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On some k-Oresme hybrid numbers including negative indices

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Abstract. In this study, we define k-Oresme hybrid numbers including negative indices and examine their properties, using the theory of number systems created by choosing the coefficients from unique number sets. Moreover, we derive some fundamental identities related to these numbers.

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1. Introduction

In 1961, Nicole Oresme defined rational sequences and studied the powers of these rational numbers [10]. In 1974, A. F. Horadam reconsidered the sequence Nicole Oresme noticed while examining rational sequences, and named this sequence as Oresme sequence in memory of Oresme [7]. Horadam detailed many algebraic properties of this sequence, which is denoted by $\{O_n\}_{n\geq 1}$ and whose general term is $\left\{\frac{n}{2^n}\right\}$ [8]. The author also gave the recurrence relation of this sequence and different representations for this relation. In 2004, C. K. Cook variously presented the properties of Oresme numbers and their generalization [2]. He gave some identities similar to those in Horadam's work [4]. In 2019, G. Cerda-Morales studied a generalization of Oresme numbers with a new set of numbers called Oresme polynomials [1]. In 2019, T. Goy discussed some families of Toeplitz-Hessenberg determinants whose elements are Oresme numbers [3]. Since the sum formulas and generating function formulas of Oresme numbers in Horadam's study are the fundamental equations related to these numbers, these equations should be reminded.

$$\sum_{j=0}^{n-1} O_j = 4\left(\frac{1}{2} - O_{n+1}\right), \tag{1}$$

$$\sum_{j=0}^{n-1} O_{2j} = \frac{4}{9}\left(2 + O_{2n-1} - 5O_{2n}\right), \tag{2}$$

$$\sum_{j=0}^{n} O_{2j+1} = \frac{1}{9}\left(10 + 5O_{2n-1} - 16O_{2n}\right), \tag{3}$$

$$O_{n+1}O_{n-1} - (O_n)^2 = -\left(\frac{1}{4}\right)^n, \tag{4}$$

$$\sum_{n=0}^{\infty} \frac{n}{2^n}x^n = \frac{2x}{(2-x)^2}. \tag{5}$$

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k-Oresme sequence $\{O_n^{(k)}\}_{n\geq 2}$ which is a type of generalization of Oresme numbers is given by

$$O_n^{(k)} = O_{n-1}^{(k)} - \frac{1}{k^2}O_{n-2}^{(k)}, k \geq 2 \tag{6}$$

with initial conditions $O_1^{(k)} = \frac{1}{k}, O_0^{(k)} = 0$ [2]. In the case $k = 2$, this sequence is reduced to the classical Oresme sequence [2]. The characteristic equation of the recurrence relation in (6) is $x^2 - x + \frac{1}{k^2} = 0$. For $k^2 - 4 > 0$, the roots of this equation are

$$\alpha = \frac{k + \sqrt{k^2-4}}{2k}, \beta = \frac{k - \sqrt{k^2-4}}{2k},$$

respectively [8]. From the recurrence relation, the Binet's formula is given by

$$O_n^{(k)} = \frac{1}{\sqrt{k^2-4}}\left[\left(\frac{k+\sqrt{k^2-4}}{2k}\right)^n - \left(\frac{k-\sqrt{k^2-4}}{2k}\right)^n\right]. \tag{7}$$

In 2022, Halici et *al.* k-Oresme polynomials examined [5]. k-Oresme polynomials sequence $\{O_n^{(k)}(x)\}_{n\geq 2}$ which is a type of generalization of Oresme numbers is given by

$$O_{n+2}^{(k)}(x) = O_{n+1}^{(k)}(x) - \frac{1}{k^2x^2}O_n^{(k)}(x), x \geq 1 \tag{8}$$

with initial conditions $O_1^{(k)}(x) = \frac{1}{kx}, O_0^{(k)}(x) = 0$. In the case $k = 1$ and $x = 1$ this sequence is reduced to the classical Oresme sequence [5].
In 2021, Gurses et *al.* presented two new types of Oresme numbers [12]. And they investigated special linear recurrence relations and summation properties for DGC Oresme numbers of these types. The DGC numbers here are Dual-Hyperbolic Oresme numbers. In 2022, Halici et *al.* examined k-Oresme numbers with negative indices [4]. This sequence is denoted by $\{O_{-n}^{(k)}\}_{n\geq 0}$ and defined as

$$O_{-n}^{(k)} = k^2\left(O_{-n+1}^{(k)} - O_{-n+2}^{(k)}\right), k \geq 2 \tag{9}$$

with initial conditions $O_{-1}^{(k)} = -k, O_0^{(k)} = 0$. The nth term of this sequence is defined by

$$O_{-n}^{(k)} = -k^{2n}\frac{(\alpha^n - \beta^n)}{\sqrt{k^2-4}}. \tag{10}$$

On the other hand, in 2018, Ozdemir defined a non-commutative number system and called it hybrid numbers [11]. The author examined in detail the algebraic and geometrical properties of the new number system, which he described. In 2018, Szynal-Liana introduced Horadam hybrid numbers and examined their special cases [13]. Hybrid numbers and Horadam hybrid numbers are defined by

$$\mathbb{K} = \{z = a + b\mathbf{i} + c\boldsymbol{\epsilon} + d\mathbf{h}; a, b, c, d \in \mathbb{R}\} \tag{11}$$

and

$$H_n = Wn + W_{n+1}\mathbf{i} + W_{n+2}\boldsymbol{\epsilon} + W_{n+3}\mathbf{h}, \tag{12}$$

respectively [11, 13]. The relations provided between the three different base elements used in the set $\mathbb{K}$ are

$$\mathbf{i}^2 = -1, \boldsymbol{\epsilon}^2 = 0, \mathbf{h}^2 = 1, \mathbf{ih} = -\mathbf{hi} = \boldsymbol{\epsilon} + \mathbf{1}. \tag{13}$$

The conjugate of any hybrid number z is defined as $\overline{z} = a - b\mathbf{i} - c\boldsymbol{\epsilon} - d\mathbf{h}$.
The character value of element z, used in important identities, is

$$C(z) = z\overline{z} = \overline{z}z = a^2 + (b-c)^2 - c^2 - d^2. \tag{14}$$

This value of the hybrid number is often used to determine the generalized norm of a hybrid number. Depending on the selection of the coefficients a, b, c and d, different norms are obtained and they are examining.

$$\begin{array}{c|l} c = d = 0 & N(z) = \sqrt{a^2+b^2}\ , \\ c = b = 0 & N(z) = \sqrt{|a^2-d^2|}\ , \\ b = d = 0 & N(z) = |a|. \end{array}$$

In [14], Oresme hybrid numbers were defined and discussed by Syznal et *al.* For any positive number *n,* the *nth* Oresme hybrid number is

$$OH_n = O_n + O_{n+1}\mathbf{i} + O_{n+2}\boldsymbol{\epsilon} + O_{n+3}\mathbf{h} \tag{15}$$

and the k-Oresme hybrid number is

$$OH_n^{(k)} = O_n^{(k)} + O_{n+1}^{(k)}\mathbf{i} + O_{n+2}^{(k)}\boldsymbol{\epsilon} + O_{n+3}^{(k)}\mathbf{h} \tag{16}$$

[14]. In addition, in [14], the authors gave some fundamental identities with the help of iterative relation including k-Oresme hybrid numbers, but many important identities are not given in this study (see [14], Thr. 2.4). They also defined Oresme hybrationals for a nonzero real variable x and $n \geq 0$

$$OH_n(x) = On(x) + O_{n+1}(x)\mathbf{i} + O_{n+2}(x)\boldsymbol{\epsilon} + O_{n+3}(x)\mathbf{h}. \tag{17}$$

In [9], Gurses et *al.* defined Pentanacci and Pentanacci-Lucas hybrid numbers.
In this current study, we define and studied k-Oresme numbers with negative indices. For $k \geq 2$, $n \geq 0$ we give some important identities, such as the Cassini identity, which have various applications in the literature and include these numbers.

2. K-Oresme Hybrid Numbers Including Negative Indices

In [6], we introduced k-Oresme hybrid numbers and investigate their fundamental properties.
In this section, we constructed the theory of k-Oresme hybrid numbers with negative indices.

Definition 1. *By the aid of the hybrid numbers and the Oresme numbers, let us define k-Oresme hybrid numbers with negative indices as follows.*

$$OH_{-n}^{(k)} = O_{-n}^{(k)} + O_{-n+1}^{(k)}\mathbf{i} + O_{-n+2}^{(k)}\boldsymbol{\epsilon} + O_{-n+3}^{(k)}\mathbf{h}, n \geq 0. \tag{18}$$

The algebraic operations of the numbers we have just defined here are done by considering the algebraic operations of both Oresme numbers and hybrid numbers. Since we use the terms of this new number sequence $\{OH_{-n}^{(k)}\}_{n\geq 0}$ later, it is appropriate to write some of its terms.

$$\left\{..., \left(-k^3 - k\mathbf{i} + \frac{1}{k}\mathbf{h}\right), \left(-k + \frac{1}{k}\boldsymbol{\epsilon} + \frac{1}{k}\mathbf{h}\right), \left(\frac{1}{k} + \frac{1}{k}\boldsymbol{\epsilon} + \frac{(k^2-1)}{k^3}\mathbf{h}\right), ...\right\}.$$

In the following theorem, we give the Binet formula which provides the derivation of many important identities.

Theorem 1. *For the sequence $\{OH_{-n}^{(k)}\}_{n\geq 0}$, the Binet formula is*

$$OH_{-n}^{(k)} = \frac{-k^{2n}}{\sqrt{k^2-4}}\left(\alpha^n\tilde{\beta} - \beta^n\tilde{\alpha}\right), \tag{19}$$

where, $\tilde{\alpha} = 1 + \alpha\mathbf{i} + \alpha^2\boldsymbol{\epsilon} + \alpha^3\mathbf{h}$ *and* $\tilde{\beta} = 1 + \beta\mathbf{i} + \beta^2\boldsymbol{\epsilon} + \beta^3\mathbf{h}$.

Proof. From the equality (10),

$$OH_{-n}^{(k)} = -\frac{k^{2n}}{\sqrt{k^2-4}}\left[(\alpha^n - \beta^n)\right] - \frac{k^{2n}}{\sqrt{k^2-4}}\left[k^{-2}(\alpha^{n-1} - \beta^{n-1})\mathbf{i}\right]$$
$$-\frac{k^{2n}}{\sqrt{k^2-4}}\left[k^{-4}(\alpha^{n-2} - \beta^{n-2})\boldsymbol{\epsilon}\right] - \frac{k^{2n}}{\sqrt{k^2-4}}\left[k^{-6}(\alpha^{n-3} - \beta^{n-3})\mathbf{h}\right].$$

Then, we get

$$LHS = \frac{-k^{2n}}{\sqrt{k^2-4}}\left[\alpha^n\left(1 + \frac{1}{k^2\alpha}\mathbf{i} + \left(\frac{1}{k^2\alpha}\right)^2\boldsymbol{\epsilon} + \left(\frac{1}{k^2\alpha}\right)^3\mathbf{h}\right)\right]$$
$$+\frac{k^{2n}}{\sqrt{k^2-4}}\beta^n\left[\left(1 + \frac{1}{k^2\beta}\mathbf{i} + \left(\frac{1}{k^2\beta}\right)^2\boldsymbol{\epsilon} + \left(\frac{1}{k^2\beta}\right)^3\mathbf{h}\right)\right],$$

$$LHS = \frac{-k^{2n}}{\sqrt{k^2-4}}\left[\alpha^n\left(1 + \beta\mathbf{i} + \beta^2\boldsymbol{\epsilon} + \beta^3\mathbf{h}\right) - \beta^n\left(1 + \alpha\mathbf{i} + \alpha^2\boldsymbol{\epsilon} + \alpha^3\mathbf{h}\right)\right].$$

If we complete the necessary algebraic operations, then we obtain

$$OH_{-n}^{(k)} = \frac{-k^{2n}}{\sqrt{k^2-4}}\left(\alpha^n\tilde{\beta} - \beta^n\tilde{\alpha}\right)$$

which is desired result. Thus, the proof is completed. □

In the next theorem, we give the recurrence relation provided by the elements of the newly defined sequence.

Theorem 2. *For $n \in \mathbb{Z}$, the following equality is satisfied.*

$$OH_{-n+1}^{(k)} = k^2\left(OH_{-n+2}^{(k)} - OH_{-n+3}^{(k)}\right). \tag{20}$$

Proof. The proof of this equality is easily seen using induction. □

Theorem 3. *The character value for elements of the sequence $\{OH_{-n}^{(k)}\}_{n\geq 0}$ is*

$$C(z) = \left(O_{-n+1}^{(k)}\right)^2\left(\frac{k^8+k^4-1}{k^4}\right) - 2O_{-n+1}^{(k)}O_{-n+2}^{(k)}\left(\frac{k^6+k^2-1}{k^2}\right) \tag{21}$$

$$+\left(O_{-n+2}^{(k)}\right)^2\left(k^4-1\right).$$

Proof. From the definition in the equation (20), $C(z)$ is

$$C(z) = \left(O_{-n}^{(k)}\right)^2 + \left(O_{-n+1}^{(k)} - O_{-n+2}^{(k)}\right)^2 - \left(O_{-n+2}^{(k)}\right)^2 - \left(O_{-n+3}^{(k)}\right)^2,$$

$$LHS = \left[k^2\left(O_{-n+1}^{(k)} - O_{-n+2}^{(k)}\right)\right]^2 + \left(O_{-n+1}^{(k)}\right)^2$$

$$-2\left(O_{-n+1}^{(k)} - O_{-n+2}^{(k)}\right) - \left(O_{-n+2}^{(k)} - \frac{1}{k^2}O_{-n+1}^{(k)}\right)^2,$$

$$LHS = \left(O_{-n+1}^{(k)}\right)^2\left(\frac{k^8+k^4-1}{k^4}\right) - 2O_{-n+1}^{(k)}O_{-n+2}^{(k)}\left(\frac{k^6+k^2-1}{k^2}\right)$$

$$+\left(O_{-n+2}^{(k)}\right)^2\left(k^4-1\right).$$

So, the proof is completed. □

Theorem 4. *For elements of the sequence $\{OH_{-n}^{(k)}\}_{n\geq 0}$, we have*

$$i)\ OH_{-n}^{(k)} + \overline{OH_{-n}^{(k)}} = 2O_{-n}^{(k)}. \tag{22}$$

$$ii)\ C(OH_{-n}^{(k)}) = 2OH_{-n}^{(k)}.O_{-n}^{(k)} - \left(O_{-n}^{(k)}\right)^2. \tag{23}$$

Proof. Since the first equality in this theorem can be seen immediately from the definition (18), we consider the second equality.

$$\left(OH_{-n}^{(k)}\right)^2 = \left(O_{-n}^{(k)}\right)^2 - \left(O_{-n+1}^{(k)}\right)^2 + \left(O_{-n+3}^{(k)}\right)^2 + 2\left(O_{-n+1}^{(k)}O_{-n+2}^{(k)}\right)$$

$$+2\left(O_{-n}^{(k)}O_{-n+1}^{(k)}\mathbf{i} + O_{-n}^{(k)}O_{-n+2}^{(k)}\boldsymbol{\epsilon} + O_{-n}^{(k)}O_{-n+3}^{(k)}\mathbf{h}\right),$$

$$\left(OH_{-n}^{(k)}\right)^2 = 2O_{-n}^{(k)}OH_{-n}^{(k)} - \left(O_{-n}^{(k)}\right)^2 - \left(O_{-n+1}^{(k)}\right)^2 + \left(O_{-n+3}^{(k)}\right)^2 + 2O_{-n+1}^{(k)}O_{-n+2}^{(k)}.$$

Thus, we get,

$$\left(OH_{-n}^{(k)}\right)^2 = 2O_{-n}^{(k)}OH_{-n}^{(k)} - C(z),$$

$$C(z) = 2O_{-n}^{(k)}OH_{-n}^{(k)} - \left(OH_{-n}^{(k)}\right)^2.$$

□

In the next theorem, we give the Cassini identity which includes k-Oresme hybrid numbers with negative indices.

Theorem 5. *For elements of sequence* $\{OH_{-n}^{(k)}\}_{n\geq 0}$, *we have*

$$OH_{-n+1}^{(k)}OH_{-n-1}^{(k)}-\left(OH_{-n}^{(k)}\right)^2=-\frac{k^{2n}}{k^2-4}\left[\tilde{\beta}\tilde{\alpha}\left((k\alpha)^{-2}-2\right)+\tilde{\alpha}\tilde{\beta}\,(k\alpha)^2\right] \tag{24}$$

Proof. From, $OH_{-n+1}^{(k)}, OH_{-n-1}^{(k)}$ and $OH_{-n}^{(k)}$, we can write the left-hand side of the desired equation. Here, $\tilde{\alpha}\tilde{\beta}\neq\tilde{\beta}\tilde{\alpha}$. That is,

$$LHS=\left[\frac{-k^{2n-2}}{\sqrt{k^2-4}}\left(\alpha^{n-1}\tilde{\beta}-\beta^{n-1}\tilde{\alpha}\right)\right]\left[\frac{-k^{2n+2}}{\sqrt{k^2-4}}\left(\alpha^{n+1}\tilde{\beta}-\beta^{n+1}\tilde{\alpha}\right)\right] -\left[\frac{-k^{2n}}{\sqrt{k^2-4}}\left(\alpha^{n}\tilde{\beta}-\beta^{n}\tilde{\alpha}\right)\right]^2,$$

$$LHS=\frac{k^{4n}}{k^2-4}\left[\left(\alpha^{n-1}\tilde{\beta}-\beta^{n-1}\tilde{\alpha}\right)\left(\alpha^{n+1}\tilde{\beta}-\beta^{n+1}\tilde{\alpha}\right)-\left(\alpha^{n}\tilde{\beta}-\beta^{n}\tilde{\alpha}\right)^2\right],$$

$$LHS=-\frac{k^{4n}}{k^2-4}\left[\alpha^{n-1}\beta^{n+1}\tilde{\beta}\tilde{\alpha}+\alpha^{n+1}\beta^{n-1}\tilde{\alpha}\tilde{\beta}-2(\alpha\beta)^n\tilde{\beta}\tilde{\alpha}\right],$$

$$LHS=-\frac{k^{4n}}{k^2-4}(\alpha\beta)^n\left[\tilde{\beta}\tilde{\alpha}\left(\frac{\beta}{\alpha}-2\right)+\tilde{\alpha}\tilde{\beta}\left(\frac{\alpha}{\beta}\right)\right],$$

If we write the values $\alpha\beta$, $\frac{\beta}{\alpha}$ and $\frac{\alpha}{\beta}$, then we get

$$LHS=-\frac{k^{2n}}{k^2-4}\left[\tilde{\beta}\tilde{\alpha}\left((k\alpha)^{-2}-2\right)+\tilde{\alpha}\tilde{\beta}\,(k\alpha)^2\right].$$

Thus, we complete the proof. □

In the following theorem, we give the Catalan identity provided by k-Oresme hybrid numbers with the negative indices.

Theorem 6. *For $n\geq r$, the elements of sequence* $\{OH_{-n}^{(k)}\}_{n\geq 0}$ *provide.*

$$OH_{-n+r}^{(k)}OH_{-n-r}^{(k)}-\left(OH_{-n}^{(k)}\right)^2=-\frac{k^{2n}}{k^2-4}\left[\tilde{\beta}\tilde{\alpha}\left((k\alpha)^{-2r}-2\right)+\tilde{\alpha}\tilde{\beta}\,(k\alpha)^{2r}\right]. \tag{25}$$

Proof. $OH_{-n+r}^{(k)}OH_{-n-r}^{(k)}-\left(OH_{-n}^{(k)}\right)^2$ is equal to this:

$$LHS=\frac{k^{4n}}{k^2-4}\left[2(\alpha\beta)^n\tilde{\beta}\tilde{\alpha}-\alpha^{n-r}\beta^{n+r}\tilde{\beta}\tilde{\alpha}-\alpha^{n+r}\beta^{n-r}\tilde{\alpha}\tilde{\beta}\right],$$

$$LHS=-\frac{k^{4n}}{k^2-4}(\alpha\beta)^n\left[\tilde{\beta}\tilde{\alpha}\left(\left(\frac{\beta}{\alpha}\right)^r-2\right)+\tilde{\alpha}\tilde{\beta}\left(\frac{\alpha}{\beta}\right)^r\right],$$

If relations valid between the roots of characteristic equation of the sequence are used the relations

$$OH_{-n+r}^{(k)}OH_{-n-r}^{(k)}-\left(OH_{-n}^{(k)}\right)^2=-\frac{k^{2n}}{k^2-4}\left[\tilde{\beta}\tilde{\alpha}\left((k\alpha)^{-2r}-2\right)+\tilde{\alpha}\tilde{\beta}\,(k\alpha)^{2r}\right]$$

is obtained which is the desired result. □

In the case of $r=1$, it is obvious that this equation is reduced to the Cassini identity.
In the following theorem, we give the identity d'Ocagne containing the elements of the sequence $\{OH_{-n}^{(k)}\}_{n\geq 0}$.

Theorem 7. *For $m,n\in\mathbb{Z}$, the elements $OH_{-n}^{(k)}$ satisfy the following identity.*

$$OH_{-n+1}^{(k)}OH_{-m}^{(k)}-OH_{-n}^{(k)}OH_{-m+1}^{(k)}=\frac{k^{2m}}{\sqrt{k^2-4}}\left(\beta^{m-n}\tilde{\beta}\tilde{\alpha}-\alpha^{m-n}\tilde{\alpha}\tilde{\beta}\right). \tag{26}$$

Proof. If we use definition of the numbers $OH_{-n+1}^{(k)}, OH_{-m}^{(k)}, OH_{-n}^{(k)}$ and $OH_{-m+1}^{(k)}$, then we can write the left-hand side of the desired equation as follows.

$$OH_{-n+1}^{(k)}OH_{-m}^{(k)} - OH_{-n}^{(k)}OH_{-m+1}^{(k)} = \frac{k^{2(m+n-1)}}{k^2-4}\left(A+B\right),$$

where

$$A = \left[\alpha^{m+n-1}(\tilde{\beta})^2 - \alpha^{n-1}\beta^m\tilde{\beta}\tilde{\alpha} - \beta^{n-1}\alpha^m\tilde{\alpha}\tilde{\beta} + \beta^{m+n-1}(\tilde{\alpha})^2\right],$$

$$B = \left[-\alpha^{m+n-1}(\tilde{\beta})^2 + \alpha^{m-1}\beta^n\tilde{\alpha}\tilde{\beta} + \beta^{m-1}\alpha^n\tilde{\beta}\tilde{\alpha} - \beta^{m+n-1}(\tilde{\alpha})^2\right].$$

In the last equation we obtained, we can write the following equations as a result of simplification and some calculations.

$$OH_{-n+1}^{(k)}OH_{-m}^{(k)} - OH_{-n}^{(k)}OH_{-m+1}^{(k)} = \frac{k^{2(m+n-1)}}{k^2-4}\left[\tilde{\beta}\tilde{\alpha}\alpha^n\beta^m\left(\frac{1}{\beta}-\frac{1}{\alpha}\right) + \tilde{\alpha}\tilde{\beta}\alpha^m\beta^n\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\right],$$

$$OH_{-n+1}^{(k)}OH_{-m}^{(k)} - OH_{-n}^{(k)}OH_{-m+1}^{(k)} = \frac{k^{2(m+n-1)}}{k^2-4}\left(\frac{\alpha-\beta}{\alpha\beta}\right)\left(\alpha^n\beta^m\tilde{\beta}\tilde{\alpha} - \tilde{\alpha}\tilde{\beta}\alpha^m\beta^n\right),$$

$$OH_{-n+1}^{(k)}OH_{-m}^{(k)} - OH_{-n}^{(k)}OH_{-m+1}^{(k)} = \frac{k^{2m}}{\sqrt{k^2-4}}\left(\beta^{m-n}\tilde{\beta}\tilde{\alpha} - \alpha^{m-n}\tilde{\alpha}\tilde{\beta}\right).$$

Thus, the proof is completed. □

In the next theorem, we give the identity Honsberger's identity involving the elements of the sequence $\{OH_{-n}^{(k)}\}_{n\geq 0}$.

Theorem 8. *For $m, n \in \mathbb{Z}$, the following equation is true.*

$$OH_{-(m+n)}^{(k)} = kO_{-n}^{(k)}OH_{-m+1}^{(k)} - \frac{1}{k}O_{-n-1}^{(k)}OH_{-m}^{(k)}. \tag{27}$$

Proof. We write the following equation to see correctness of the desired equation.

$$OH_{-(m+n)}^{(k)} = k_1O_{-n}^{(k)} - k_2O_{-(n+1)}^{(k)}.$$

We should see that the equations $k_1 = kOH_{-m+1}^{(k)}$ and $k_2 = \frac{1}{k}OH_{-m}^{(k)}$ are true. From the Binet formula (19),

$$k^{2m}\left(\alpha^{m+n}\tilde{\beta} - \beta^{m+n}\tilde{\alpha}\right) = \beta^n\left(-k_1 + k_2k^2\alpha\right) + \alpha^n\left(k_1 - k_2k^2\beta\right),$$

can be written. Also, we get

$$-k^{2m}\beta^m\tilde{\alpha} = -k_1 + k_2k^2\alpha,$$

$$k^{2m}\alpha^m\tilde{\alpha} = k_1 - k_2k^2\beta.$$

And so, $k_2 = k^{2m-2}\frac{(\alpha^m\tilde{\beta}-\beta^m\tilde{\alpha})}{\alpha-\beta} = -\frac{1}{k}OH_{-m}$. If we substitute this value in the equation, then

$$k^{2m}\beta^m\tilde{\alpha} = -k_1 + \alpha k^{2m}\frac{\left(\alpha^m\tilde{\beta} - \beta^m\tilde{\alpha}\right)}{\alpha-\beta},$$

$$k^{2m}\beta^{m+1}\tilde{\alpha} - k^{2m}\alpha^{m+1}\tilde{\beta} = k_1(\alpha-\beta),$$

$$k_1 = -\frac{k^{2m}(\alpha^{m+1}\tilde{\beta} - \beta^{m+1}\tilde{\alpha})}{(\alpha-\beta} = kOH_{-m+1}.$$

Then, we obtain that

$$OH_{-(m+n)}^{(k)} = kO_{-n}^{(k)}OH_{-m+1}^{(k)} - \frac{1}{k}O_{-n-1}^{(k)}OH_{-m}^{(k)}.$$

□

In the following theorem, we give the generating function of k-Oresme hybrid numbers with negative indices.

Theorem 9. *For the sequence $\{OH_{-n}^{(k)}\}_{n\geq 0}$, the generating function is*

$$\sum_{i\geq 0} OH_i^{(k)} z^i = \frac{OH_0^{(k)}(1-zk^2) + zOH_{-1}^{(k)}}{1-zk^2+z^2k^2}. \tag{28}$$

Proof. The following equaitons are calculated respectively.

$$f(z) = OH_0^{(k)} + zOH_{-1}^{(k)} + z^2OH_{-2}^{(k)} + z^3OH_{-3}^{(k)} \cdots$$

$$zk^2 f(z) = zk^2OH_0^{(k)} - z^2k^2OH_{-1}^{(k)} - z^3k^2OH_{-2}^{(k)} + \cdots$$

$$z^2k^2 f(z) = z^2k^2OH_0^{(k)} + z^3k^2OH_{-1}^{(k)} + z^4k^2OH_{-2}^{(k)} + \cdots$$

$$f(z)\left(1-zk^2+z^2k^2\right) = OH_0^{(k)} + z\left(OH_{-1}^{(k)} - k^2OH_0^{(k)}\right).$$

Then, we get

$$f(z) = \frac{OH_0^{(k)}(1-zk^2) + zOH_{-1}^{(k)}}{1-zk^2+z^2k^2}.$$

□

In the following theorem, we derive the formula for the sum of k-Oresme hybrid numbers with negative indices.

Theorem 10. *For the elements of sequence $\{OH_{-n}^{(k)}\}_{n\geq 0}$, the following is satisfied.*

$$\sum_{k=1}^{n} OH_{-n}^{(k)} = k^2\left(OH_{-n}^{(k)} - OH_1^{(k)}\right) - \left(OH_0^{(k)} + OH_{-n-1}^{(k)}\right). \tag{29}$$

Proof. If we use the Binet formula (19), then we write

$$\sum_{k=1}^{n} OH_n^{(k)} = \frac{-1}{\sqrt{k^2-4}}\left[\tilde{\beta}\sum_{k=1}^{n}\left(k^2\alpha\right)^n - \tilde{\alpha}\sum_{k=1}^{n}\left(k^2\beta\right)^n\right],$$

Here, $\tilde{\beta}\sum_{k=1}^{n}\left(k^2\alpha\right)^n$ and $\tilde{\alpha}\sum_{k=1}^{n}\left(k^2\beta\right)^n$ are

$$\tilde{\beta}\sum_{k=1}^{n}\left(k^2\alpha\right)^n = \tilde{\beta}\left(1-k^2\beta-k^{2n+2}\alpha^{n+1}+k^{2n+4}\alpha^{n+1}\beta\right)$$

and

$$\tilde{\alpha}\sum_{k=1}^{n}\left(k^2\beta\right)^n = \tilde{\alpha}\left(1-k^2\alpha-k^{2n+2}\beta^{n+1}+k^{2n+4}\beta^{n+1}\alpha\right),$$

respectively. If we substitute these calculated values, then we obtain

$$\sum_{k=1}^{n} OH_{-n}^{(k)} = -k^2OH_1^{(k)} + k^2OH_{-n}^{(k)} + OH_0^{(k)} - OH_{-n-1}^{(k)},$$

$$\sum_{k=1}^{n} OH_{-n}^{(k)} = k^2\left(OH_{-n}^{(k)} - OH_1^{(k)}\right) + \left(OH_0^{(k)} - OH_{-n-1}^{(k)}\right).$$

Thus, the proof is completed. □

Theorem 11. *For the elements of sequence $\{OH_{-n}^{(k)}\}_{n\geq 0}$ the following is satisfied.*

$$\sum_{j=1}^{n} OH_{-j}^{(k)} = -k^2\left(OH_2^{(k)} - OH_{-n+1}^{(k)}\right). \tag{30}$$

Proof. From the equality [4], $\sum_{i=1}^{n} O_{-i}^{(k)} = -k\left(1 - O_{-n+1}^{(k)}\right)$ and by induction

$$\sum_{j=0}^{n+1} OH_{-j}^{(k)} = OH_0^{(k)} + OH_{-1}^{(k)} + OH_{-2}^{(k)} + \cdots + OH_{-(n+1)}^{(k)},$$

can be written. Thus,

$$\sum_{j=0}^{n} OH_{-j}^{(k)} = \left(O_0^{(k)} + O_{-1}^{(k)}\mathbf{i} + O_{-2}^{(k)}\boldsymbol{\epsilon} + O_{-3}^{(k)}\mathbf{h}\right) + \left(O_{-1}^{(k)} + O_0^{(k)}\mathbf{i} + O_1^{(k)}\boldsymbol{\epsilon} + O_2^{(k)}\mathbf{h}\right) + \cdots,$$

$$\sum_{j=0}^{n} OH_{-j}^{(k)} = \left[-k - k\mathbf{i} + \frac{\mathbf{i}}{k} + \frac{2\boldsymbol{\epsilon}}{k} - \epsilon k + \frac{(k^2-1)}{k^3}\mathbf{h} + \frac{2\mathbf{h}}{k} - \mathbf{h}\right] + k^2 OH_{-n+1}^{(k)},$$

$$\sum_{j=0}^{n} OH_{-j}^{(k)} = \left[-k + \frac{(1-k^2)}{k}\mathbf{i} + \frac{(2-k^2)}{k}\boldsymbol{\epsilon} + \frac{(3k^2-k^4-1)}{k^3}\mathbf{h}\right] + k^2 OH_{-n+1}^{(k)},$$

$$\sum_{j=0}^{n} OH_{-j}^{(k)} = -k^2\left[\frac{1}{k} + \frac{(k^2-1)}{k^3}\mathbf{i} + \frac{(k^2-2)}{k^3}\boldsymbol{\epsilon} + \frac{(k^4-3k^2+1)}{k^5}\mathbf{h}\right] + k^2 OH_{-n+1}^{(k)}.$$

$$\sum_{j=0}^{n} OH_{-j}^{(k)} = -k^2\left(OH_2^{(k)} - OH_{-n+1}^{(k)}\right)$$

is obtained which is the proof is completed. □

Theorem 12. *For the elements of sequence* $\{OH_{-n}^{(k)}\}_{n\geq 0}$ *the following is satisfied.*

$$\sum_{j=1}^{n} (-1)^i OH_{-i}^{(k)} = \frac{1}{2k^2+1}\left(k + (-1)^n\left(k^2 OH_{-n}^{(k)} + OH_{-n-1}^{(k)}\right)\right). \tag{31}$$

Proof. It can be seen that the equality claimed in the statement of the theorem is true with the help of the induction method. □

Theorem 13. *For the elements of sequence* $\{OH_{-n}^{(k)}\}_{n\geq 0}$ *the following is satisfied.*

$$T = \sum_{n=1}^{t}\left(OH_{-n}^{(k)}\right)^2 = \frac{k^4\left(OH_1^{(k)} - \left(OH_{t+1}^{(k)}\right)^2\right) - OH_0^{(k)} + \left(OH_t^{(k)}\right)^2}{1+2k^2}$$

$$+\frac{2\left(\frac{1-x^t}{1-x}\right)\left[\frac{\tilde{\beta}\tilde{\alpha}\left((k\alpha)^{-2}-2\right)+\tilde{\alpha}\tilde{\beta}(k\alpha)^2}{k^2-4}\right]}{1+2k^2}. \tag{32}$$

Proof. We have $T = \left(OH_{-1}^{(k)}\right)^2 + \left(OH_{-2}^{(k)}\right)^2 + \left(OH_{-3}^{(k)}\right)^2 + \cdots$. From the equality (20),

$$OH_{-n}^{(k)} = \frac{k^2 OH_{-n-1}^{(k)} + OH_{-n+1}^{(k)}}{k^2}.$$

$$T = \sum_{n=1}^{t}\left(OH_{-n}^{(k)}\right)^2 = \sum_{n=1}^{t}\left[\frac{k^2 OH_{-n-1}^{(k)} + OH_{-n+1}^{(k)}}{k^2}\right]^2,$$

$$k^4 T = k^4\sum_{n=1}^{t}\left(OH_{-n+1}^{(k)}\right)^2 + \sum_{n=1}^{t}\left(OH_{-n-1}^{(k)}\right)^2 + 2k^2\sum_{n=1}^{t} OH_{-n+1}^{(k)} OH_{-n-1}^{(k)}.$$

If we also use the Cassini identity,
$OH_{n+1}^{(k)}OH_{n-1}^{(k)} = \left(OH_{-n}^{(k)}\right)^2 - \frac{k^{2n}}{k^2-4}\left[\tilde{\beta}\tilde{\alpha}\left((k\alpha)^{-2}-2\right)+\tilde{\alpha}\tilde{\beta}\,(k\alpha)^2\right]$, then we get

$$k^4T = k^4\left(T-\left(OH_1^{(k)}\right)^2+\left(OH_{t+1}^{(k)}\right)^2\right)+\left(T-\left(OH_0^{(k)}\right)^2+\left(OH_t^{(k)}\right)^2\right)$$

$$+2k^2\sum_{n=1}^{t}\left(OH_n^{(k)}\right)^2-\frac{k^{2n}}{k^2-4}\left[\tilde{\beta}\tilde{\alpha}\left((k\alpha)^{-2}-2\right)+\tilde{\alpha}\tilde{\beta}\,(k\alpha)^2\right],$$

$$k^4T = k^4\left(T-\left(OH_1^{(k)}\right)^2+\left(OH_{t+1}^{(k)}\right)^2\right)+\left(T+\left(OH_0^{(k)}\right)^2-\left(OH_t^{(k)}\right)^2\right)$$

$$+2k^2T-2\sum_{n=1}^{t}\frac{k^{2n+2}}{k^2-4}\left[\tilde{\beta}\tilde{\alpha}\left((k\alpha)^{-2}-2\right)+\tilde{\alpha}\tilde{\beta}\,(k\alpha)^2\right],$$

$$k^4T = T(k^4+2k^2+1)-k^4\left(OH_1^{(k)}\right)^2+k^4\left(OH_{t+1}^{(k)}\right)^2+\left(OH_0^{(k)}\right)^2-\left(OH_t^{(k)}\right)^2$$

$$-2\left(\frac{1-x^t}{1-x}\right)\left[\frac{\tilde{\beta}\tilde{\alpha}\left((k\alpha)^{-2}-2\right)+\tilde{\alpha}\tilde{\beta}\,(k\alpha)^2}{k^2-4}\right].$$

Thus, we obtain

$$T=\sum_{n=1}^{t}\left(OH_{-n}^{(k)}\right)^2=\frac{k^4\left(OH_1^{(k)}-\left(OH_{t+1}^{(k)}\right)^2\right)-OH_0^{(k)}+\left(OH_t^{(k)}\right)^2}{1+2k^2}$$

$$+\frac{2\left(\frac{1-x^t}{1-x}\right)\left[\frac{\tilde{\beta}\tilde{\alpha}\left((k\alpha)^{-2}-2\right)+\tilde{\alpha}\tilde{\beta}(k\alpha)^2}{k^2-4}\right]}{1+2k^2}.$$

Which is the desired result. □

Theorem 14. *For the elements of sequence* $\{OH_{-n}^{(k)}\}_{n\geq 0}$ *the following is satisfied.*

$$i)\ \sum_{k=1}^{n}OH_{-2n}^{(k)}=\left(OH_0^{(k)}-OH_{-2n}^{(k)}\right)-k^4\left(OH_2^{(k)}-OH_{-(2n-2)}^{(k)}\right). \qquad (33)$$

$$ii)\ \sum_{k=1}^{n}OH_{-2n-1}^{(k)}=\left(OH_0^{(k)}-OH_{-(2n-1)}^{(k)}\right)+k^2\left(OH_1^{(k)}-OH_{-(2n-2)}^{(k)}\right). \qquad (34)$$

Proof.

$$i)\ \sum_{k=1}^{n}OH_{-2n}^{(k)}=\frac{-1}{\sqrt{k^2-4}}\left[\tilde{\beta}\sum_{k=1}^{n}(\alpha^2k^4)^n-\tilde{\alpha}\sum_{k=1}^{n}(\beta^2k^4)^n\right],$$

$$\sum_{k=1}^{n}OH_{-2n}^{(k)}=\frac{-1}{\sqrt{k^2-4}}\left[\tilde{\beta}\left(\frac{1-(k^4\alpha^2)^n}{1-k^4\alpha^2}\right)-\tilde{\alpha}\left(\frac{1-(k^4\beta^2)^n}{1-k^4\beta^2}\right)\right],$$

$$\sum_{k=1}^{n}OH_{-2n}^{(k)}=\frac{-1}{\sqrt{k^2-4}}\left[(\tilde{\beta}-\tilde{\alpha})-k^{4n}(\tilde{\beta}\alpha^{2n}-\tilde{\alpha}\beta^{2n})\right]$$

$$-\frac{1}{\sqrt{k^2-4}}\left[k^4(\tilde{\alpha}\alpha^2-\tilde{\beta}\beta^2)+k^{4n}(\tilde{\beta}\alpha^{2n-2}-\tilde{\alpha}\beta^{2n-2})\right],$$

$$\sum_{k=1}^{n} OH_{-2n}^{(k)} = \left(OH_0^{(k)} - OH_{-2n}^{(k)}\right) - k^4 \left(OH_2^{(k)} - OH_{-(2n-2)}^{(k)}\right).$$

Thus, the sum of k-Oresme hybrid numbers with the negative even indices is given. Similarly, the sum of terms with odd indices can be obtained. □

3. Conclusion

In this study, we inspired by the theory of number systems created by choosing coefficients from special number sets and defined at the negative indices k-Oresme hybrid numbers. We examined these newly identified numbers in detail. In particular, we obtained the fundamental and important identities provided by the elements of this sequence and frequently encountered in the literature.

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