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Araştırma Makalesi / Research Article

Solution of Some Integral Equations by Point-Collocation Method

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Abstract

In several engineering or physics problems, particularly those involving electromagnetic theory, thermal and radiation effects, acoustics, elasticity, and some fluid mechanics, it is not always easy or possible to find the analytical solution of integral equations that describe them. For this reason, numerical techniques are used. In this study, Point-collocation method was applied to linear and nonlinear, Volterra and Fredholm type integral equations and the performance and accuracy of the method was compared with several other methods that seem to be popular choices. As the base functions, a suitably chosen family of polynomials were employed. The convergence of the method was verified by increasing the number of polynomial base functions. The results demonstrate that the collocation method performs well even with a relatively low number of base functions and is a good candidate for solving a wide variety of integral equations. Nonlinear problems take longer to calculate approximate solution coefficients than linear problems. Furthermore, it is necessary to use the real and smallest coefficients found in order to obtain a suitable approximate solution to these problems.

Keywords: Collocation method, Nonlinear integral equations, Volterra equations, Fredholm equations, Approximate solution method.

Bazı İntegral Denklemlerin Nokta Kollokasyon Yöntemiyle Çözümü

Öz

Çeşitli mühendislik veya fizik problemlerinde, özellikle elektromanyetik teori, termal ve radyasyon etkileri, akustik, elastisite ve akışkanlar mekaniğinde, bunları tanımlayan integral denklemlerin analitik çözümünü bulmak her zaman kolay veya mümkün değildir. Bu yüzden sayısal teknikler kullanılır. Bu çalışmada temel bilimlerde ve mühendislikte karşılaşılan integral denklemlerin sayısal çözümleri için kullanılabilecek polinom temelli kollokasyon yöntemi sunulmuştur. Yöntem, doğrusal veya doğrusal olmayan Volterra ve Fredholm integral denklemlerine uygulanacak şekilde formüle edilmiştir. Doğrusal olmayan denklemlerin kollokasyon noktalarında cebirsel denklemlere indirgenmesi ve meydana gelen denklem sisteminin çözümü mümkün olmuştur. İncelenen örneklerin sayısal sonuçları, önerilen bu yöntemin iyi çalıştığını ve az sayıda kollokasyon noktası alındığında bile polinom seçiminin yaklaşık çözüm için uygun olduğunu göstermektedir. Ayrıca, yöntemin performansı farklı polinom mertebeleri için karşılaştırılmıştır. Doğrusal olmayan problemlerin yaklaşık çözüm katsayılarını hesaplamak doğrusal problemlere göre daha uzun sürmektedir. Ayrıca bu problemlere uygun yaklaşık çözüm elde edebilmek için bulunan gerçek ve en küçük katsayıların kullanılması gerekmektedir.

Anahtar Kelimeler: Kollokasyon yöntemi, Doğrusal olmayan integral denklemler, Volterra denklemleri, Fredholm denklemleri, Yaklaşık çözüm yöntemi

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1. Introduction

Integral equations are useful in modelling a wide range of problems in science and engineering. For instance, integral equations define mathematical models relevant to engineering studies concerning electromagnetic theory, thermal and radiation effects, acoustics, elasticity, fluid mechanics, and some mechanics problems. A number of approximate methods for solving differential equations first formulate the differential problem in the form of an integral equation; a well-known example for this is the method of boundary integral equations.

Volterra integral equations are usually encountered in evolution type physical problems (Krasnov et al., 1971). Fredholm integral equations can model, for example, fluid-solid interactions; a recent interesting example is (Daddi-Moussa-Ider et al., 2019) where the hydrodynamic interactions between a cell membrane and nanoparticles were investigated. The third type of integral equations, Volterra-Fredholm equations (Wazwaz, 2011) are encountered in parabolic models employed in various physical and biological problems. Various diffusion, population development, nerve behaviour, viscoelastic material, materials with memory problems can also be modeled based on integral equations (Guo, 2020).

There are many approximate solution methods for integral equations. In addition to classical methods, some of the contemporary methods are Adomian decomposition (Wazwaz, 2011), homotopy analysis (Adawi et al., 2009), modified Taylor's method (Matoog et al., 2023), Runge-Kutta (Brunner et al., 1982), differential transform (Arikoglu and Ozkol, 2008), variational iteration (Shakeri et al., 2009; Prajapati et al., 2011), series solutions (Wazwaz and Khuri, 1996), and homotopy perturbation method (S. Abbasbandy, 2006; Biazar and Eslami, 2010).

On the contrary, the Adomian decomposition method, the collocation method does not need to use any special polynomials. In this method, it is not necessary to rely on the auxiliary parameter of the series solution for solution convergence as in the homotopy analysis method. Furthermore, there is no need to linearize when collocation is applied to nonlinear problems.

In the present study, some of the problems solved by various authors with a selection of above methods will be solved by the collocation method and compared in terms of accuracy and simplicity of the method of solution.

2. Material and method

2.1. Weighted residual and collocation methods

Collocation method is a type of weighted residual method. To shortly describe both, consider an equation (differential or integral, linear or nonlinear) involving a single variable function $y(x)$

$$L[y(x)] = 0 \quad (1a)$$

where L is a linear or nonlinear operator and appropriate boundary conditions

$$B[y(x)] = 0 \quad (1b)$$

Denote an approximate solution of the equation as

$$\hat{y}(x, C_1, C_2, \dots, C_N) \quad (1c)$$

Here the form of this solution is to be chosen and involves free parameters $C_1, C_2, \dots, C_N$ to be found so that the approximate solution is as close to the exact solution as possible.

$\hat{y}(x, C_1, C_2, \dots, C_N)$ can be chosen to satisfy the boundary conditions

$$B[\hat{y}(x, C_1, C_2, \dots, C_N)] = 0 \quad (1d)$$

But it will not satisfy the equation; when substituted, the result will be the residual

$$R(x, C_1, C_2, \dots, C_N) = L[\hat{y}(x, C_1, C_2, \dots, C_N)] \neq 0 \quad (2)$$

In equation 2, the expression R is called the residual. In the weighted residual method, the free parameters within the approximate solution are determined from

$$\int R(x, C_1, C_2, \dots, C_N)\, W_n(x) dx = 0 \qquad n = 1,2, \dots, N \quad (3)$$

where the integration is over the solution domain and $W_n(x)$ are weight functions, also to be chosen. Various weighted residual methods exist according to the choice of weight functions.

In the collocation method, the weight functions are taken as

$$W_n(x) = \delta(x - x_n) \tag{4}$$

Hence Eq. 3 becomes

$$\int R(x, C_1, C_2, \dots, C_N)\, \delta(x - x_n)\ dx = 0 \qquad n = 1,2, \dots, N \tag{5}$$

In the equation, δ denotes the delta function and possesses the following attribute in solution region

$$\int_{\Omega} \delta(x - x_0)\theta(x)\, d\Omega = \theta(x_0) \tag{6}$$

and x_n are chosen collocation points within the solution domain Ω. This makes (3) a system of N equations for the parameters $C_1, C_2, \dots, C_N$

$$R(x_n, C_1, C_2, \dots, C_N) = 0 \qquad n = 1,2, \dots, N \tag{7}$$

In principle, the form of the approximate solution can be chosen freely. In the following, the approximate solution will be taken as a linear combination of polynomials

$$\hat{y}(x, C_1, C_2, \dots, C_N) = \sum_{n=1}^{N} C_n x^n \tag{8}$$

2.2. Application to integral equations

The following Volterra equation was solved in (Wazwaz, 2011) by Adomian Decomposition Method

$$y(x) = 1 - \int_0^x \mathrm{y}(t)\, dt \tag{9}$$

The exact solution is $= e^{-x}$. Since (9) gives $y(0) = 1$, the approximate solution is taken as

$$\hat{y}(x) = 1 + \sum_{n=1}^{N} C_n x^n \tag{10}$$

involving $N = 5$ free parameters. The residual is

$$R(x) = \hat{y}(x) - \int_0^x \hat{y}(t)\, dt \tag{11}$$

To simplify notation, the dependence on C_n is suppressed. Substituting (10) into (11) gives

$$R(x) = 1 + \sum_{n=1}^{5} C_n x^n - 1 + \int_0^x (1 + \sum_{n=1}^{5} C_n t^n)\, dt \tag{12}$$

Collocation points will be equally spaced within the solution domain $0 < x < 1$, i.e.,

$$x_n = \frac{n}{N+1} \qquad n = 1,2,..,N \tag{13}$$

Thus the collocation equations

$$R(x_n) = 0 \qquad n = 1,2,..,N \tag{14}$$

Become

$$\begin{array}{c} 0.167 + 0.181C_1 + 0.029C_2 + 0.004C_3 + 0.0007C_4 + 0.0001C_5 = 0 \\ 0.333 + 0.389C_1 + 0.123C_2 + 0.040C_3 + 0.0013C_4 + 0.004C_5 = 0 \\ 0.500 + 0.625C_1 + 0.292C_2 + 0.140C_3 + 0.0068C_4 + 0.033C_5 = 0 \\ 0.667 + 0.889C_1 + 0.543C_2 + 0.345C_3 + 0.223C_4 + 0.146C_5 = 0 \\ 0.833 + 1.1806C_1 + 0.887C_2 + 0.669C_3 + 0.562C_4 + 0.457C_5 = 0 \end{array} \tag{15}$$

After solving this linear system of equations for C_n, the approximate solution becomes

$$\hat{y}(x) = 1 + \sum_{n=1}^{5} C_n x^n \tag{16a}$$

The solution was also carried out with $N = 10$ and $N = 15$, the exact solution, approximate solutions and errors are shown in Table 1.

Table 1. Collocation solution of linear Volterra integral equation (9).

x	Exact	Approximate Solution N=5	N=10	N=15	Absolute Errors in Eq (9) N=5	N=10	N=15
0.0	1.00000	1.000000	1.000000	1.000000	0.00000	0.00000	0.00000
0.1	0.90483	0.904838	0.904837	0.904837	$2.1x10^{-7}$	$2.6x10^{-15}$	$5.5x10^{-16}$
0.2	0.81873	0.818731	0.818731	0.818731	$1.0x10^{-7}$	$2.2x10^{-16}$	$2.2x10^{-16}$
0.3	0.74081	0.740818	0.740818	0.740818	$6.5x10^{-8}$	$1.1x10^{-15}$	0.000000
0.4	0.67032	0.670320	0.670320	0.670320	$4.8x10^{-8}$	$4.4x10^{-16}$	$1.1x10^{-16}$
0.5	0.60653	0.606531	0.606531	0.606531	$2.0x10^{-8}$	$7.7x10^{-16}$	$1.1x10^{-16}$
0.6	0.54881	0.548812	0.548812	0.548812	$1.1x10^{-7}$	$1.1x10^{-16}$	$2.2x10^{-16}$
0.7	0.49658	0.496585	0.496585	0.496585	$1.0x10^{-7}$	$1.3x10^{-15}$	$1.1x10^{-16}$
0.8	0.44932	0.449329	0.449329	0.449329	$2.5x10^{-7}$	$2.0x10^{-15}$	$2.2x10^{-16}$
0.9	0.40657	0.406568	0.406569	0.406570	$2.1x10^{-6}$	$1.4x10^{-14}$	$1.9x10^{-15}$
1.0	0.36787	0.367866	0.367879	0.367879	$1.3x10^{-5}$	$2.1x10^{-12}$	$4.6x10^{-13}$

Approximate solutions for $N = 10$ and $N = 15$ are respectively

$$y(x) = 1 - x + 0.5x^2 - 0.166x^3 + 0.041x^4 - 0.008x^5 + 0.001x^6 - 1 \times 10^{-4}x^7 + 2 \times 10^{-5} \times x^8 - 2 \times 10^{-6}x^9 + 1.75 \times 10^{-7}x^{10} \tag{16b}$$

$$y(x) = 1 - x + 0.5x^2 - 0.166x^3 + 0.041x^4 - 0.008x^5 + 0.001x^6 - 2 \times 10^{-4}x^7 + 2.9 \times 10^{-5} \times x^8 - 1.2 \times 10^{-5}x^9 + 1.4 \times 10^{-5}x^{10} - 1.51 \times 10^{-7}x^{11} + 1.15 \times 10^{-7}x^{12} - 5.95 \times 10^{-6}x^{13} + 1.83 \times 10^{-6}x^{14} - 2.57 \times 10^{-7}x^{15} \tag{16c}$$

The following nonlinear Volterra equation was solved in (Darania et al., 2006) by Linearization method.

$$y(x) = e^x - \frac{1}{2}(e^{2x} - 1) + \int_0^x y^2(t)\,dt \tag{17}$$

The exact solution is $y = e^x$. Since (17) gives $y(0) = 1$ the approximate solution involving 5 terms is taken as

$$\hat{y}(x) = 1 + \sum_{n=1}^{5} C_n x^n \tag{18}$$

and the residual is

$$R(x) = \hat{y}(x) - e^x + \frac{1}{2}(e^{2x} - 1) - \int_0^x \hat{y}^2(t)\,dt \tag{19}$$

Collocation points are chosen as before; and the resulting system of equations is nonlinear for this problem. There are multiple solutions to this nonlinear system; some are complex numbers. It turns out that the real solutions with the smallest absolute values give the correct approximate solution. This results in

$$y(x) = 1 + 1.0004x + 0.499538x^2 + 0.168836x^3 + 0.0370243x^4 + 0.0128194x^5 \quad (20)$$

The solution was also carried out for $N = 10$, and Table 2 displays the present solutions together with the solution given in (Darania et al., 2006), exact solution and errors made.

Table 2. Absolute Errors for Nonlinear Volterra İntegral Equation (17)

	Collocation Point Number						
	(Darania et al., 2006), (h=0.0001)	**N=5**	**N=6**	**N=7**	**N=8**	**N=9**	**N=10**
0.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.1	$2.0x10^{-8}$	$7.6x10^{-7}$	$2.6x10^{-8}$	$7.1x10^{-10}$	$1.5x10^{-11}$	$2.7x10^{-13}$	$2.8x10^{-15}$
0.2	$5.8x10^{-8}$	$5.6x10^{-8}$	$5.1x10^{-10}$	$4.1x10^{-11}$	$5.6x10^{-12}$	$2.9x10^{-13}$	$1.0x10^{-14}$
0.3	$8.5x10^{-8}$	$1.2x10^{-7}$	$1.1x10^{-8}$	$5.1x10^{-10}$	$1.5x10^{-11}$	$3.9x10^{-13}$	$1.0x10^{-14}$
0.4	$1.1x10^{-7}$	$5.0x10^{-7}$	$1.7x10^{-8}$	$4.6x10^{-10}$	$1.4x10^{-11}$	$5.1x10^{-13}$	$1.5x10^{-14}$
0.5	$1.8x10^{-7}$	$5.1x10^{-7}$	$1.4x10^{-8}$	$6.8x10^{-10}$	$2.5x10^{-11}$	$7.1x10^{-13}$	$2.0x10^{-14}$
0.6	$2.5x10^{-7}$	$4.3x10^{-7}$	$3.0x10^{-8}$	$1.0x10^{-9}$	$2.9x10^{-11}$	$9.9x10^{-13}$	$3.0x10^{-14}$
0.7	$3.8x10^{-7}$	$1.1x10^{-6}$	$4.5x10^{-8}$	$1.1x10^{-9}$	$5.3x10^{-11}$	$1.4x10^{-12}$	$4.1x10^{-14}$
0.8	$6.2x10^{-7}$	$2.3x10^{-6}$	$2.6x10^{-8}$	$3.1x10^{-9}$	$6.1x10^{-11}$	$2.1x10^{-12}$	$7.3x10^{-14}$
0.9	$9.3x10^{-7}$	$1.7x10^{-6}$	$2.2x10^{-7}$	$6.2x10^{-10}$	$1.5x10^{-10}$	$4.0x10^{-12}$	$6.1x10^{-14}$
1.0	$1.6x10^{-6}$	$2.7x10^{-5}$	$1.9x10^{-6}$	$8.3x10^{-8}$	$4.0x10^{-9}$	$1.4x10^{-10}$	$5.37x10^{-12}$

The absolute error is seen to be smaller than that of (Darania et al., 2006) for all the cases with 7 or more collocation points.

The next problem from (Abbasbandy and Shivanian, 2011) is a linear Fredholm equation

$$y(x) = x + Sin(x) - \int_0^{\pi/2} xty(t)\, dt \quad (21)$$

with the exact solution $y = \sin x$. 10 terms are taken in the approximate solution

$$\hat{y}(x) = \sum_{n=1}^{10} C_n x^n \quad (22)$$

Residual is

$$R(x) = \hat{y}(x) - x - Sin(x) + \int_0^{\pi/2} xt\hat{y}(t)\, dt \tag{23}$$

Resulting linear system of 10 equations is solved and gives

$$y(x) = 1.00x - 5.1 \times 10^{-9}x^2 - 0.16x^3 - 2.03 \times 10^{-7}x^4 + 0.008x^5 - 0.0000010879403651713 34x^6 - 0.0001x^7 - 0.000001x^8 + 0.000003x^9 - 1.7 \times 10^{-7}x^{10} \tag{24}$$

Table 3 shows absolute errors. (Abbasbandy and Shivanian, 2011) gives errors as graphics and their method reproduces the exact solution for certain values of their "convergence control parameter". But this is due to the peculiarity of this specific problem having a simple exact solution.

Table 3. Collocation and Exact values for Linear Fredholm equation (21) in (Abbasbandy and Shivanian, 2011)

x	Exact	Collocation	Absolute Error
0.000000	0.000000	0.000000	0.000000
0.157080	0.156434	0.156434	$1.7\text{x}10^{-12}$
0.314159	0.309017	0.309017	$2.7\text{x}10^{-12}$
0.471239	0.453990	0.453990	$4.3\text{x}10^{-12}$
0.628319	0.587785	0.587785	$5.6\text{x}10^{-12}$
0.785398	0.707107	0.707107	$7.2\text{x}10^{-12}$
0.942478	0.809017	0.809017	$8.5\text{x}10^{-12}$
1.099560	0.891007	0.891007	$1.0\text{x}10^{-11}$
1.256640	0.951057	0.951057	$1.1\text{x}10^{-11}$
1.413720	0.987688	0.987688	$1.5\text{x}10^{-11}$
1.570800	1.000000	1.000000	$3.3\text{x}10^{-10}$

Our last example is a nonlinear Fredholm equation from (Maturi, 2019)

$$y(x) = Cos(x) - \frac{\pi^2}{48} + \frac{1}{12}\int_0^{\pi/2} ty^2(t)\, dt \tag{25}$$

With the exact solution $y = \cos x$. The approximate solution is in the form

$$\hat{y} = \sum_{n=0}^{N} C_n x^n \tag{26}$$

and calculations were carried out for $N = 6,7,8,9$ and 10. Table 4 shows the results

$$R(x) = \hat{y}(x) - Cos(x) + \frac{\pi^2}{48} + \int_0^x t\hat{y}^2(t)\,dt \tag{27}$$

For example, the approximate solution for $N = 6$ is

$$y(x) = 1.001 - 0.0104x - 0.475111x^2 - 0.030x^3 + 0.0627557928x^4 - 0.0079x^5 \tag{28}$$

Table 4. Absolute Errors for Nonlinear Fredholm equation (25) in (Maturi, 2019)

x	Ref [16]	N=6	N=7	N=8	N=9	N=10
0.1	$6.59x10^{-4}$	$9.89x10^{-8}$	$7.23x10^{-8}$	$1.60x10^{-7}$	$1.15x10^{-7}$	$1.63x10^{-11}$
0.2	$6.59x10^{-4}$	$5.04x10^{-8}$	$3.36x10^{-8}$	$6.02x10^{-6}$	$3.81x10^{-6}$	$4.33x10^{-10}$
0.3	$6.59x10^{-4}$	$2.42x10^{-8}$	$1.40x10^{-8}$	$2.12x10^{-6}$	$1.14x10^{-6}$	$1.29x10^{-10}$
0.4	$6.59x10^{-4}$	$1.17x10^{-8}$	$5.78x10^{-7}$	$1.03x10^{-6}$	$5.92x10^{-11}$	$1.05x10^{-10}$
0.5	$6.59x10^{-4}$	$6.94x10^{-7}$	$3.56x10^{-7}$	$1.00x10^{-6}$	$7.29x10^{-11}$	$1.33x10^{-10}$
0.6	$6.59x10^{-4}$	$6.12x10^{-7}$	$4.02x10^{-7}$	$1.21x10^{-6}$	$9.38x10^{-11}$	$1.49x10^{-10}$
0.7	$6.59x10^{-4}$	$6.88x10^{-7}$	$5.24x10^{-7}$	$1.37x10^{-6}$	$1.03x10^{-6}$	$1.51x10^{-10}$
0.8	$6.59x10^{-4}$	$7.93x10^{-7}$	$6.25x10^{-7}$	$1.43x10^{-6}$	$1.03x10^{-6}$	$1.48x10^{-10}$
0.9	$6.59x10^{-4}$	$8.70x10^{-7}$	$6.73x10^{-7}$	$1.41x10^{-6}$	$9.96x10^{-11}$	$1.46x10^{-10}$
1.0	$6.59x10^{-4}$	$9.04x10^{-7}$	$6.71x10^{-7}$	$1.38x10^{-6}$	$9.62x10^{-11}$	$1.57x10^{-10}$

The method (called successive approximation) used in (Maturi, 2019) results in a slightly shifted form of the exact solution cos x. Therefore, their error values (column 2 in Table 4) are the same at each point. This is again a peculiarity due to this problem having a simple closed form exact solution. It is seen that the collocation method gives better solutions even for $N = 6$, and it is likely to give good results for $N = 5$, 4, may be even 3.

3. Findings and Discussion

Two linear and two nonlinear Volterra and Fredholm integral equations were solved by the point collocation method. These equations were taken from (Wazwaz, 2011; Darania et al., 2006; Abbasbandy and Shivanian, 2011; Maturi, 2019). (Abbasbandy and Shivanian, 2011) is a linear and (Maturi, 2019). is a nonlinear Fredholm equation, while (Wazwaz, 2011) and (Darania et al., 2006) are linear and nonlinear Volterra equations, respectively. All equations have the solution domain $0 < x < 1$, and all of them were constructed to yield simple analytical solutions to facilitate comparison with numerical methods. It can be seen that about 5 collocation points give more than enough accuracy for all cases. In (Wazwaz, 2011), linear Volterra equation was solved by Adomian Decomposition method; in (Darania et al., 2006) nonlinear problem (Volterra equation) was solved by Linearization method; in (Abbasbandy and Shivanian, 2011) linear Fredholm equation was solved by Homotopy analysis method, and in (Maturi, 2019). nonlinear equation was solved by Successive approximation method. Although all of these methods give good results, the present study shows that collocation method gives similar results with possibly less work.

4. Conclusions and Recommendations

Using the collocation method, it was possible to find polynomial based solutions to the problems studied. Thanks to this method, the solution of a system of equations is all that is needed to solve integral equations. If the collocation method's base functions are selected as polynomials, there is no requirement to compute numerical integrals in the examined problems.

This approach significantly reduces the absolute errors in approximate solutions obtained with less complex algorithms. The technique is particularly useful to other researchers working in the field of numerical analysis, such as integral and integro differential, etc. It allows approximate solutions of equations to be found more easily.

For example; Collocation, viscoelasticity-induced heat modelling (Yang et al., 2023), flexoelectricity (Tannhäuser, 2023), vibration of functionally graded structures under water (Xi, et al., 2024), thermal buckling of nanocomposite plates (Huang et al., 2023) etc., was used in engineering topics. In the field of health, the method was used to examine the new Coronavirus (SARS CoV-2) spread model (Yüzbaşı and Yıldırım, 2023).

Since the equations of motion of plates and shells subject to flow-induced vibration are in integro-differential form, the collocation method can be used to solve these problems. In addition, the presented collocation method is a reliable method that can be applied to systems of integral and/or integro-differential equations.

In future studies, the following points should be taken into consideration when using the collocation method;

1. Unless the test functions are polynomials, it is necessary to select functions that are easy to integrate.

2. When examining a nonlinear problem, the computation time for calculating approximate solution coefficients increases. In order to reduce this time, various numerical analysis methods and problem-specific computation algorithms should be developed.

3. It is essential to examine the contrast between the values of the coefficients obtained from N collocation points and from $N+1$ collocation points when faced with a problem for which there is no analytical or numerical solution. Setting an acceptable criterion for this value will save significant time, especially in nonlinear studies.

Table 5. List of Symbols

Symbol	Symbol Name
$y(x)$	single variable function
$\hat{y}(x, C_1, C_2, \dots, C_N)$	approximate solution
L	differential or integral operator
B	boundary
R	residual
$C_1, C_2, \dots, C_N$	coefficients of the approximate solution
$W_n(x)$	weight function
δ	the delta function
x_n	collocation points

Authors' Contributions

All authors contributed equally to the study.

Statement of Conflicts of Interest

The authors declare no conflicts of interest.

Statement of Research and Publication Ethics

The authors declare that this study complies with Research and Publication Ethics.

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