Some Symmetry Properties of Almost S-Manifolds

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Abstract

Manifold theory is an important topic in differential geometry. Riemannian manifolds are a wide class of differentiable manifolds. Riemannian manifolds consist of two fundamental class, as contact manifolds and complex manifolds. The notion of globally framed metric f-manifold is a generalization of these fundamental classes. Almost S-manifolds which are globally framed metric f-manifold generalize some contact manifolds carrying their dimension to $(2n+s)$. On the other hand, classification is important for Riemannian manifolds with respect to some intrinsic and extrinsic tools as well as all sciences. Moreover, symmetric manifolds play an important role in differential geometry. There are a lot of symmetry type for Riemannian manifolds with respect to different arguments. Under these considerations, in the present paper we study some symmetry conditions on almost S-manifolds. We investigate weak symmetries and φ-symmetries of these type manifolds. We obtain some necessary and sufficient conditions to characterize of their structures. Firstly, we prove that the existence of weakly symmetric and weakly Ricci symmetric almost S-manifolds under some special conditions. Then, we show that every φ-symmetric almost S-manifold verifying the (κ, μ)-nullity distribution is an η-Einstein manifold of globally framed type. Finally, we get some necessary and sufficient condition for a φ-Ricci symmetric almost S-manifold verifying the (κ, μ)-nullity distribution to be an η-Einstein manifold of globally framed type.

Keywords — η-Einstein Manifold, φ-Ricci Symmetric Manifolds, Almost S-Manifold, Globally Framed Metric f-Manifold, S-Structure, Weakly Symmetric Manifold.

1 Introduction

An extensive research about contact geometry is done in recent years. In the present paper, we are concerned with weak symmetries and Ricci symmetries of almost S-manifolds. We recall the price definitions.

Let M be a $(2n+s)$-dimensional manifold and φ is a non-null $(1,1)$-tensor field on M. If φ satisfies

$$\varphi^3 + \varphi = 0,$$

then φ is called an f-structure and M is called an f-manifold [1].

Let M be a $(2n+s)$-dimensional manifold. If there are given on M an f-structure φ, s global vector fields $\xi_1, \ldots, \xi_s$ and 1-forms $\eta^1, \ldots \eta^s$ on M satisfying the

$$\varphi^2 = -Id + \sum_{j=1}^{s} \eta^j \otimes \xi_j,$$

following conditions $\varphi \xi_i = 0, \quad \eta^j \circ \varphi = 0,$ \hfill 1.1)

$$\eta^i(\xi_j) = \delta_j{}^i,$$

for all $i, j \in \{1, \ldots s\}$, then (φ, ξ_i, η^i) is called a glob-

ally framed f-structure and M is called a globally framed f-manifold [2]. On a globally framed f-manifold M, if there exists a Riemannian metric which satisfies

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{k=1}^{s} \eta^k(X)\eta^k(Y),$$

$$g(X, \xi_k) = \eta^k(X),$$

(1.2)

for all vector fields X and Y on M, then M is called a globally framed metric f-manifold [2]. On a globally framed metric f-manifold, fundamental 2-form Φ is defined by $\Phi(X, Y) = g(X, \varphi Y)$ for all vector fiels X, Y [2]. For a globally framed metric f-manifold,

$$N_\varphi + \sum_{k=1}^{s} d\eta^k \otimes \xi_k = 0,$$

is satisfied, then M is called normal globally framed metric f-manifold, where N_φ denotes the Nijenhuis torsion tensor of φ [3].

A globally framed metric f-manifold is said to be an almost S-manifold if and only if $d\eta^1 = \ldots = d\eta^s = \Phi$ and $d\Phi = 0$. An almost S-manifold which is normal is called S-manifold [4].

The study of globally framed metric f-manifold was started by Blair [4], Goldberg and Yano [5], Vanzura [6]. Almost S-structures were studied, without being precisely named, by Cabrerizo et al. [7]. Then Duggal et al. [8] also studied such manifolds and gave them the name almost S-manifold.

On an almost S-manifold, we can define a $(1,1)$ tensor fields

$$h_i := \frac{1}{2} L_{\xi_i} \varphi,$$

for $i = 1, \ldots, s$, where L denotes the Lie derivative in [7]. We use extensively the properties of these tensor fields in the present paper. In particular, these operators are self-adjoint, traceless, anticommute with φ and, we have

$$h_i \xi_j = 0, \quad \eta^i \circ h_j = 0.$$

(1.3)

for $i, j \in \{1, \ldots s\}$ [7]. Moreover, the following identities hold

$$\nabla_X \xi_i - \varphi X - \varphi h_i X,$$

$$\nabla_{\xi_i} \varphi = 0, \quad \nabla_{\xi_i} \xi_j = 0,$$

(1.4)

where ∇ is the Levi Civita connection of g, $X \in \Gamma(TM)$ and $i, j \in \{1, \ldots s\}$ [8]. In 1995, Blair et al. [9] studied contact metric manifolds such that the characteristic vector field belongs to the (κ, μ)-nullity distribution. This concept was generalized by Cappelletti-Montano and Di Terlizzi in [10].

2 Preliminaries

In this section, we give some fundamental informations which we use in the next part from [10].

Definition 2.1. Let M be an almost S-manifold and let κ, μ be real constants. We say that M verifies the (κ, μ)-nullity condition if and only if for each $i \in \{1, \ldots, s\}$ and $X, Y \in \Gamma(TM)$, the following identity holds

$$R(X, Y)\xi_i =$$

$$+ \kappa \left\{ \sum_{k=1}^{s} \eta^k(X)\varphi^2 Y - \eta^k(Y)\varphi^2 X \right\}$$

(2.1)

$$+ \mu \left\{ \sum_{k=1}^{s} \eta^k(Y) h_i X - \eta^k(X) h_i Y \right\}.$$

Lemma 2.1. Let M be an almost S-manifold verifying the (κ, μ)-nullity condition. Then we have

$i)$ $h_i \circ h_j = h_j \circ h_i$ for each $i, j \in \{1, \ldots s\}$,

$ii)$ $\kappa \leq 1$,

$iii)$ if $\kappa < 1$ then for each $i \in \{1, \ldots s\}$, h_i has eigenvalues $0, \pm\sqrt{1 - \kappa}$.

$iv)$ $h_i^2 = (\kappa - 1)\varphi^2$.

Propositions 2.1. Let M be an almost S-manifold verifying the (κ, μ)-nullity condition. Then

$$h_1 = \ldots = h_s.$$

(2.2)

Remark 2.1. Throughout all this paper whenever (2.1) holds we put $h := h_1 = \ldots = h_s$. Then (2.1) becomes

$$R(X,Y)\xi_i =$$
$$+ \kappa \left\{ \sum_{k=1}^{s} \eta^k(X)\varphi^2 Y - \eta^k(Y)\varphi^2 X \right\} \qquad (2.3)$$
$$+ \mu \left\{ \sum_{k=1}^{s} \eta^k(Y)hX - \eta^k(X)hY \right\}.$$

Proposition 2.2. Let M be an almost S-manifold verifying the (κ,μ)-nullity condition. Then M is an S-manifold if and only if $\kappa = 1$.

3 Weakly Symmetric Almost S-manifolds
Firstly, we recall the definition of weakly symmetric manifolds.

Definition 3.1. A non-flat n-dimensional differential manifold (M,g), $n > 3$, is called weakly symmetric if there exists a vector field P and 1-forms $\alpha, \beta, \gamma, \delta$ on M such that
$$(\nabla_X R)(Y,Z)W = \alpha(X)R(Y,Z)W$$
$$+ \beta(Y)R(X,Z)W + \gamma(Z)R(Y,X)W \qquad (3.1)$$
$$+ \delta(W)R(Y,Z)X + g(R(Y,Z)W, X)P,$$
holds for all vector fields $X,Y,Z,W \in \Gamma(TM)$. A weakly symmetric manifold M is pseudo-symmetric if
$$\beta = \gamma = \delta = \frac{1}{2}\alpha \qquad \text{and} \qquad P = A, \qquad \text{where}$$
$g(X,A) = \alpha(A)$ and M is locally symmetric if $\alpha = \beta = \gamma = \delta = 0$ and $P = 0$. A weakly symmetric manifold is said to be proper if at least one of the 1-forms $\alpha, \beta, \gamma, \delta$ are not zero or $P \neq 0$ [11, 12].

From (3.1), an easy calculation shows that if M is weakly symmetric then we have
$$(\nabla_X S)(Z,W) = \alpha(X)S(Z,W)$$
$$+ \beta(R(X,Z)W) + \gamma(Z)S(X,W) \qquad (3.2$$
$$+ \delta(W)S(X,Z) + P(R(X,W)Z),$$
where P is defined by $\rho(X) = g(X,P)$ for any vector field $X \in \Gamma(TM)$ [11].

Now, we consider this definition for almost S-manifold verifying the (κ,μ)-nullity condition.

Theorem 3.1. There exists no weakly symmetric $(2n+s)$

-dimensional almost S-manifold M verifying the (κ,μ)-nullity condition, if
$$\alpha(X) + \gamma(X) + \delta(X) \neq \frac{X(\kappa)}{\kappa},$$
$X \in \Gamma(TM)$, where κ is a smooth function and $\dim(M) > 3$.

Proof. Assume that M is a weakly symmetric almost S-manifold verifying the (κ,μ)-nullity condition. Putting $W = \xi_i$ in (3.2), then we get
$$(\nabla_X S)(Z,\xi_i) = \alpha(X)S(Z,\xi_i)$$
$$+ \beta(R(X,Z)\xi_i) + \gamma(Z)S(X,\xi_i) \qquad (3.3)$$
$$+ \delta(\xi_i)S(X,Z) + P(R(X,\xi_i)Z).$$

By using $Q\xi_i = 2n\kappa \sum_{k=1}^{s} \xi_k$ then we conclude
$$(\nabla_X S)(Z,\xi_i) = 2n\kappa\alpha(X)\sum_{k=1}^{s} \eta^k(Z)$$
$$+ \beta(R(X,Z)\xi_i) + 2n\kappa\gamma(Z)\sum_{k=1}^{s} \eta^k(X) \qquad (3.4)$$
$$+ \delta(\xi_i)S(X,Z) + P(R(X,\xi_i)Z).$$

By the covariant differentiation of the Ricci tensor S, the left side of the above equation can be written as
$$(\nabla_X S)(Z,\xi_i) = \nabla_X S(Z,\xi_i)$$
$$- S(\nabla_X Z,\xi_i) - S(Z,\nabla_X \xi_i).$$

Using (1.4) and $Q\xi_i = 2n\kappa \sum_{k=1}^{s} \xi_k$ and in view of the parallelity of the metric tensor g, then we derive
$$(\nabla_X S)(Z,\xi_i) = 2nX(\kappa)\sum_{k=1}^{s} \eta^k(Z)$$
$$- 2n\kappa g(Z,\varphi X) - 2n\kappa g(Z,\varphi hX) \qquad (3.5)$$
$$+ S(Z,\varphi X) + S(Z,\varphi hX).$$

Comparing the right hand sides of the (3.4) and (3.5), we see that

$$2nX(\kappa)\sum_{k=1}^{s}\eta^{k}(Z) - 2n\kappa g(Z,\varphi X)$$

$$- 2n\kappa g(Z,\varphi h X) + S(Z,\varphi X)$$

$$+ S(Z,\varphi h X) = 2n\kappa\alpha(X)\sum_{k=1}^{s}\eta^{k}(Z) \qquad (3.6)$$

$$+ \beta(R(X,Z)\xi_{i}) + 2n\kappa\gamma(Z)\sum_{k=1}^{s}\eta^{k}(X)$$

$$+ \delta(\xi_{i})S(X,Z) + P(R(X,\xi_{i})Z).$$

Now, putting $X = Z = \xi_{i}$ in (3.6) and using (1.1), (1.2) and $Q\xi_{i} = 2n\kappa\sum_{k=1}^{s}\xi_{k}$, then we get

$$2n\xi_{i}(\kappa) = 2n\kappa\alpha(\xi_{i}) + 2n\kappa\gamma(\xi_{i}) + 2n\kappa\delta(\xi_{i}).$$

Since $\dim M > 3$ and $\kappa \neq 0$, then we have

$$\alpha(\xi_{i}) + \gamma(\xi_{i}) + \delta(\xi_{i}) = \frac{\xi_{i}(\kappa)}{\kappa}. \qquad (3.7)$$

Now, we will show that

$$\alpha(X) + \gamma(X) + \delta(X) = \frac{X(\kappa)}{\kappa}$$

for any vector field X on M.

Setting $Z = \xi_{i}$ in (3.2), similar to the previous calculations, it follows that

$$2nX(\kappa)\sum_{k=1}^{s}\eta^{k}(W) - 2n\kappa g(W,\varphi X)$$

$$- 2n\kappa g(W,\varphi h X) + S(W,\varphi X)$$

$$+ S(W,\varphi h X) = 2n\kappa\alpha(X)\sum_{k=1}^{s}\eta^{k}(W) \qquad (3.8)$$

$$+ \beta(R(X,\xi_{i})W) + \gamma(\xi_{i})S(X,W)$$

$$+ 2n\kappa\delta(W)\sum_{k=1}^{s}\eta^{k}(X) + P(R(X,W)\xi_{i}).$$

Replacing W by ξ_{i} in (3.8) and using (1.1) and

$Q\xi_{i} = 2n\kappa\sum_{k=1}^{s}\xi_{k}$, then we obtain

$$2nX(\kappa) = 2n\kappa\alpha(X)$$

$$+ \beta(R(X,\xi_{i})\xi_{i}) + 2n\kappa\gamma(\xi_{i})\sum_{k=1}^{s}\eta^{k}(X) \qquad (3.9)$$

$$+ 2n\kappa\delta(\xi_{i})\sum_{k=1}^{s}\eta^{k}(X) + P(R(X,\xi_{i})\xi_{i}).$$

Again choosing $X = \xi_{i}$ in (3.8) and by making use of (1.1), (2.3) and $Q\xi_{i} = 2n\kappa\sum_{k=1}^{s}\xi_{k}$, then we deduce

$$2n\xi_{i}(\kappa)\sum_{k=1}^{s}\eta^{k}(W) = 2n\kappa\alpha(\xi_{i})\sum_{k=1}^{s}\eta^{k}(W)$$

$$+ 2nk\gamma(\xi_{i})\sum_{k=1}^{s}\eta^{k}(W) + 2n\kappa\delta(W) \qquad (3.10)$$

$$+ P(R(\xi_{i},W)\xi_{i}).$$

Replacing W by X in (3.10) and taking summation with (3.10), in view of (3.7)

$$2nX(\kappa) + 2n\xi_{i}(\kappa)\sum_{k=1}^{s}\eta^{k}(X) = 2n\kappa\alpha(X)$$

$$+ \beta(R(X,\xi_{i})\xi_{i}) + 2n\kappa\gamma(\xi_{i})\sum_{k=1}^{s}\eta^{k}(X)$$

$$+ 2n\kappa\delta(\xi_{i})\sum_{k=1}^{s}\eta^{k}(X) + 2n\kappa\alpha(\xi_{i})\sum_{k=1}^{s}\eta^{k}(X) \qquad (3.11)$$

$$+ 2nk\gamma(\xi_{i})\sum_{k=1}^{s}\eta^{k}(X) + 2n\kappa\delta(X).$$

Setting $X = \xi_{i}$ in (3.6), then we have

$$2n\xi_{i}(\kappa)\sum_{k=1}^{s}\eta^{k}(Z) = 2n\kappa\alpha(\xi_{i})\sum_{k=1}^{s}\eta^{k}(Z)$$

$$+ \beta(R(\xi_{i},Z)\xi_{i}) + 2n\kappa\delta(\xi_{i})\sum_{k=1}^{s}\eta^{k}(Z) \qquad (3.12)$$

$$+ 2n\kappa\gamma(Z).$$

By replacing Z with X in (3.12) and taking summation with (3.11) and using (3.7), then we conclude

$$2n\kappa\{\alpha(X) + \gamma(X) + \delta(X)\} = 2nX(\kappa).$$

Since $\dim M > 3$ and $\kappa \neq 0$, then we get

$$\alpha(X) + \gamma(X) + \delta(X) = \frac{X(\kappa)}{\kappa},$$

which completes the proof.

Corollary 3.1. There exists no weakly symmetric almost S-manifolds M verifying (κ,μ)-nullity distribution, if $\alpha + \gamma + \delta \neq 0$, where κ is a constant and $\dim M > 3$.

4 Weakly Ricci Symmetric Almost S-manifolds

Firstly, we recall the definition of weakly Ricci symmetric manifold and then we consider this definition for almost S

-manifolds.

Definition 4.1. An n-dimensional differential manifold (M, g), $n > 3$, is called weakly Ricci symmetric if there exist 1-forms $\varepsilon, \sigma, \rho$ on M such that

$$(\nabla_X S)(Y, Z) = \varepsilon(X)S(Y, Z) + \sigma(Y)S(X, Z) + \rho(Z)S(X, Y) \quad (4.1)$$

holds for all vector fields $X, Y, Z \in \Gamma(TM)$ [12, 13]. If $\varepsilon = \sigma = \rho$, then M is called pseudo Ricci symmetric [14, 15].

Theorem 4.1. There exists no weakly Ricci symmetric almost S-manifold M verifying the (κ, μ)-nullity condition,

if $\varepsilon(X) + \sigma(X) + \rho(X) \neq \dfrac{X(\kappa)}{\kappa}$,

$X \in \Gamma(TM)$, where κ is a smooth function and $\dim(M) > 3$.

Proof. Let M be a weakly Ricci symmetric almost S-manifold verifying the (κ, μ)-nullity condition. Putting $Z = \xi_i$ in (4.1) and using $Q\xi_i = 2n\kappa \sum_{k=1}^{s} \xi_k$, then we obtain

$$(\nabla_X S)(Y, \xi_i) = 2n\kappa\varepsilon(X)\sum_{k=1}^{s}\eta^k(Y)$$
$$+ 2n\kappa\sigma(Y)\sum_{k=1}^{s}\eta^k(X) + \rho(\xi_i)S(X, Y). \quad (4.2)$$

Replacing Z by Y in (3.5) and comparing the right hand sides of the equations (4.2) and (3.5), then we get

$$2nX(\kappa)\sum_{k=1}^{s}\eta^k(Y) - 2n\kappa g(Y, \varphi X)$$
$$- 2n\kappa g(Y, \varphi hX) + S(Y, \varphi X)$$
$$+ S(Y, \varphi hX) = 2n\kappa\varepsilon(X)\sum_{k=1}^{s}\eta^k(Y) \quad (4.3)$$
$$+ 2n\kappa\sigma(Y)\sum_{k=1}^{s}\eta^k(X) + \rho(\xi_i)S(X, Y).$$

Taking $X = Z = \xi_i$ in (4.3) and using (1.1), (1.2) and $Q\xi_i = 2n\kappa \sum_{k=1}^{s} \xi_k$ then we have

$$2n\kappa\{\varepsilon(\xi_i) + \sigma(\xi_i) + \rho((\xi_i))\} = 2n\xi_i(\kappa).$$

Since $\dim M > 3$ and $\kappa \neq 0$, we derive

$$\varepsilon(\xi_i) + \sigma(\xi_i) + \rho((\xi_i)) = \dfrac{\xi_i(\kappa)}{\kappa}. \quad (4.4)$$

Putting $X = \xi_i$ in (4.3), then we get

$$2n\xi_i(\kappa)\sum_{k=1}^{s}\eta^k(Y) = 2n\kappa\varepsilon(\xi_i)\sum_{k=1}^{s}\eta^k(Y)$$
$$+ 2n\kappa\sigma(Y) + 2n\kappa\rho(\xi_i)\sum_{k=1}^{s}\eta^k(Y).$$

Thus, from (4.4), this yields

$$2n\xi_i(\kappa)\sum_{k=1}^{s}\eta^k(Y) = 2n\kappa\left\{\varepsilon(\xi_i)\sum_{k=1}^{s}\eta^k(Y)\right.$$
$$\left. + \sigma(Y) + \rho(\xi_i)\sum_{k=1}^{s}\eta^k(Y)\right\},$$

which gives us

$$\sigma(Y) = \sigma(\xi_i)\sum_{k=1}^{s}\eta^k(Y). \quad (4.5)$$

Similarly, setting $Y = \xi_i$ in (4.3), we also obtain

$$2nX(\kappa) = 2n\kappa\varepsilon(X)$$
$$+ 2n\kappa\sigma(\xi_i)\sum_{k=1}^{s}\eta^k(X)$$
$$+ 2n\kappa\rho(\xi_i)\sum_{k=1}^{s}\eta^k(X).$$

Applying (4.4) into the last equation, then we deduce

$$\varepsilon(X) = \dfrac{X(\kappa)}{\kappa} - \sum_{k=1}^{s}\eta^k(X)\left\{\dfrac{\xi_i(\kappa)}{\kappa} - \varepsilon(\xi_i)\right\}. \quad (4.6)$$

Takiing $X = Y = \xi_i$ in (4.1), it follows that

$$2n\xi_i(\kappa)\sum_{k=1}^{s}\eta^k(Z) = \quad (4.7)$$
$$2n\kappa\{\varepsilon(\xi_i) + \sigma(\xi_i) + \rho(Z)\}.$$

By making use of (4.4), the equation (4.7) reduces to

$$\rho(Z) = \sum_{k=1}^{s}\eta^k(Z)\rho(\xi_i). \quad (4.8)$$

Replacing Y by X, Z by X in (4.5) and in (4.8), respectively and taking summation with (4.6), then we have

$$\varepsilon(X) + \sigma(X) + \rho(X) = \dfrac{X(\kappa)}{\kappa},$$

for all vector field X. Hence, the proof is completed.

Corollary 4.1. There exists no weakly Ricci symmetric almost S-manifolds M verifying (κ,μ)-nullity distribution, if $\varepsilon+\sigma+\rho\neq 0$, where κ is a constant and $\dim M>3$.

5 φ-symmetric Almost S-manifolds

Definition 5.1. An almost S-manifold M, which verifies the (κ,μ)-nullity condition, is said to be locally φ-symmetric if the

$$\varphi^2\big(\nabla_W R(X,Y)Z\big)=0, \tag{5.1}$$

for all vector fields X,Y,Z,W orthogonal to ξ_i, for each $i\in\{1,\ldots,s\}$. If the equation (5.1) is satisfied for arbitrary vector fields X,Y,Z,W, then M is said to be φ-symmetric.

These notions were introduced for Sasakian manifolds by Takahashi in [11].

Theorem 5.1. A φ-symmetric almost S-manifold M verifying the (κ,μ)-nullity condition is an η-Einstein manifold of globally framed type.

Proof. By using (1.1) and (5.1), then we get

$$-\big(\nabla_W R\big)(X,Y)Z$$
$$+\sum_{k=1}^{s}\eta^k\big((\nabla_W R)(X,Y)Z\big)\xi_k=0, \tag{5.2}$$

from which it follows that

$$-g\big((\nabla_W R)(X,Y)Z,U\big)$$
$$+\sum_{k=1}^{s}\eta^k\big((\nabla_W R)(X,Y)Z\big)\eta^k(U)=0, \tag{5.3}$$

for any vector field U on M. Let

$$\{e_1,\ldots,e_{2n},e_{2n+1}=\xi_1,\ldots,e_{2n+s}=\xi_s\}$$

be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X=U=e_i$ in (5.3) and taking summation over $1\le i\le 2n+s$, then we conclude

$$-\big(\nabla_W S\big)(Y,Z)$$
$$+\sum_{j=1}^{2n+s}\eta^j\big((\nabla_W R)(e_i,Y)Z\big)\eta^j(e_i)=0. \tag{5.4}$$

The second term of (5.4) by putting $Z=\xi_j$ takes the form

$$\eta^j\big((\nabla_W R)(e_i,Y)\xi_k\big)\eta^j(e_i)$$
$$=g\big((\nabla_W R)(e_i,Y)\xi_j,\xi_j\big)g(e_i,\xi_j), \tag{5.5}$$

which vanishes identically. On the other hand, we have

$$g\big((\nabla_W R)(e_i,Y)\xi_j,\xi_j\big)=$$
$$+g\big(\nabla_W R(e_i,Y)\xi_j,\xi_j\big)$$
$$-g\big(R(\nabla_W e_i,Y)\xi_j,\xi_j\big)$$
$$-g\big(R(e_i,\nabla_W Y)\xi_j,\xi_j\big)$$
$$-g\big(R(e_i,Y)\nabla_W\xi_j,\xi_j\big), \tag{5.6}$$

at $p\in M$. Since the basis is orthonormal, thus we have $\nabla_W e_i=0$ at $p\in M$. Using (1.1), (1.3) and (2.3), then we get

$$g\big(R(e_i,\nabla_W Y)\xi_j,\xi_j\big)=0. \tag{5.7}$$

Taking into account of (5.7) in (5.6), then we derive

$$g\big((\nabla_W R)(e_i,Y)\xi_j,\xi_j\big)=$$
$$+g\big(\nabla_W R(e_i,Y)\xi_j,\xi_j\big)$$
$$-g\big(R(e_i,Y)\nabla_W\xi_j,\xi_j\big), \tag{5.8}$$

Since $\quad g\big(R(e_i,Y)\xi_j,\xi_j\big)=-g\big(R(\xi_j,\xi_j)Y,e_i\big)=0$

hence we get

$$g\big(\nabla_W R(e_i,Y)\xi_j,\xi_j\big)+g\big(R(e_i,Y)\xi_j,\nabla_W\xi_j\big)=0.$$

Using the last equation in (5.8), then we have

$$g\big((\nabla_W R)(e_i,Y)\xi_j,\xi_j\big)=-g\big(R(e_i,Y)\xi_j,\nabla_W\xi_j\big)$$
$$-g\big(R(e_i,Y)\nabla_W\xi_j,\xi_j\big).$$

From (1.4) and first Bianchi identity, we obtain

$$g\big((\nabla_W R)(e_i,Y)\xi_j,\xi_j\big)=g\big(R(e_i,Y)\xi_j,\varphi W\big)$$
$$+g\big(R(e_i,Y)\xi_j,\varphi hW\big)+g\big(R(e_i,Y)\varphi W,\xi_j\big)$$
$$+g\big(R(e_i,Y)\varphi hW,\xi_j\big)=0,$$

i.e.,

$$g\big((\nabla_W R)(e_i,Y)\xi_j,\xi_j\big)=0. \tag{5.9}$$

Using (5.9) in (5.4), then we deduce

$$\big(\nabla_W S\big)(Y,\xi_j)=0. \tag{5.10}$$

It is well-known that

$$\big(\nabla_W S\big)(Y,\xi_j)=\nabla_W S(Y,\xi_j)$$
$$-S(\nabla_W Y,\xi_j)-S(Y,\nabla_W\xi_j).$$

From (3.5), it follows that

$$\big(\nabla_W S\big)(Y,\xi_i)=2nW(\kappa)\sum_{k=1}^{s}\eta^k(Y)$$
$$-2n\kappa g(Y,\varphi W)-2n\kappa g(Y,\varphi hW)$$
$$+S(Y,\varphi W)+S(Y,\varphi hW). \tag{5.11}$$

Replacing Y by φY in (5.11), then we have

$$S(Y,W) = \{s(2(n-1)+\mu)(\kappa-1)g(Y,W)\}$$
$$+ 4n\kappa\sum_{k=1}^{s}\eta^k(Y)\eta^k(W). \quad (5.12)$$

This completes the proof.

6 φ-Ricci symmetric Almost S-manifolds

Definition 6.1. An almost S-manifold M verifying the (κ,μ)-nullity condition is said to be φ-Ricci symmetric if the Ricci operator Q satisfies

$$\varphi^2(\nabla_X Q)Y = 0, \quad (6.1)\text{ or}$$

any vector fields X and Y on M and $S(X,Y) = g(QX,Y)$. If X and Y are orthogonal to ξ_i for each $i \in \{1,\ldots,s\}$, then M is called locally φ-Ricci symmetric.

Now, we give the definition of Einstein manifold of globally framed type.

Definition 6.2. An almost S-manifold M verifying the (κ,μ)-nullity condition is said to be η-Einstein manifold of globally framed type if its Ricci tensor S is of the form

$$S(X,Y) = Ag(X,Y) + B\sum_{k=1}^{s}\eta^k(X)\eta^k(Y),$$

where A and B are smooth function and X and Y are vector fields on M.

Theorem 6.1. An almost S-manifold M verifying the (κ,μ)-nullity condition is an η-Einstein manifold of globally framed type, if

$$\mu = 2\left(1 - \frac{1}{n} + \frac{\kappa}{s}\right) \text{ or the } (1,1)\text{-tensor field } h$$

vanishes identically.

Proof. In view of the assumption, using (1.1) in (6.1), then we have

$$-(\nabla_X Q)Y + \sum_{k=1}^{s}\eta^k((\nabla_X Q)Y)\xi_k = 0. \quad (6.2)$$

Taking the inner product of (6.2) with Z, then we obtain

$$-g((\nabla_X Q)Y,Z) + \sum_{k=1}^{s}\eta^k((\nabla_X Q)Y)\eta^k(Z) = 0,$$

which on simplifying gives us

$$-g(\nabla_X QY,Z) + S(\nabla_X Y,Z)$$
$$+ \sum_{k=1}^{s}\eta^k((\nabla_X Q)Y)\eta^k(Z) = 0. \quad (6.3)$$

Taking $Y = \xi_i$ in (6.3), then we derive

$$-g(\nabla_X Q\xi_i,Z) + S(\nabla_X \xi_i,Z)$$
$$+ \sum_{k=1}^{s}\eta^k((\nabla_X Q)\xi_i)\eta^k(Z) = 0. \quad (6.4)$$

Using (1.4) in (6.4), then we get

$$-2nX(\kappa)\sum_{k=1}^{s}\eta^k(Z) + 2n\kappa g(Z,\varphi X)$$
$$-2n\kappa g(Z,\varphi hX) - S(Z,\varphi X) \quad (6.5)$$
$$-S(Z,\varphi hX) + \sum_{k=1}^{s}\eta^k((\nabla_X Q)\xi_i)\eta^k(Z) = 0.$$

Replacing Z by φZ in (6.5) and using

$$S(\varphi X,\varphi Z) = -S(X,Z) + 2n\kappa\sum_{k=1}^{s}\eta^k(X)\eta^k(Z),$$

then we conclude

$$S(X,Z) = -\{2n\kappa - s(2(1-n)+\mu n)\}g(hX,Z)$$
$$-\{2s(2(n-1)+\mu)(1-\kappa)+2n\kappa\}g(X,Z)$$
$$+\{4n\kappa + s(2(1-n)+\mu n)\}\sum_{k=1}^{s}\eta^k(X)\eta^k(Z).$$

Thus, we complete the proof.

7 Conclusion
Symmetric manifolds play an important role in differential geometry. Thus, classification of manifolds with respect to symmetry properties helps the researcher for their advanced studies. We prove some existence theorems for some weakly symmetric manifolds. We give a characterization of φ-symmetric spaces.

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