibit a high degree of consistency. Consequently, this method can adapt to different situations by adjusting the values of (p, q), expanding its applicability beyond existing methods to address the complexities of multi-attribute decision-making problems. This is the primary advantage of the method over others.

Table 3. Ranking order of the alternatives provided in Example 5.1

Method	Operator	Ranking
Deli and Keleş [13]	S_i	$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$
Uluçay et al. [25]	$TFMG_w$	$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$
Şahin et al. [27]	D_w	$z_2 \prec z_4 \prec z_1 \prec z_5 \prec z_3$
Kesen and Deli [40]	$TFMBHM_w^{(1,1)}$	$z_4 \prec z_2 \prec z_5 \prec z_3 \prec z_1$
Uluçay [47]	S_w	$z_2 \prec z_5 \prec z_1 \prec z_3 \prec z_4$
Proposed method	$TFMBAM_w^{(1,1)}$	$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$
Proposed method	$TFMBAM_w^{(1,2)}$	$z_2 \prec z_1 \prec z_4 \prec z_3 \prec z_5$

7. Conclusion

To get a solution of the multiple attribute decision-making problem within the context of TFM-numbers, this research introduced a novel aggregation method known as the TFM-Bonferroni arithmetic mean operator to combine the TFM information. Then, its properties and special cases were analyzed. Furthermore, a methodology was formulated to handle multiple attribute decision-making problems within the context of TFM environments. Moreover, the suggested approach was applied to multi-criteria decision-making problems within the scope of TFM environments. To get to the main advantage of the paper, it provided a useful operator that is quite flexible to decision-makers. Decision-makers can adjust their preferences by changing p and q values. Then, the operator is a good tool to see the interrelationship among aggregated arguments. However, the operator lacks of finding interrelationship among three or more aggregated arguments. To surpass this disadvantages, our research will be extended to TFM generalized Bonferroni arithmetic mean operator containing SWARA, ANP, ENTROPY, and ELECTRE III methods. In addition, the operator will be extended to trapezoidal intuitionistic fuzzy-multi numbers.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the second author's master's thesis supervised by the first author. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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