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On Some Properties of Bihyperbolic Numbers of The Lucas Type

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Abstract
To date, many authors in the literature have worked on special arrays in various computational systems. In this article, Lucas type bihyperbolic numbers were defined and their algebraic properties were examined. Bihyperbolic Lucas numbers were studied by Azak in 2021. Therefore, we only examined bihyperbolic Jacobsthal-Lucas and Pell-Lucas numbers. We also gave properties of bihyperbolic Jacobstal-Lucas and bihyperbolic Pell-Lucas numbers such as recursion relation, derivation function, Binet formula, D'Ocagne identity, Cassini identity and Catalan identity.

Keywords: Bihyperbolic number, Bihyperbolic Jacobsthal-Lucas number, Bihyperbolic Pell-Lucas number, Jacobsthal-Lucas number, Lucas number, Pell-Lucas number
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1. Introduction

In 1843, Hamilton [1] discovered quaternions. After Hamilton's discovery of real quaternions Cockle [2] revealed the tessarine numbers in 1848. The difference between Tessarine numbers and quaternions is that they have the property of change. The quaternions are not commutative. After Cockle's work on Tessarines in 1892, Segre [3] obtained bicomplex numbers by replacing the quaternions found by Hamilton and Clifford with complex numbers with real coefficients and formed an isomorphic algebra with Tessarine numbers. With the discovery of bicomplex numbers, a new number system has been found which is called a system of real Tesssarines and defined as follows

$$\{a+jc \mid a,c\in\mathbb{R},\ j^2=1, j\notin\mathbb{R}\}$$

The real Tessarine numbers are called hyperbolic numbers [6]. These new numbers are called generalized commutative hypercomplex numbers as follows

$$\{q=q_0+iq_1+jq_2+kq_3 \mid q_0,q_1,q_2,q_3\in\mathbb{R}\}$$

where

$$i^2=k^2=\alpha,\ j^2=1,\ ij=ji=k.$$

These new numbers are called elliptic, parabolic, or hyperbolic commutative quaternion, respectively, according to which alpha is $\alpha < 0, \alpha = 0$ or $\alpha > 0$. In particular, bicomplex numbers are the $\alpha = -1$ case. The bicomplex numbers are generalized by Catoni et al.[4].
In [5], Price introduced the set of bicomplex numbers, which can be represented as

$$\mathbb{BC} = \{ q = (q_1 + i q_2) + j (q_3 + i q_4) \mid q_1, q_2, q_3, q_4 \in \mathbb{R} \}$$

where

$$i^2 = -1, \; j^2 = -1, \; i j = j i.$$

Recently, many authors have considered special number sequences with different number systems.
The bihyperbolic numbers are numbers that can be written as a linear combination of pairs of hyperbolic number. These numbers allow to establish a connection between bicomplex numbers and Euclidean 4-space. In 2008, Pogorui et al.[6] bihyperbolic numbers set is defined by as follows

$$\mathscr{B}h = \{ q = a_0 + a_1 j_1 + a_2 j_2 + a_3 j_3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}; j_1, j_2, j_3 \notin \mathbb{R} \}$$

where j_1, j_2 and j_3 satisfy the conditions

$$j_1^2 = j_2^2 = j_3^2 = 1, \; j_1 j_2 = j_2 j_1 = j_3, \, j_1 j_3 = j_3 j_1 = j_2, \, j_2 j_3 = j_3 j_2 = j_1.$$

The addition and subtraction of two bihyperbolic numbers can be expressed as follows:

$$\begin{aligned} q \pm r = & \; (a_0 + a_1 j_1 + a_2 j_2 + a_3 j_3) \pm (b_0 + b_1 j_1 + b_2 j_2 + b_3 j_3) \\ = & \; (a_0 \pm b_0) + j_1 (a_1 \pm b_1) + j_2 (a_2 \pm b_2) + j_3 (a_3 \pm b_3) \end{aligned}$$

The multiplication of two bihyperbolic numbers can be expressed as follows:

$$\begin{aligned} q \times r = & \; (a_0 + a_1 j_1 + a_2 j_2 + a_3 j_3) \times (b_0 + b_1 j_1 + b_2 j_2 + b_3 j_3) \\ = & \; (a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3) + j_1 (a_0 b_1 + a_1 b_0 + a_2 b_3 + a_3 b_2) \\ & + j_2 (a_0 b_2 + a_1 b_3 + a_2 b_0 + a_3 b_1) + j_3 (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0) \end{aligned}$$

Bihyperbolic numbers have three different conjugations and represented as follows:

$$\bar{q}^{j_1} = a_0 + j_1 a_1 - j_2 a_2 - j_3 a_3,$$

$$\bar{q}^{j_2} = a_0 - j_1 a_1 + j_2 a_2 - j_3 a_3,$$

$$\bar{q}^{j_3} = a_0 - j_1 a_1 - j_2 a_2 + j_3 a_3.$$

In 2002, Olariu [7] introduced commutative hypercomplex numbers of different dimensions, and in his book he called these numbers in 4-dimensional circular fourcomplex numbers or hyperbolic fourcomplex numbers if $\alpha = -1$ or $\alpha = 1$, respectively. In 2008, hyperbolic fourcomplex numbers are called bihyperbolic numbers by Pogorui et al.[6] and they studied the roots of bihyperbolic polynomials. In 2020, the algebraic properties of these numbers were studied by Bilgin and Ersoy [8]. In 2021, Gürses et al. have studied dual-generalized complex and hyperbolic generalized complex numbers. Moreover, for $J = j$ and $p = 1$, they have obtained bihyperbolic numbers [9]. In 2021, [10] Brod et al. have introduced identities and summation formulas of bihyperbolic Fibonacci, Pell and Jacobsthal numbers as follows:

$$\mathscr{B}hF_n = F_n + F_{n+1} \, j_1 + F_{n+2} \, j_2 + F_{n+3} \, j_3 \,,$$

$$\mathscr{B}hJ_n = J_n + J_{n+1} \, j_1 + J_{n+2} \, j_2 + J_{n+3} \, j_3 \,,$$

$$\mathscr{B}hP_n = P_n + P_{n+1} \, j_1 + P_{n+2} \, j_2 + P_{n+3} \, j_3 \,,$$

where j_1, j_2 and j_3 satisfy the conditions

$$j_1^2 = j_2^2 = j_3^2 = 1, \; j_1 j_2 = j_2 j_1 = j_3, \, j_1 j_3 = j_3 j_1 = j_2, \, j_2 j_3 = j_3 j_2 = j_1 \,.$$

and in [11] Brod et al. were studied on a new generalization of bihyperbolic Pell numbers.
In 2021,[12] Azak defined bihyperbolic Lucas and bihyperbolic generalized Fibonacci numbers and given some new identities of these numbers as follows:

$$\mathscr{B}hL_n = L_n + L_{n+1}\, j_1 + L_{n+2}\, j_2 + L_{n+3}\, j_3$$

where j_1, j_2 and j_3 satisfy the conditions

$$j_1^2 = j_2^2 = j_3^2 = 1,\ j_1 j_2 = j_2 j_1 = j_3,\ j_1 j_3 = j_3 j_1 = j_2,\ j_2 j_3 = j_3 j_2 = j_1.$$

In 2022, Szynal-Liana et al.[13] introduced on certain bihypernomials related to Pell and Pell-Lucas numbers. In 2023, Gökbaş [24] introduced Gaussian-bihyperbolic numbers containing Pell and Pell-Lucas numbers.

In 1996, Horadam [14] introduced the Jacobsthal and Jacobsthal-Lucas sequences recurrence relation $\{J_n\}$ and $\{j_n\}$ are defined by the recurrence relations

$$J_0 = 0,\ \ J_1 = 1,\ \ J_n = J_{n-1} + 2J_{n-2},\ \ for\ n \geq 2,,$$

$$j_0 = 2,\ \ j_1 = 1,\ \ j_n = j_{n-1} + 2\,j_{n-2},\ \ for\ n \geq 2 \tag{1.1}$$

respectively.

In 1996, [14] Horadam studied on the Jacobsthal and Jacobsthal-Lucas sequences and in 1997, [15] he gave Cassini-like formulas as follows

$$J_{n+1}J_{n-1} - {J_n}^2 = (-1)^n.2^{n-1}, \tag{1.2}$$

$$j_{n+1}j_{n-1} - {j_n}^2 = 3^2.(-1)^{n+1}.2^{n-1}.$$

The first eleven terms of Jacobsthal sequence $\{J_n\}$ are $\{0,1,1,3,5,11,21,43,85,171,341\}$.
This sequence is given by the formula

$$J_n = \frac{2^n - (-1)^n}{3}. \tag{1.3}$$

The first eleven terms of Jacobsthal-Lucas sequence $\{j_n\}$ are $\{2,1,5,7,17,31,65,127,257,511,1025\}$.
This sequence is given by the formula

$$j_n = 2^n + (-1)^n.$$

Besides the n-th Jacobsthal and Jacobsthal-Lucas number are formulized as $J_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $j_n = \alpha^n + \beta^n$, where $\alpha = 2$, $\beta = -1$.

Also, for Jacobsthal and Jacobsthal-Lucas numbers the following properties hold:

$$\begin{cases} J_n + j_n = 2J_{n+1}, \\ 3J_n + j_n = 2^{n+1}, \\ j_n J_n = J_{2n}, \\ J_m j_n + J_n j_m = 2J_{m+n}, \\ J_m j_n - J_n j_m = (-1)^n 2^{n+1} J_{m-n}, \\ j_{n+1} + j_n = 3(J_{n+1} + J_n) = 3.2^n, \\ j_n J_{m+1} + 2j_{n-1} J_m = j_{m+n}, \\ j_{n+1} - j_n = 3(J_{n+1} - J_n) + 4(-1)^{n+1} = 2^n + 2(-1)^{n+1}, \\ j_{n+r} - j_{n-r} = 3(J_{n+r} - J_{n-r}) = 2^{n-r}(2^{2r} - 1), \\ j_{n+r} + j_{n-r} = 3(J_{n+r} + J_{n-r}) + 4(-1)^{n-r}. \end{cases}$$

and summation formulas

$$\begin{cases} \sum\limits_{i=2}^{n} J_i = \frac{J_{n+2}-3}{2}, \\ \sum\limits_{i=1}^{n} j_i = \frac{j_{n+2}-5}{2}. \end{cases}$$

In 2018, [16] Torunbalcı Aydın were studied on the generalizations of the Jacobsthal sequence. In 2018, [17] gave a new generalization for Jacobsthal and Jacobsthal-Lucas sequences. In 2019, [18] Al-Kateeb gave a generalization of the Jacobsthal and Jacobsthal-Lucas numbers. In 2022, [19] Brod et al. were studied on generalized Jacobsthal and Jacobsthal-Lucas numbers.

In 1971, [20] Horadam studied on the Pell P_n and Pell-Lucas p_n sequences and Pell identities. The n-th Pell and n-th Pell-Lucas numbers is defined by respectively as follows

$$P_n = 2P_{n-1} + P_{n-2}\ , P_0 = 0, P_1 = 1,$$

$$p_n = 2p_{n-1} + p_{n-2}\ , p_0 = 2, p_1 = 2.$$

In 1985, Horadam and Mahon obtained some Pell P_n and Pell-Lucas p_n identities and summation formulas respectively as follows [21]

$$\begin{cases} P_{m-1}\, p_n + P_m\, p_{n+1} = p_{m+n}\, , \\ p_{n+1}\, p_{n-1} - p_n^2 = 8\,(-1)^{n+1}\, , \\ p_m\, p_n - p_{m+r}\, p_{n-r} = 8\,(-1)^{n-r+1}\, P_{m+r-n}\, P_r\, . \end{cases}$$

$$\begin{cases} \sum\limits_{r=1}^{n} p_r = \frac{(p_{n+1}+p_n-4)}{2}\, , \\ \sum\limits_{r=1}^{n} p_{2r} = \frac{(p_{2n+1}-2)}{2}\, , \\ \sum\limits_{r=1}^{n} p_{2r-1} = \frac{(p_{2n}-2)}{2}\, . \end{cases}$$

Besides the n-th Pell and Pell-Lucas number are formulized as $P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $p_n = \alpha^n + \beta^n$, where $\alpha = 1 + \sqrt{2}$, $\beta = 1 - \sqrt{2}$.

In 2006, some properties of sums involving Pell numbers were studied by Santana Falcon [22]. In 2018, Torunbalcı Aydın introduced bicomplex Pell and Pell-Lucas numbers [25].

Our subject of study is the combinatorial properties of bihyperbolic numbers of Lucas type, but since the article on bihyperbolic Lucas numbers was previously reviewed by Azak [6], only bihyperbolic Jacobsthal-Lucas and bihyperbolic Pell-Lucas numbers were examined in this study.

2. The Bihyperbolic Jacobsthal-Lucas Numbers

In this section, we define the bihyperbolic Jacobsthal-Lucas numbers. Then, we obtain the generating function, Binet's formula, d'Ocagne's identity, Cassini's identity, Catalan's identity and Honsberger identity.

Definition.2.1. For $n \geq 1$, the n-th bihyperbolic Jacobsthal-Lucas number $\mathscr{B}hj_n$ are defined by using the Jacobsthal-Lucas numbers as follows

$$\mathscr{B}hj_n = j_n + j_{n+1}\, j_1 + j_{n+2}\, j_2 + j_{n+3}\, j_3, \tag{2.1}$$

where j_1, j_2 and j_3 satisfy the conditions

$$j_1^2 = j_2^2 = j_3^2 = 1,\;\; j_1 j_2 = j_2 j_1 = j_3,\, j_1 j_3 = j_3 j_1 = j_2,\, j_2 j_3 = j_3 j_2 = j_1.$$

Theorem 2.1. *Let $\mathscr{B}hj_n$ be the n-th bihyperbolic Jacobsthal-Lucas number. For any integer $n \geq 0$,*

$$\mathscr{B}hj_n = \mathscr{B}hj_{n-1} + 2\,\mathscr{B}hj_{n-2} \tag{2.2}$$

Proof. (2.2): By using Eq.(1.1) in Eq.(2.1) we obtain that,

$$\begin{aligned}
\mathscr{B}hj_n &= j_n + j_{n+1}\, j_1 + j_{n+2}\, j_2 + j_{n+3}\, j_3 \\
&= (j_{n-1} + 2\, j_{n-2}) + j_1\,(j_n + 2\, j_{n-1}) \\
&\quad + j_2\,(j_{n+1} + 2\, j_n) + j_3\,(j_{n+2} + 2\, j_{n+1}) \\
&= (j_{n-1} + j_1\, j_n + j_2\, j_{n+1} + j_3\, j_{n+2}) \\
&\quad + 2\,(j_{n-2} + j_1\, j_{n-1} + j_2\, j_n + j_3\, j_{n+1}) \\
&= \mathscr{B}hj_{n-1} + 2\,\mathscr{B}hj_{n-2}
\end{aligned}$$

Also, initial values are $\mathscr{B}hj_0 = 2 + j_1 + 5\, j_2 + 7\, j_3,\;\; \mathscr{B}hj_1 = 1 + 5\, j_1 + 7\, j_2 + 17\, j_3$.

Let $\mathscr{B}hj_n$ and $\mathscr{B}hj_m$ be two bihyperbolic Jacobsthal-Lucas numbers such that

$$\mathscr{B}hj_n = j_n + j_1\, j_{n+1} + j_2\, j_{n+2} + j_3\, j_{n+3}$$

and

$$\mathscr{B}hj_m = j_m + j_1\, j_{m+1} + j_2\, j_{m+2} + j_3\, j\mathscr{B}hj_{m+3}$$

Then, the addition and subtraction of two bihyperbolic Jacobsthal-Lucas numbers are defined in the obvious way,

$$\begin{aligned}
\mathscr{B}hj_n \pm \mathscr{B}hj_m = &\quad (j_n + j_1\, j_{n+1} + j_2\, j_{n+2} + j_3\, j_{n+3}) \\
&\quad \pm (j_m + j_1\, j_{m+1} + j_2\, j_{m+2} + j_3\, j_{m+3}) \\
= &\quad (j_n \pm j_m) + j_1\,(j_{n+1} \pm j_{m+1}) \\
&\quad + j_2\,(j_{n+2} \pm j_{m+2}) + j_3\,(j_{n+3} \pm j_{m+3}).
\end{aligned}$$

The multiplication of two bihyperbolic Jacobsthal-Lucas numbers is defined by

$$\begin{aligned}
\mathscr{B}hj_n \times \mathscr{B}hj_m = &\quad (j_n + j_1\, j_{n+1} + j_2\, j_{n+2} + j_3\, j_{n+3}) \\
&\quad (j_m + j_1\, j_{m+1} + j_2\, j_{m+2} + j_3\, j_{m+3}) \\
= &\quad (j_n j_m + j_{n+1} j_{m+1} + j_{n+2} j_{m+2} + j_{n+3} j_{m+3}) \\
&\quad + j_1\,(j_{n+1} j_m + j_n j_{m+1} + j_{n+3} j_{m+2} + j_{n+2} j_{m+3}) \\
&\quad + j_2\,(j_{n+2} j_m + j_n j_{m+2} + j_{n+3} j_{m+1} + j_{n+1} j_{m+3}) \\
&\quad + j_3\,(j_{n+3} j_m + j_n j_{m+3} + j_{n+1} j_{m+2} + j_{n+2} j_{m+1}) \\
= &\quad \mathscr{B}hj_m \times \mathscr{B}hj_n\,.
\end{aligned} \tag{2.3}$$

Three kinds of conjugation can be defined for bihyperbolic numbers [6]. Therefore, conjugation of the bihyperbolic Jacobsthal-Lucas number is defined in three different ways as follows

$$\overline{\mathscr{B}hj}_n^{\,j_1} = j_n + j_1\, j_{n+1} - j_2\, j_{n+2} - j_3\, j_{n+3},$$

$$\overline{\mathscr{B}hj}_n^{\,j_2} = j_n - j_1\, j_{n+1} + j_2\, j_{n+2} - j_3\, j_{n+3}, \tag{2.4}$$

$$\overline{\mathscr{B}hj}_n^{\,j_3} = j_n - j_1\, j_{n+1} - j_2\, j_{n+2} + j_3\, j_{n+3}. \tag{2.5}$$

In the following theorem, some properties related to the conjugations of the bihyperbolic Jacobsthal-Lucas numbers are given.

Theorem 2.2. *Let $\overline{\mathscr{B}hj}_n^{j_1}$, $\overline{\mathscr{B}hj}_n^{j_2}$ and $\overline{\mathscr{B}hj}_n^{j_3}$, be three kinds of conjugation of the bihyperbolic Jacobsthal-Lucas number $\mathscr{B}hj_n$. In this case, we can give the following relations:*

$$\begin{aligned}
\mathscr{B}hj_n\overline{\mathscr{B}hj}_n^{j_1} &= j_n^2+j_{n+1}^2-j_{n+2}^2-j_{n+3}^2 \\
&+2\,j_1\,(j_nj_{n+1}-j_{n+2}j_{n+3}),\\
\mathscr{B}hj_n\overline{\mathscr{B}hj}_n^{j_2} &= j_n^2-j_{n+1}^2-j_{n+2}^2+j_{n+3}^2 \\
&+2\,j_2\,(j_nj_{n+2}-j_{n+1}j_{n+3}),\\
\mathscr{B}hj_n\overline{\mathscr{B}hj}_n^{j_3} &= j_n^2-j_{n+1}^2-j_{n+2}^2+j_{n+3}^2 \\
&+2\,j_3\,(j_nj_{n+3}-j_{n+1}j_{n+2}).
\end{aligned}$$

Proof. The proof can be easily done using equations Eq.(2.4-2.5) .

In the following theorems, some properties related to the bihyperbolic Jacobsthal-Lucas numbers are given.

Theorem 2.3. *Let $\mathscr{B}hj_n$ be the n-th bihyperbolic Jacobsthal-Lucas number. For any integer $n\geq 0$, summation formula as follows:*

$$\sum_{k=0}^{n}\mathscr{B}hj_k=\frac{1}{2}\left(\mathscr{B}hj_{n+2}-\mathscr{B}hj_2\right). \tag{2.6}$$

Proof. (2.6): Using the summation formula Eq.(1.3), we obtain

$$\begin{aligned}
\sum_{k=1}^{n}\mathscr{B}hj_k &= (\sum_{k=1}^{n}|_k+j_1\sum_{k=1}^{n}|_{k+1}+j_2\sum_{k=1}^{n}|_{k+2}+j_3\sum_{k=1}^{n}|_{k+3})\\
&=(\tfrac{j_{n+2}-5}{2})+j_1\,(\tfrac{j_{n+3}-7}{2})+j_2\,(\tfrac{j_{n+4}-17}{2})+j_3\,(\tfrac{j_{n+5}-31}{2})\\
&=\tfrac{1}{2}\left[\mathscr{B}hj_{n+2}-(5+7\,j_1+17\,j_2+31\,j_3)\right]\\
&=\tfrac{1}{2}\left[\mathscr{B}hj_{n+2}-(\mathscr{B}hj_2)\right].
\end{aligned}$$

where $\mathscr{B}hj_2=(5+7\,j_1+17\,j_2+31\,j_3)$.

Theorem 2.4. ***(Generating function)***
Let $\mathscr{B}hj_n$ be the n-th bihyperbolic Jacobsthal-Lucas number. For the generating function of the bihyperbolic Jacobsthal-Lucas numbers is as follows:

$$\begin{aligned}
g_{\mathscr{B}hj_n}(t)=\sum_{n=0}^{\infty}\mathscr{B}hj_n\,t^n &= \frac{\mathscr{B}hj_0+(\mathscr{B}hj_1-\mathscr{B}hj_0)\,t}{1-t-2t^2}\\
&=\frac{(2+j_1+5\,j_2+7\,j_3)+t\,(-1+4\,j_1+2\,j_2+10\,j_3)}{1-t-2t^2}
\end{aligned}$$

Proof. (2.8): Using the definition of generating function, we obtain

$$g_{\mathscr{B}hj_n}(t)=\mathscr{B}hj_0+\mathscr{B}hj_1\,t+\ldots+\mathscr{B}hj_n\,t^n+\ldots. \tag{2.7}$$

Multiplying $(1-t-2t^2)$ both sides of Eq.(2.7) and using Eq.(2.2), we have

$$\begin{aligned}
(1-t-2t^2)\,g_{\mathscr{B}hj_n}(t) &= \mathscr{B}hj_0+(\mathscr{B}hj_1-\mathscr{B}hj_0)\,t\\
&+(\mathscr{B}hj_2-\mathscr{B}hj_1-2\,\mathscr{B}hj_0)\,t^2\\
&+(\mathscr{B}hj_3-\mathscr{B}hj_2-2\,\mathscr{B}hj_1)\,t^3+\ldots\\
&+(\mathscr{B}hj_{k+1}-\mathscr{B}hj_k-2\,\mathscr{B}hj_{k-1})\,t^{k+1}+\ldots
\end{aligned}$$

where $\mathscr{B}hj_1-\mathscr{B}hj_0=-1+4\,j_1+2\,j_2+10\,j_3$, $\mathscr{B}hj_2-\mathscr{B}hj_1-2\,\mathscr{B}hj_0=0$ and $\mathscr{B}hj_3-\mathscr{B}hj_2-2\,\mathscr{B}hj_1=0\ldots=0$.

Theorem 2.5. ***(Binet's formula)*** *Let $\mathscr{B}hj_n$ be the n-th bihyperbolic Jacobsthal-Lucas number. For any integer $n \geq 0$, the Binet's formula for these numbers is as follows:*

$$\mathscr{B}hj_n = \hat{\alpha}\,\alpha^n + \hat{\beta}\,\beta^n\,. \tag{2.8}$$

where

$$\hat{\alpha} = 1 + j_1\,\alpha + j_2\,\alpha^2 + j_3\,\alpha^3, \quad \alpha = 2\,,$$

$$\hat{\beta} = 1 + j_1\,\beta + j_2\,\beta^2 + j_3\,\beta^3, \quad \beta = -1\,,$$

$$\hat{\alpha}\,\hat{\beta} = \hat{\beta}\,\hat{\alpha}\,.$$

Proof. Using the Binet's formula of Jacobsthal-Lucas number [15] and Eq.(2.1) we obtain that,

$$\begin{aligned}
\mathscr{B}hj_n = &\quad j_n + j_1\,j_{n+1} + j_2\,j_{n+2} + j_3\,j_{n+3} \\
= &\quad (\alpha^n + \beta^n) + j_1\,(\alpha^{n+1} + \beta^{n+1}) \\
&\quad + j_2\,(\alpha^{n+2} + \beta^{n+2}) + j_3\,(\alpha^{n+3} + \beta^{n+3}) \\
= &\quad \alpha^n(1 + j_1\,\alpha + j_2\,\alpha^2 + j_3\,\alpha^3) \\
&\quad + \beta^n(1 + j_1\,\beta + j_2\,\beta^2 + i\,j_3\,\beta^3) \\
= &\quad \hat{\alpha}\,\alpha^n + \hat{\beta}\,\beta^n\,.
\end{aligned}$$

Here, Binet's formula of the Jacobsthal-Lucas number sequence, $j_n = \alpha^n + \beta^n$ is used.

Theorem 2.6. ***(D'Ocagne's identity)*** *Let $\mathscr{B}hj_n$ be the n-th bihyperbolic Jacobsthal-Lucas number. For $m \geq n+1$, the following equality holds:*

$$\begin{aligned}
\mathscr{B}hj_m\,\mathscr{B}hj_{n+1} - \mathscr{B}hj_{m+1}\,\mathscr{B}hj_n = &\quad (-2)^n\,(-9)\,J_{m-n}\,[-5 + 5\,j_1 - 5\,j_2 + 5\,j_3\,] \\
= &\quad -3\,(\hat{\alpha}\,\hat{\beta}\,)\,(-2)^n\,(\alpha - \beta)\,J_{m-n}\,.
\end{aligned} \tag{2.9}$$

Proof. (2.9): Considering Eq.(2.3), using the commutative property of bihyperbolic numbers and d'Ocagne's identity of Jacobsthal-Lucas numbers [16], we obtain that

$$\begin{aligned}
\mathscr{B}hj_m\,\mathscr{B}hj_{n+1} - \mathscr{B}hj_{m+1}\,\mathscr{B}hj_n \quad &= [\,(j_m j_{n+1} - j_{m+1} j_n) \\
&\quad + (j_{m+1} j_{n+2} - j_{m+2} j_{n+1}) \\
&\quad + (j_{m+2} j_{n+3} - j_{m+3} j_{n+2}) \\
&\quad + (j_{m+3} j_{n+4} - j_{m+4} j_{n+3})\,] \\
&\quad + j_1\,[(j_m j_{n+2} - j_{m+1} j_{n+1}) \\
&\quad + (j_{m+1} j_{n+1} - j_{m+2} j_n) \\
&\quad + (j_{m+2} j_{n+4} - j_{m+3} j_{n+3}) \\
&\quad + (j_{m+3} j_{n+3} - j_{m+4} j_{n+2})\,] \\
&\quad + j_2\,[(j_m j_{n+3} - j_{m+1} j_{n+2}) \\
&\quad + (j_{m+2} j_{n+1} - j_{m+3} j_n) \\
&\quad + (j_{m+1} j_{n+4} - j_{m+2} j_{n+3}) \\
&\quad + (j_{m+3} j_{n+2} - j_{m+4} j_{n+1})] \\
&\quad + j_3\,[(j_m j_{n+4} - j_{m+1} j_{n+3}) \\
&\quad + (j_{m+3} j_{n+1} - j_{m+4} j_n) \\
&\quad + (j_{m+1} j_{n+3} - j_{m+2} j_{n+2}) \\
&\quad + (j_{m+2} j_{n+2} - j_{m+3} j_{n+1})] \\
&= \quad (-2)^n\,(-9)\,J_{m-n} - 5(1 - j_1 + j_2 - j_3)\,.
\end{aligned}$$

where $\hat{\alpha}\,\hat{\beta} = \hat{\beta}\,\hat{\alpha}$ and the identities $j_m j_{n+1} - j_{m+1} j_n = (-2)^n\,(-9)\,j_{m-n}\,,\;\; -4\,j_{n-2} - j_{n+2} = -5\,j_n\;\; -8\,j_{n-3} + j_{n+3} = 7\,j_n$ and $4\,j_{n-1} - 2\,j_{n+1} = -2\,j_n$ are used [16].

Theorem 2.6.A. *Now let's prove this identity using the Binet's formula:*

$$\begin{aligned}
\mathscr{B}hj_m \mathscr{B}hj_{n+1} - \mathscr{B}hj_{m+1}\mathscr{B}hj_n &= (\hat{\alpha}\,\alpha^m + \hat{\beta}\,\beta^m)(\hat{\alpha}\,\alpha^{n+1} + \hat{\beta}\,\beta^{n+1}) \\
&\quad -(\hat{\alpha}\,\alpha^{m+1} + \hat{\beta}\,\beta^{m+1})(\hat{\alpha}\,\alpha^n + \hat{\beta}\,\beta^n) \\
&= \hat{\alpha}\hat{\beta}[\alpha^m\beta^n(-\alpha+\beta) + \alpha^n\beta^m(\alpha-\beta)] \\
&= \hat{\alpha}\hat{\beta}(\alpha\beta)^n(\alpha-\beta)[\beta^{m-n} - \alpha^{m-n}] \\
&= -(\hat{\alpha}\hat{\beta})(-2)^n(\alpha-\beta)[\alpha^{m-n} - \beta^{m-n}] \\
&= -3(\hat{\alpha}\hat{\beta})(-2)^n(\alpha-\beta)J_{m-n}.
\end{aligned}$$

where $\hat{\alpha}\hat{\beta} = -5(1-j_1+j_2-j_3)$ *and* $3J_{m-n} = \alpha^{m-n} - \beta^{m-n}$.

Theorem 2.7. ***(Cassini's identity)*** *Let $\mathscr{B}hj_n$ be the n-th bihyperbolic Jacobsthal-Lucas number. For $n \geq 1$, the following equality holds:*

$$\begin{aligned}
\mathscr{B}hj_{n-1}\mathscr{B}hj_{n+1} - \mathscr{B}hj_n\mathscr{B}hj_n &= 9(-2)^{n-1}[-5(1-j_1+j_2-j_3)] \\
&= 9(-2)^{n-1}(\hat{\alpha}\hat{\beta}).
\end{aligned} \tag{2.10}$$

Proof. (2.10): By (2.3) and using the commutative property of bihyperbolic numbers and Cassini's identity of Jacobsthal-Lucas numbers [16], we obtain that

$$\begin{aligned}
\mathscr{B}hj_{n-1}\mathscr{B}hj_{n+1} - \mathscr{B}hj_n\mathscr{B}hj_n = &\ [(j_{n-1}j_{n+1} - j_nj_n) \\
&\quad +(j_nj_{n+2} - j_{n+1}j_{n+1}) \\
&\quad +(j_{n+1}j_{n+3} - j_{n+2}j_{n+2}) \\
&\quad +(j_{n+2}j_{n+4} - j_{n+3}j_{n+3})] \\
&+ j_1[(j_{n-1}j_{n+2} - j_nj_{n+1}) \\
&\quad +(j_nj_{n+1} - j_{n+1}j_n) \\
&\quad +(j_{n+1}j_{n+4} - j_{n+2}j_{n+3}) \\
&\quad +(j_{n+2}j_{n+3} - j_{n+3}j_{n+2})] \\
&+ j_2[(j_{n-1}j_{n+3} - j_nj_{n+2}) \\
&\quad +(j_{n+1}j_{n+1} - j_{n+2}j_n) \\
&\quad +(j_nj_{n+4} - j_{n+1}j_{n+3}) \\
&\quad +(j_{n+2}j_{n+2} - j_{n+3}j_{n+1})] \\
&+ j_3[(j_{n-1}j_{n+4} - j_nj_{n+3}) \\
&\quad +(j_{n+2}j_{n+1} - j_{n+3}j_n) \\
&\quad +(j_nj_{n+3} - j_{n+1}j_{n+2}) \\
&\quad +(j_{n+1}j_{n+2} - j_{n+2}j_{n+1})] \\
= &\ 9(-2)^{n-1}[-5(1-j_1+j_2-j_3)].
\end{aligned}$$

where the identity of the Jacobsthal-Lucas numbers $j_{n-1}j_{n+1} - j_nj_n = 9(-2)^{n-1}$ is used [16].

Theorem 2.7.A. *Now let's prove this identity using the Binet's formula:*

$$\begin{aligned}
\mathscr{B}hj_{n-1}\mathscr{B}hj_{n+1} - \mathscr{B}hj_n\mathscr{B}hj_n &= (\hat{\alpha}\,\alpha^{n-1} + \hat{\beta}\,\beta^{n-1})(\hat{\alpha}\,\alpha^{n+1} + \hat{\beta}\,\beta^{n+1}) \\
&\quad -(\hat{\alpha}\,\alpha^n + \hat{\beta}\,\beta^n)(\hat{\alpha}\,\alpha^n + \hat{\beta}\,\beta^n) \\
&= \hat{\alpha}\hat{\beta}\,(\alpha\beta)^n[\tfrac{\beta}{\alpha} + \tfrac{\alpha}{\beta} - 2] \\
&= \hat{\alpha}\hat{\beta}\,(\alpha\beta)^n\,\tfrac{(\alpha^2+\beta^2-2\alpha\beta)}{\alpha\beta} \\
&= (-2)^{n-1}(\alpha-\beta)^2(\hat{\alpha}\hat{\beta}) \\
&= 9(-2)^{n-1}(\hat{\alpha}\hat{\beta}).
\end{aligned}$$

where $\hat{\alpha}\,\hat{\beta} = -5\,(1-j_1+j_2-j_3)\,.$

Theorem 2.8. ***(Catalan's identity)*** *Let* $\mathscr{B}h\,j_n$ *be the n-th bihyperbolic Jacobsthal-Lucas number. For* $n \geq 1$*, the following equality holds:*

$$\begin{aligned} \mathscr{B}h\,j_n^2 - \mathscr{B}h\,j_{n-r}\mathscr{B}h\,j_{n+r} &= (-2)^{n-r}\,[\,j_r^2 - (-2)^{r+2}\,]\,[-5(1-j_1+j_2-j_3)\,] \\ &= (-2)^{n-r}\,(\hat{\alpha}\,\hat{\beta}\,)\,[\,4\,(\alpha\beta)^2 - (\alpha^r+\beta^r)^2\,]\,. \end{aligned} \tag{2.11}$$

Proof. (2.11): By (2.3) and using the commutative property of bihyperbolic numbers Catalan's identity of Jacobsthal-Lucas numbers [16], we obtain that

$$\begin{aligned} \mathscr{B}h\,j_n\mathscr{B}h\,j_n - \mathscr{B}h\,j_{n-r}\mathscr{B}h\,j_{n+r} = \;& [\,(\,j_n j_n - j_{n-r}j_{n+r}) \\ & \quad +(\,j_{n+1}j_{n+1} - j_{n-r+1}j_{n+r+1}) \\ & \quad +(\,j_{n+2}j_{n+3} - j_{n-r+2}j_{n+r+2}) \\ & \quad +(\,j_{n+2}j_{n+3} - j_{n-r+3}j_{n+r+3})\,] \\ & + j_1\,[(\,j_n j_{n+1} - j_{n-r}j_{n+r+1}) \\ & \quad +(\,j_{n+1}j_n - j_{n-r+1}j_{n+r}) \\ & \quad +(\,j_{n+2}j_{n+3} - j_{n-r+2}j_{n+r+3}) \\ & \quad +(\,j_{n+3}j_{n+2} - j_{n-r+3}j_{n+r+2})\,] \\ & + j_2\,[(\,j_n j_{n+2} - j_{n-r}j_{n+r+2}) \\ & \quad +(\,j_{n+2}j_n - j_{n-r+2}j_{n+r}) \\ & \quad +(\,j_{n+1}j_{n+3} - j_{n-r+1}j_{n+r+3}) \\ & \quad +(\,j_{n+3}j_{n+1} - j_{n-r+3}j_{n+r+1})] \\ & + j_3\,[(\,j_n j_{n+3} - j_{n-r}j_{n+r+3}) \\ & \quad +(\,j_{n+3}j_n - j_{n-r+3}j_{n+r}) \\ & \quad +(\,j_{n+1}j_{n+2} - j_{n-r+1}j_{n+r+2}) \\ & \quad +(\,j_{n+2}j_{n+1} - j_{n-r+2}j_{n+r+1})] \\ = \;& (-2)^{n-r}\,[\,j_r^2 - (-2)^{r+2}\,]\,[-5(1-j_1+j_2-j_3)\,]\,. \end{aligned}$$

where the identities of the Jacobsthal-Lucas numbers

$$j_{n-r}\,j_{n+r} - j_n\,j_n = (-2)^{n-r}\,[\,j_r^2 - (-2)^{r+2}\,]$$

is used [16].

Theorem 2.8.A. *Now let's prove this identity using the Binet's formula:*

$$\begin{aligned} \mathscr{B}h\,j_n\mathscr{B}h\,j_n - \mathscr{B}h\,j_{n-r}\mathscr{B}h\,j_{n+r} &= (\,\hat{\alpha}\;\alpha^n + \hat{\beta}\;\beta^n)\,(\hat{\alpha}\;\alpha^n + \hat{\beta}\;\beta^n) \\ & \quad -(\hat{\alpha}\;\alpha^{n-r} + \hat{\beta}\;\beta^{n-r})\,(\hat{\alpha}\;\alpha^{n+r} + \hat{\beta}\;\beta^{n+r}) \\ &= \hat{\alpha}\,\hat{\beta}\;(\alpha\,\beta)^n\,[2 - (\tfrac{\beta}{\alpha})^r - (\tfrac{\alpha}{\beta})^r\,] \\ &= (-2)^{n-r}\,(\hat{\alpha}\,\hat{\beta}\,)\,[\,4\,(\alpha\beta)^2 - (\alpha^r+\beta^r)^2\,]\,. \end{aligned}$$

where $\alpha^{2r} + \beta^{2r} - 2(\alpha\,\beta)^r = j_r^2 - (-2)^{2r}\,.$

3. The Bihyperbolic Pell-Lucas Numbers

In this section, we define the bihyperbolic Pell-Lucas numbers. Then, we obtain the generating function, Binet's formula, d'Ocagne's identity, Cassini's identity, Catalan's identity and Honsberger identity.

Definition 3.1. *For* $n \geq 1$*, the n-th bihyperbolic Pell-Lucas number* $\mathscr{BHPL}_n$ *are defined by using the Pell-Lucas numbers as follows*

$$\mathscr{B}hp_n = p_n + j_1\,p_{n+1} + j_2\,p_{n+2} + j_3\,p_{n+3}\,. \tag{3.1}$$

where j_1, j_2 *and* j_3 *satisfy the conditions*

$$j_1^2 = j_2^2 = j_3^2 = 1,\ j_1 j_2 = j_2 j_1 = j_3,\ j_1 j_3 = j_3 j_1 = j_2,\ j_2 j_3 = j_3 j_2 = j_1.$$

Theorem 3.1. *Let $\mathscr{B}hp_n$ be the n-th bihyperbolic Pell-Lucas number. For any integer $n \geq 0$,*

$$\mathscr{B}hp_n = 2\mathscr{B}hp_{n-1} + \mathscr{B}hp_{n-2} \tag{3.2}$$

Proof. (3.2): By placing Eq.(1.2) in Eq.(3.1) we obtain that,

$$\begin{aligned}
\mathscr{B}hp_n &= p_n + j_1\, p_{n+1} + j_2\, p_{n+2} + j_3\, p_{n+3} \\
&= (2\, p_{n-1} + p_{n-2}) + j_1\,(2\, p_n + p_{n-1}) \\
&\quad + j_2\,(2\, p_{n+1} + p_n) + j_3\,(2\, p_{n+2} + p_{n+1}) \\
&= 2\,(p_{n-1} + j_1\, q_n + j_2\, p_{n+1} + j_3\, p_{n+2}) \\
&\quad + (p_{n-2} + j_1\, p_{n-1} + j_2\, p_n + j_3\, p_{n+1}) \\
&= 2\,\mathscr{B}hp_{n-1} + \mathscr{B}hp_{n-2}
\end{aligned}$$

Also, initial values are $\mathscr{B}hp_0 = 2 + 2\, j_1 + 6\, j_2 + 14\, j_3$, $\mathscr{B}hp_1 = 2 + 6\, j_1 + 14\, j_2 + 34\, j_3$.
Let $\mathscr{B}hp_n$ and $\mathscr{B}hp_m$ be two bihyperbolic Pell-Lucas numbers such that

$$\mathscr{B}hp_n = p_n + j_1\, p_{n+1} + j_2\, p_{n+2} + j_3\, p_{n+3}$$

and

$$\mathscr{B}hp_m = p_m + j_1\, p_{m+1} + j_2\, p_{m+2} + j_3\, p_{m+3}$$

Then, the addition and subtraction of two bihyperbolic Pell numbers are defined in the obvious way,

$$\begin{aligned}
\mathscr{B}hp_n \pm \mathscr{B}hp_m = \ & (p_n \pm p_m) + j_1\,(p_{n+1} \pm p_{m+1}) \\
& + j_2\,(p_{n+2} \pm p_{m+2}) + j_3\,(p_{n+3} \pm p_{m+3}).
\end{aligned}$$

Multiplication of two bihyperbolic Pell-Lucas numbers is defined by

$$\begin{aligned}
\mathscr{B}hp_n \times \mathscr{B}hp_m = \ & (p_n p_m + p_{n+1} p_{m+1} \\
& \quad + p_{n+2} p_{m+2} + p_{n+3} p_{m+3}) \\
& + j_1\,(p_{n+1} p_m + p_n p_{m+1} \\
& \quad + p_{n+3} p_{m+2} + p_{n+2} p_{m+3}) \\
& + j_2\,(p_{n+2} p_m + p_n p_{m+2} \\
& \quad + p_{n+3} p_{m+1} + p_{n+1} p_{m+3}) \\
& + j_3\,(p_{n+3} p_m + p_n p_{m+3} \\
& \quad + p_{n+1} p_{m+2} + p_{n+2} p_{m+1}) \\
= \ & \mathscr{B}hp_m \times \mathscr{B}hp_n .
\end{aligned}$$

Three kinds of conjugation can be defined for bihyperbolic numbers [6]. Therefore, conjugation of the bihyperbolic Pell-Lucas number is defined in three different ways as follows

$$\overline{\mathscr{B}hp_n}^{j_1} = p_n + j_1\, p_{n+1} - j_2\, p_{n+2} - j_3\, p_{n+3},$$

$$\overline{\mathscr{B}hp_n}^{j_2} = p_n - j_1\, p_{n+1} + j_2\, p_{n+2} - j_3\, p_{n+3},$$

$$\overline{\mathscr{B}hp_n}^{j_3} = p_n - j_1\, p_{n+1} - j_2\, p_{n+2} + j_3\, p_{n+3}.$$

In the following theorem, some properties related to the conjugations of the bihyperbolic Pell-Lucas numbers are given.

Theorem 3.2. *Let $\overline{\mathscr{B}hp_n}^{j_1}$, $\overline{\mathscr{B}hp_n}^{j_2}$ and $\overline{\mathscr{B}hp_n}^{j_3}$, be three kinds of conjugation of the bihyperbolic Pell-Lucas number $\mathscr{B}hp_n$. In this case, we can give the following relations:*

$$\mathscr{B}hp_n\, \overline{\mathscr{B}hp_n}^{j_1} = p_n^2 + p_{n+1}^2 - p_{n+2}^2 - p_{n+3}^2 + 2\, j_1\,(p_n p_{n+1} - p_{n+2} p_{n+3}). \tag{3.3}$$

$$\mathscr{B}hp_n\, \overline{\mathscr{B}hp_n}^{j_2} = p_n^2 - p_{n+1}^2 - p_{n+2}^2 + p_{n+3}^2 + 2\, j_2\,(p_n p_{n+2} - p_{n+1} p_{n+3}),$$

$$\mathscr{B}hp_n\, \overline{\mathscr{B}hp_n}^{j_3} = p_n^2 - p_{n+1}^2 - p_{n+2}^2 + p_{n+3}^2 + 2\, j_3\,(p_n p_{n+3} - p_{n+1} p_{n+2}). \tag{3.4}$$

Proof. The proof can be easily done using equations Eq.(3.3-3.4).
In the following theorems, some properties related to the bihyperbolic Pell-Lucas numbers are given.

Theorem 3.3. *Let $\mathscr{B}hp_n$ be the bihyperbolic Pell-Lucas number. For any integer $n \geq 0$, summation formulas as follows:*

$$\sum_{k=0}^{n} \mathscr{B}hp_k = \frac{1}{2}\left[\mathscr{B}hp_{n+1} + \mathscr{B}hp_n + (\mathscr{B}hp_1 - \mathscr{B}hp_2)\right], \tag{3.5}$$

$$\sum_{k=0}^{n} \mathscr{B}hp_{2k} = \frac{1}{2}\left[\mathscr{B}hp_{2n+1} - \mathscr{B}hp_1\right], \tag{3.6}$$

$$\sum_{k=0}^{n} \mathscr{B}hp_{2k-1} = \frac{1}{2}\left[\mathscr{B}hp_{2n} - \mathscr{B}hp_0\right]. \tag{3.7}$$

Proof. (3.5): Using the summation formula $\sum\limits_{r=1}^{n} p_r = \frac{(p_{n+1}+p_n-4)}{2}$ in Eq.(1.3) , we obtain

$$\begin{aligned}
\sum_{k=1}^{n} \mathscr{B}hp_k &= \Big(\sum_{k=1}^{n} p_k + j_1 \sum_{k=1}^{n} p_{k+1} + j_2 \sum_{k=1}^{n} p_{k+2} + j_3 \sum_{k=1}^{n} p_{k+3}\Big) \\
&= \left(\tfrac{p_{n+1}+p_n-4}{2}\right) + j_1\left(\tfrac{p_{n+2}+p_{n+1}-8}{2}\right) + j_2\left(\tfrac{p_{n+3}+p_{n+2}-20}{2}\right) + j_3\left(\tfrac{p_{n+4}+p_{n+3}-48}{2}\right) \\
&= \tfrac{1}{2}\left[\mathscr{B}hp_{n+1} + \mathscr{B}hp_n - (4 + 8\,j_1 + 20\,j_2 + 48\,j_3)\right] \\
&= \tfrac{1}{2}\left[\mathscr{B}hp_{n+1} + \mathscr{B}hp_n + (\mathscr{B}hp_1 - \mathscr{B}hp_2)\right].
\end{aligned}$$

where $\mathscr{B}hp_2 = (6 + 14\,j_1 + 34\,j_2 + 82\,j_3)$.

(3.6): Using the summation formula $\sum\limits_{r=1}^{n} p_{2r} = \frac{(p_{2n+1}-2)}{2}$ in Eq.(1.3) , we obtain

$$\begin{aligned}
\sum_{k=1}^{n} \mathscr{B}hp_{2k} &= \Big(\sum_{k=1}^{n} p_{2k} + j_1 \sum_{k=1}^{n} p_{2k+1} + j_2 \sum_{k=1}^{n} p_{2k+2} + j_3 \sum_{k=1}^{n} p_{2k+3}\Big) \\
&= \left(\tfrac{p_{2n+1}-2}{2}\right) + j_1\left(\tfrac{p_{2n+2}-6}{2}\right) + j_2\left(\tfrac{p_{2n+3}-14}{2}\right) + j_3\left(\tfrac{p_{2n+4}-34}{2}\right) \\
&= \tfrac{1}{2}\left[\mathscr{B}hp_{2n+1} - (2 + 6\,j_1 + 14\,j_2 + 34\,j_3)\right] \\
&= \tfrac{1}{2}\left[\mathscr{B}hp_{2n+1} - \mathscr{B}hp_1\right].
\end{aligned}$$

(3.7): Using the summation formula $\sum\limits_{r=1}^{n} p_{2r-1} = \frac{(p_{2n}-2)}{2}$ in Eq.(1.3) , we obtain

$$\begin{aligned}
\sum_{k=1}^{n} \mathscr{B}hp_{2k-1} &= \Big(\sum_{k=1}^{n} p_{2k-1} + j_1 \sum_{k=1}^{n} p_{2k} + j_2 \sum_{k=1}^{n} p_{2k+1} + j_3 \sum_{k=1}^{n} p_{2k+2}\Big) \\
&= \left(\tfrac{p_{2n}-2}{2}\right) + j_1\left(\tfrac{p_{2n+1}-2}{2}\right) + j_2\left(\tfrac{p_{2n+2}-6}{2}\right) + j_3\left(\tfrac{p_{2n+3}-14}{2}\right) \\
&= \tfrac{1}{2}\left[\mathscr{B}hp_{2n} - (2 + 2\,j_1 + 6\,j_2 + 14\,j_3)\right] \\
&= \tfrac{1}{2}\left[\mathscr{B}hp_{2n} - \mathscr{B}hp_0\right].
\end{aligned}$$

Theorem 3.4. ***(Generating function)***
Let $\mathscr{B}hp_n$ be the n-th bihyperbolic Pell-Lucas number. For the generating function for the bihyperbolic Pell-Lucas numbers is

as follows:

$$
\begin{aligned}
g_{\mathscr{B}hp_n}(t) = \sum_{n=1}^{\infty} \mathscr{B}hp_n t^n = & \quad \tfrac{\mathscr{B}hp_0+(\mathscr{B}hp_1-2\mathscr{B}hp_0)t}{1-2t-t^2} \\
= & \quad \tfrac{(2+2j_1+6j_2+14j_3)+t(-2+2j_1+2j_2+6j_3)}{1-2t-t^2}.
\end{aligned} \tag{3.8}
$$

where $\mathscr{B}hp_0 = 2+2j_1+6j_2+14j_3$, $\mathscr{B}hp_1 = 2+6j_1+14j_2+34j_3$ *and* $\mathscr{B}hp_2 = 6+14j_1+34j_2+82j_3$.

Proof. (3.8): Using the definition of generating function, we obtain

$$
g_{\mathscr{B}hp_n}(t) = \mathscr{B}hp_0 + \mathscr{B}hp_1 t + \ldots + \mathscr{B}hp_n t^n + \ldots \tag{3.9}
$$

Multiplying $(1-2t-t^2)$ both sides of Eq.(3.9) and using Eq.(3.2), we have

$$
\begin{aligned}
(1-2t-t^2)\, g_{\mathscr{B}hp_n}(t) \quad &= \mathscr{B}hp_0 + (\mathscr{B}hp_1 - 2\mathscr{B}hp_0)\,t \\
&\quad +(\mathscr{B}hp_2 - 2\mathscr{B}hp_1 - \mathscr{B}hp_0)\,t^2 \\
&\quad +(\mathscr{B}hp_3 - 2\mathscr{B}hp_2 - \mathscr{B}hp_1)\,t^3 + \ldots \\
&\quad +(\mathscr{B}hp_{k+1} - 2\mathscr{B}hp_k - \mathscr{B}hp_{k-1})\,t^{k+1} + \ldots
\end{aligned}
$$

where $\mathscr{B}hp_1 - 2\mathscr{B}hp_0 = -2+2j_1+2j_2+6j_3$, $\mathscr{B}hp_2 - 2\mathscr{B}hp_1 - \mathscr{B}hp_0 = 0$, and $\mathscr{B}hp_3 - 2\mathscr{B}hp_2 - \mathscr{B}hp_1 = 0 \ldots = 0$.

Theorem 3.5. ***(Binet's formula)*** *Let $\mathscr{B}hp_n$ be the n-th bihyperbolic Pell-Lucas number. For any integer $n \geq 0$, the Binet's formula for these numbers is as follows:*

$$
\mathscr{B}hp_n = \hat{\alpha}\, \alpha^n + \hat{\beta}\, \beta^n. \tag{3.10}
$$

where

$$
\hat{\alpha} = 1 + j_1\,\alpha + j_2\,\alpha^2 + j_3\,\alpha^3, \quad \alpha = 1+\sqrt{2},
$$

$$
\hat{\beta} = 1 + j_1\,\beta + j_2\,\beta^2 + j_3\,\beta^3, \quad \beta = 1-\sqrt{2}.
$$

Proof. Using the Binet's formula of Pell-Lucas number [20, 21] and Eq.(3.1) we obtain that,

$$
\begin{aligned}
\mathscr{B}hp_n = & \quad p_n + j_1\,p_{n+1} + j_2\,p_{n+2} + j_3\,p_{n+3} \\
= & \quad (\alpha^n+\beta^n) + j_1\,(\alpha^{n+1}+\beta^{n+1}) \\
& \quad + j_2\,(\alpha^{n+2}+\beta^{n+2}) + j_3\,(\alpha^{n+3}+\beta^{n+3}) \\
= & \quad \alpha^n(1+j_1\,\alpha+j_2\,\alpha^2+j_3\,\alpha^3) \\
& \quad +\beta^n(1+j_1\,\beta+j_2\,\beta^2+i\,j_3\,\beta^3) \\
= & \quad \hat{\alpha}\,\alpha^n + \hat{\beta}\,\beta^n.
\end{aligned}
$$

Here, Binet's formula of the Pell-Lucas number sequence, $p_n = \alpha^n + \beta^n$ is used.

Theorem 3.6. ***(D'Ocagne's identity)*** *Let $\mathscr{B}hp_n$ be the n-th bihyperbolic Pell-Lucas number. For $m \geq n+1$, the following equality holds:*

$$
\mathscr{B}hp_m\,\mathscr{B}hp_{n+1} - \mathscr{B}hp_{m+1}\,\mathscr{B}hp_n = \quad (-1)^{n-1}\,\hat{\alpha}\hat{\beta}\,(\alpha-\beta)\,[\alpha^{m-n}-\beta^{m-n}]. \tag{3.11}
$$

Proof. (3.11): let's prove this identity using the Binet's formula Eq.(3.10):

$$
\begin{aligned}
\mathscr{B}hp_m\,\mathscr{B}hp_{n+1} - \mathscr{B}hp_{m+1}\,\mathscr{B}hp_n \quad &= (\hat{\alpha}\,\alpha^m + \hat{\beta}\,\beta^m)(\hat{\alpha}\,\alpha^{n+1} + \hat{\beta}\,\beta^{n+1}) \\
&\quad -(\hat{\alpha}\,\alpha^{m+1} + \hat{\beta}\,\beta^{m+1})(\hat{\alpha}\,\alpha^n + \hat{\beta}\,\beta^n) \\
&= \hat{\alpha}\hat{\beta}\,(\alpha\beta)^n\,[\alpha^{m-n}(\beta-\alpha) + \beta^{m-n}(\alpha-\beta)] \\
&= (-1)^{n-1}\,\hat{\alpha}\hat{\beta}\,(\alpha-\beta)\,[\alpha^{m-n}-\beta^{m-n}].
\end{aligned}
$$

Theorem 3.7. ***(Cassini's identity)*** *Let $\mathscr{B}hp_n$ be the n-th bihyperbolic Pell-Lucas number. For $n \geq 1$, the following equality holds:*

$$\begin{aligned} \mathscr{B}hp_{n-1}\,\mathscr{B}hp_{n+1} - \mathscr{B}hp_n\,\mathscr{B}hp_n &= (-1)^{n-1}\,\hat{\alpha}\,\hat{\beta}\,(\alpha^2+\beta^2-2\,\alpha\,\beta) \\ &= 8\,(-1)^{n-1}\,\hat{\alpha}\,\hat{\beta}\,. \end{aligned} \tag{3.12}$$

Proof. (3.12): let's prove this identity using the Binet's formula Eq.(3.10):

$$\begin{aligned} \mathscr{B}hp_{n-1}\,\mathscr{B}hp_{n+1} - \mathscr{B}hp_n\,\mathscr{B}hp_n &= (\,\hat{\alpha}\,\alpha^{n-1} + \hat{\beta}\,\beta^{n-1})\,(\,\hat{\alpha}\,\alpha^{n+1} + \hat{\beta}\,\beta^{n+1}) \\ &\quad -(\,\hat{\alpha}\,\alpha^{n} + \hat{\beta}\,\beta^{n})\,(\,\hat{\alpha}\,\alpha^{n} + \hat{\beta}\,\beta^{n}) \\ &= \hat{\alpha}\,\hat{\beta}\,(\alpha\,\beta)^n\,[\,\tfrac{\beta}{\alpha} + \tfrac{\alpha}{\beta} - 2\,] \\ &= (-1)^{n-1}\,\hat{\alpha}\,\hat{\beta}\,(\alpha^2+\beta^2-2\,\alpha\,\beta) \\ &= 8\,(-1)^{n-1}\,\hat{\alpha}\,\hat{\beta}\,. \end{aligned}$$

Theorem 3.8. ***(Catalan's identity)*** *Let $\mathscr{B}hp_n$ be the n-th bihyperbolic Pell-Lucas number. For $n \geq 1$, the following equality holds:*

$$\mathscr{B}hp_n^2 - \mathscr{B}hp_{n-r}\,\mathscr{B}hp_{n+r} \quad = (-1)^n\,\hat{\alpha}\,\hat{\beta}\,[\,(\alpha-\beta)^2\,]^r\,. \tag{3.13}$$

Proof. (3.13): Let's prove this identity using the Binet's formula Eq.(3.10):

$$\begin{aligned} \mathscr{B}hp_n\,\mathscr{B}hp_n - \mathscr{B}hp_{n-r}\,\mathscr{B}hp_{n+r} &= (\,\hat{\alpha}\,\alpha^{n} + \hat{\beta}\,\beta^{n})\,(\,\hat{\alpha}\,\alpha^{n} + \hat{\beta}\,\beta^{n}) \\ &\quad -(\,\hat{\alpha}\,\alpha^{n-r} + \hat{\beta}\,\beta^{n-r})\,(\,\hat{\alpha}\,\alpha^{n+r} + \hat{\beta}\,\beta^{n+r}) \\ &= \hat{\alpha}\,\hat{\beta}\,(\alpha\,\beta)^n\,[\,2 - (\,\tfrac{\beta}{\alpha})^r - (\,\tfrac{\alpha}{\beta})^r\,] \\ &= (-1)^n\,\hat{\alpha}\,\hat{\beta}\,[\,(\alpha-\beta)^2\,]^r\,. \end{aligned}$$

4. Conclusion

In this paper, we introduced some properties of Lucas-type bihyperbolics. We gave the definition of bihyperbolic Jacobsthal-Lucas and bihyperbolic Pell-Lucas numbers and examined their algebraic properties. Additionally, by using the relationship of these numbers with Jacobsthal-Lucas and Pell-Lucas numbers, we obtained the Binet formula, generating function, d'Ocagne, Cassini and Catalan identities of bihyperbolic Jacobsthal-Lucas and bihyperbolic Pell-Lucas numbers.

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