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Application of Double Aboodh-Shehu Decomposition Method to Solve Linear and Nonlinear System of Partial Differential Equations

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Abstract. In this work, we develop a new method to obtain approximate solutions of linear and nonlinear coupled partial differential equations with the help of Double Aboodh-Shehu decomposition method (DASDM). The nonlinear term can easily be handled with the help of Adomian polynomials. The results of the present technique have closed agreement with approximate solutions obtained with the help of Adomian decomposition method (ADM).

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1. Introduction

The topic of partial differential equations is one of the most important subjects in mathematics and other sciences. Therefore, it is very important to know methods to solve such partial differential equations. In the literature, several different transforms are introduced and applied to find the solution of partial differential equations such as Laplace transform [8], Shehu transform [10], Aboodh transform [2], and so on. Two of the most popular methods for solving partial differential equations are the integral transforms method and Adomian decomposition method [13]. The decomposition method has been shown to solve efficiently, easily and accurately a large class of linear and nonlinear ordinary, partial, deterministic or stochastic differential equations [7, 16]. The method is very well suited to physical problems since it does not require unnecessary linearization, perturbation, discretization, or any unrealistic assumptions. The Adomian decomposition method is relatively easy to implement, and it can be used with other methods. It can also be used to solve both initial value problems and boundary value problems. In [9], the authors used Laplace transform with Adomian decomposition method to solve nonlinear coupled partial differential equations.

The main objective of this paper is to obtained the exact solutions of coupled linear and nonlinear partial differential equations with initial value problems by using double Aboodh-Shehu transform algorithm based on decomposition method.

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2. Preliminaries

Definition 2.1. The single Aboodh transform of the real function $f(x)$ of exponential order is defined over the set of functions

$$\mathcal{M} = \left\{ f(x) : \exists K, \tau_1, \tau_2 > 0, |f(x)| < Ke^{|x|\tau_i}, \quad x \in (-1)^i \times [0, \infty), \quad i = 1, 2 \right\},$$

by the following integral

$$A[f(x)] = F(r) = \frac{1}{r} \int_0^\infty e^{-rx} f(x) dx, \quad \tau_1 \leq r \leq \tau_2.$$

And the inverse Aboodh transform is

$$A^{-1}[F(r)] = f(x) = \frac{1}{2\pi i} \int_{\omega - i\infty}^{\omega + i\infty} re^{rx} F(r) dr, \quad \omega \geq 0.$$

Aboodh transform was introduced by K. Aboodh [2] in 2013 to facilitate the process of solving ordinary and partial differential equations in the time domain. For further details and properties of the Aboodh transform and its derivatives we refer to [1, 5].

Definition 2.2. The single Shehu transform of the function $f(t)$ of exponential order is defined over the set of functions,

$$\mathcal{B} = \left\{ f(t) : \exists N, \rho_1, \rho_2 > 0, | f(t) |< Ne^{\frac{|t|}{\rho_j}}, t \in (-1)^j \times [0, \infty), \quad j = 1, 2 \right\},$$

by the following integral

$$\mathbb{S}[f(t)] = F(s, \mathfrak{u}) = \int_0^\infty e^{\frac{-st}{\mathfrak{u}}} f(t) dt, \quad s > 0, \quad \mathfrak{u} > 0.$$

Moreover, the inverse Shehu transform is given by

$$f(t) = \mathbb{S}^{-1}[F(s, \mathfrak{u})] = \frac{1}{2\pi i} \int_{w - i\infty}^{w + i\infty} \frac{1}{\mathfrak{u}} e^{\frac{st}{\mathfrak{u}}} F(s, \mathfrak{u}) ds,$$

where s and $\mathfrak{u}$ are the Shehu transform variables, and w is a real constant and the integral in Eq. (2.2) is taken along $s = w$ in the complex plane $s = x + iy$.

For further details and properties of Shehu transform and its derivatives we refer to [3, 4, 10, 12].

Definition 2.3. The double Aboodh-Shehu transform of the continuous function $f(x, t)$, $x, t > 0$ is denoted by the operator $A_x \mathbb{S}_t[f(x, t)] = F(r, s, \mathfrak{u})$ and defined by

$$\begin{aligned} A_x \mathbb{S}_t[f(x, t)] = F(r, s, \mathfrak{u}) &= \frac{1}{r} \int_0^\infty \int_0^\infty e^{-(rx + \frac{st}{\mathfrak{u}})} f(x, t) dx dt \\ &= \frac{1}{r} \lim_{a \to \infty, b \to \infty} \int_0^a \int_0^b e^{-(rx + \frac{st}{\mathfrak{u}})} f(x, t) dx dt. \end{aligned}$$

It converges if the limit of the integral exists, and diverges if not.
The inverse of double Aboodh-Shehu transform is defined by

$$f(x, t) = A_x^{-1} \mathbb{S}_t^{-1} [F(r, s, \mathfrak{u})] = \frac{1}{(2\pi i)^2} \int_{\rho_1 - i\infty}^{\rho_1 + i\infty} re^{rx} \left\{ \int_{\rho_2 - i\infty}^{\rho_2 + i\infty} \frac{1}{\mathfrak{u}} e^{\frac{st}{\mathfrak{u}}} F(r, s, \mathfrak{u}) ds \right\} dr,$$

where ρ_1 and ρ_2 are real constants.

Double Aboodh-Shehu transform for second partial derivatives property

$$\begin{aligned} A_x \mathbb{S}_t[\frac{\partial^2 f(x, t)}{\partial x^2}] &= r^2 F(r, s, \mathfrak{u}) - \mathbb{S}[f(0, t)] - \frac{1}{r} \mathbb{S}[f_x(0, t)], \\ A_x \mathbb{S}_t[\frac{\partial^2 f(x, t)}{\partial t^2}] &= \frac{s^2}{\mathfrak{u}^2} F(r, s, \mathfrak{u}) - \frac{s}{\mathfrak{u}} A[f(x, 0)] - A[f_t(x, 0)], \\ A_x \mathbb{S}_t[\frac{\partial^2 f(x, t)}{\partial x \partial t}] &= \frac{sr}{\mathfrak{u}} F(r, s, \mathfrak{u}) - rA[f(x, 0)] - \frac{1}{r} \mathbb{S}[f_t(0, t)], \end{aligned}$$

where $A[.]$ and $\mathbb{S}[.]$ denote to single Aboodh transform and single Shehu transform, respectively.

Recently, in 2022, the authors in [11] discussed some theorems and properties about the double Aboodh-Shehu transform and gave the double Aboodh-Shehu transform of some elementary functions.

We consider the general inhomogeneous nonlinear partial differential equation with initial conditions given below:

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = f(x,t), \tag{2.1}$$

$$u(x,0) = f_1(x), \qquad u_t(x,0) = f_2(x), \tag{2.2}$$

where $L = \frac{\partial^2}{\partial t^2}$ is the second order derivative which is assumed to be easily invertible, R is the remaining linear differential operator, Nu represents the nonlinear terms and $f(x,t)$, $f_1(x)$ and $f_2(x)$ are known functions.
The methodology consists of applying double Aboodh-Shehu transform first on both sides of Eq. (2.1)

$$A_x\mathbb{S}_t\{Lu(x,t) + Ru(x,t) + Nu(x,t)\}= A_x\mathbb{S}_t\{f(x,t)\}. \tag{2.3}$$

By linearity property of double Aboodh-Shehu transform, Eq. (2.3) becomes

$$A_x\mathbb{S}_t[Lu(x,t)] + A_x\mathbb{S}_t[Ru(x,t)] + A_x\mathbb{S}_t[Nu(x,t)] = A_x\mathbb{S}_t[f(x,t)]. \tag{2.4}$$

Using the property of partial derivative of double Aboodh-Shehu transform for Eq. (2.4), we have

$$\begin{aligned} \frac{s^2}{\mathfrak{u}^2}U(r,s,\mathfrak{u}) - \frac{s}{\mathfrak{u}}A[u(x,0)] - A[u_t(x,0)] + A_x\mathbb{S}_t[Ru(x,t)] \\ + A_x\mathbb{S}_t[Nu(x,t)] = A_x\mathbb{S}_t[f(x,t)]. \end{aligned} \tag{2.5}$$

Using given initial conditions and arrangement, Eq. (2.5) becomes

$$\begin{aligned} U(r,s,\mathfrak{u}) = \frac{\mathfrak{u}}{s}A[f_1(x)] + \frac{\mathfrak{u}^2}{s^2}A[f_2(x)] + \frac{\mathfrak{u}^2}{s^2}A_x\mathbb{S}_t[f(x,t)] \\ - \frac{\mathfrak{u}^2}{s^2}A_x\mathbb{S}_t[Ru(x,t)] - \frac{\mathfrak{u}^2}{s^2}A_x\mathbb{S}_t[Nu(x,t)]. \end{aligned} \tag{2.6}$$

Application of inverse double Aboodh-Shehu transform to (2.6) leads to

$$\begin{aligned} u(x,t) = A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}}{s}A[f_1(x)] + \frac{\mathfrak{u}^2}{s^2}A[f_2(x)] + \frac{\mathfrak{u}^2}{s^2}A_x\mathbb{S}_t\big[f(x,t)\big]\Big] \\ - A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}^2}{s^2}A_x\mathbb{S}_t[Ru(x,t)]+\frac{\mathfrak{u}^2}{s^2}A_x\mathbb{S}_t[Nu(x,t)]\Big]. \end{aligned} \tag{2.7}$$

The second step in double Aboodh-Shehu decomposition method is that we represent solution as an infinite series:

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t), \tag{2.8}$$

and the nonlinear term can be decomposed as

$$Nu(x,t) = \sum_{i=0}^{\infty} A_i, \tag{2.9}$$

where A_i are the Adomian polynomials [15] of $u_0, u_1, u_2, ..., u_n$ and it can be calculated via the general formula given below

$$A_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\Big[N\sum_{i=0}^{\infty}\lambda^i u_i\Big]_{\lambda=0}.$$

Substituting Eq. (2.8) and Eq. (2.9) in Eq. (2.7), we get

$$\begin{aligned} \sum_{i=0}^{\infty} u_i(x,t) = A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}}{s}A[f_1(x)] + \frac{\mathfrak{u}^2}{s^2}A[f_2(x)] + \frac{\mathfrak{u}^2}{s^2}A_x\mathbb{S}_t[f(x,t)]\Big] \\ - A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}^2}{s^2}A_x\mathbb{S}_t\Big[R\sum_{i=0}^{\infty}u_i(x,t)\Big]+\frac{\mathfrak{u}^2}{s^2}A_x\mathbb{S}_t\Big[\sum_{i=0}^{\infty}A_i\Big]\Big]. \end{aligned} \tag{2.10}$$

On comparing both sides of the Eq. (2.10) and by using standard Adomian decomposition method (ADM), we then define the recurrence relations as

$$u_0(x,t) = A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}}{s}A[f_1(x)] + \frac{\mathfrak{u}^2}{s^2}A[f_2(x)] + \frac{\mathfrak{u}^2}{s^2}A_x\mathbb{S}_t[f(x,t)]\Big],$$
$$u_1(x,t) = -A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}^2}{s^2}A_x\mathbb{S}_t[Ru_0(x,t)] + \frac{\mathfrak{u}^2}{s^2}A_x\mathbb{S}_t[A_0]\Big],$$
$$u_2(x,t) = -A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}^2}{s^2}A_x\mathbb{S}_t[Ru_1(x,t)] + \frac{\mathfrak{u}^2}{s^2}A_x\mathbb{S}_t[A_1]\Big].$$

In more general, the recursive relation is given by

$$u_{i+1}(x,t) = -A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}^2}{s^2}A_x\mathbb{S}_t[Ru_i(x,t)] + \frac{\mathfrak{u}^2}{s^2}A_x\mathbb{S}_t[A_i]\Big], \quad i \geq 0.$$

The recurrence relation generates the solution of (2.1) in series form given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots + u_i(x,t) + \cdots.$$

3. Applications

In order to illustrate the applicability and efficiency of the double Aboodh-Shehu decomposition method, we apply this method to solve some equations.

Example 3.1. Consider the following nonlinear partial differential equation

$$u_{xx}(x,t) + \frac{1}{4}u_t^2(x,t) = u(x,t), \tag{3.1}$$

subject to the initial conditions

$$u(0,t) = 1 + t^2, \qquad u_x(0,t) = 1.$$

Applying double Aboodh-Shehu transform first on both sides of Eq. (3.1), we have

$$A_x\mathbb{S}_t[u_{xx}(x,t)] + A_x\mathbb{S}_t[\frac{1}{4}u_t^2(x,t)] = A_x\mathbb{S}_t[u(x,t)].$$

Using the differentiation property of double Aboodh-Shehu transform, we have

$$r^2U(r,s,\mathfrak{u}) - \mathbb{S}[u(0,t)] - \frac{1}{r}\mathbb{S}[u_x(0,t)] + A_x\mathbb{S}_t[\frac{1}{4}u_t^2] = A_x\mathbb{S}_t[u].$$

Rearranging the terms and using given initial conditions, we have

$$\begin{aligned} U(r,s,\mathfrak{u}) &= \frac{1}{r^2}\mathbb{S}[1+t^2] + \frac{1}{r^3}\mathbb{S}[1] + \frac{1}{r^2}A_x\mathbb{S}_t[u] - \frac{1}{4r^2}A_x\mathbb{S}_t[u_t^2] \\ &= \frac{\mathfrak{u}}{r^2s} + \frac{2!\mathfrak{u}^3}{r^2s^3} + \frac{\mathfrak{u}}{r^3s} + \frac{1}{r^2}A_x\mathbb{S}_t[u] - \frac{1}{4r^2}A_x\mathbb{S}_t[u_t^2]. \end{aligned} \tag{3.2}$$

Application of inverse double Aboodh-Shehu transform to (3.2) leads to

$$u(x,t) = 1 + t^2 + x + A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{1}{r^2}A_x\mathbb{S}_t[u] - \frac{1}{4r^2}A_x\mathbb{S}_t[u_t^2]\Big]. \tag{3.3}$$

The double Aboodh-Shehu decomposition method assumes a series solution of the function $u(x,t)$ is given by

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t). \tag{3.4}$$

Using Eq. (3.4) into Eq. (3.3) we get

$$\sum_{i=0}^{\infty} u_i(x,t) = 1 + t^2 + x + A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{1}{r^2}A_x\mathbb{S}_t\Big[\sum_{i=0}^{\infty} u_i\Big] - \frac{1}{4r^2}A_x\mathbb{S}_t\Big[\sum_{i=0}^{\infty} A_i(u)\Big]\Big], \tag{3.5}$$

where A_i are Adomian polynomials that represents nonlinear terms.
So, Adomian polynomials are given as follows:

$$\sum_{i=0}^{\infty} A_i(u) = u_t^2(x,t). \tag{3.6}$$

The few components of the Adomian polynomials are given as follow:

$$A_0(u) = u_{0t}^2, \quad A_1(u) = 2u_{0t}u_{1t}, \quad ..., \quad A_i(u) = \sum_{r=0}^{i} u_{rt}u_{(i-r)t}.$$

From Eqs. (3.5) and (3.6) we obtain

$$\begin{aligned} u_0 &= 1 + t^2 + x, \\ \sum_{i=0}^{\infty} u_{i+1}(x,t) &= A_x^{-1}\mathbb{S}_t^{-1}\left[\frac{1}{r^2}A_x\mathbb{S}_t\left[\sum_{i=0}^{\infty} u_i - \frac{1}{4}\sum_{i=0}^{\infty} A_i(u)\right]\right], \; i \geq 0. \end{aligned}$$

Then, the first few components of $u_i(x,t)$ follows immediately upon setting

$$\begin{aligned} u_1(x,t) &= A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{1}{r^2}A_x\mathbb{S}_t[u_0] - \frac{1}{4r^2}A_x\mathbb{S}_t[A_0(u)]\Big] \\ &= A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{1}{r^2}A_x\mathbb{S}_t[1 + t^2 + x] - \frac{1}{4r^2}A_x\mathbb{S}_t[4t^2]\Big] \\ &= A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}}{r^4 s} + \frac{\mathfrak{u}}{r^5 s}\Big] \\ &= \frac{1}{2!}x^2 + \frac{1}{3!}x^3, \\ u_2(x,t) &= A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{1}{r^2}A_x\mathbb{S}_t[u_1] - \frac{1}{4r^2}A_x\mathbb{S}_t[A_1(u)]\Big] \\ &= A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{1}{r^2}A_x\mathbb{S}_t[\frac{1}{2!}x^2 + \frac{1}{3!}x^3] - \frac{1}{4r^2}A_x\mathbb{S}_t[0]\Big] \\ &= A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}}{r^6 s} + \frac{\mathfrak{u}}{r^7 s}\Big] \\ &= \frac{1}{4!}x^4 + \frac{1}{5!}x^5, \end{aligned}$$

and so on for other components. Therefore, the exact solution obtained by double Aboodh-Shehu decomposition method is given as follows:

$$\begin{aligned} u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) &= t^2 + 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + ... \\ &= t^2 + \Big(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + ...\Big) \\ &= t^2 + e^x. \end{aligned}$$

Which is same as solution obtained by variational iteration method (VIM) [14].

Example 3.2. Consider the following linear system of partial differential equations

$$\begin{aligned} u_t(x,t) - v_x(x,t) - (u - v) &= -2, \\ v_t(x,t) + u_x(x,t) - (u - v) &= -2, \end{aligned} \tag{3.7}$$

with initial conditions

$$\begin{aligned} u(x,0) &= 1 + e^x, \\ v(x,0) &= -1 + e^x. \end{aligned} \tag{3.8}$$

Taking the double Aboodh-Shehu transform on both sides of (3.7), then by using the differentiation property of double Aboodh-Shehu transform, we have

$$\begin{aligned}\frac{s}{\mathfrak{u}}U(r,s,\mathfrak{u})-A[u(x,0)]&=A_x\mathbb{S}_t[-2]+A_x\mathbb{S}_t[v_x]+A_x\mathbb{S}_t[u-v],\\ \frac{s}{\mathfrak{u}}V(r,s,\mathfrak{u})-A[v(x,0)]&=A_x\mathbb{S}_t[-2]-A_x\mathbb{S}_t[u_x]+A_x\mathbb{S}_t[u-v].\end{aligned}\tag{3.9}$$

Application of single Aboodh transform to (3.8) and substitute in (3.9), we have

$$\begin{aligned}U(r,s,\mathfrak{u})&=\frac{\mathfrak{u}}{r^2s}+\frac{\mathfrak{u}}{r(r-1)s}-\frac{2\mathfrak{u}^2}{r^2s^2}+\frac{\mathfrak{u}}{s}A_x\mathbb{S}_t[v_x+(u-v)],\\ V(r,s,\mathfrak{u})&=-\frac{\mathfrak{u}}{r^2s}+\frac{\mathfrak{u}}{r(r-1)s}-\frac{2\mathfrak{u}^2}{r^2s^2}-\frac{\mathfrak{u}}{s}A_x\mathbb{S}_t[u_x-(u-v)].\end{aligned}\tag{3.10}$$

Taking the inverse double Aboodh-Shehu transform in (3.10), our required recursive relation is given by

$$\begin{aligned}u(x,t)&=1+e^x-2t+A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}}{s}A_x\mathbb{S}_t[v_x+(u-v)]\Big],\\ v(x,t)&=-1+e^x-2t-A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}}{s}A_x\mathbb{S}_t[u_x-(u-v)]\Big].\end{aligned}\tag{3.11}$$

The double Aboodh-Shehu decomposition method assumes a series solution of the functions $u(x,t)$ and $v(x,t)$ are given by

$$u(x,t)=\sum_{i=0}^{\infty}u_i(x,t),\quad v(x,t)=\sum_{i=0}^{\infty}v_i(x,t).\tag{3.12}$$

Using Eq. (3.12) into Eq. (3.11), we obtain

$$\sum_{i=0}^{\infty}u_i(x,t)=1+e^x-2t+A_x^{-1}\mathbb{S}_t^{-1}\left[\frac{\mathfrak{u}}{s}A_x\mathbb{S}_t\left[\sum_{i=0}^{\infty}v_{ix}+\sum_{i=0}^{\infty}(u_i-v_i)\right]\right],\tag{3.13}$$

$$\sum_{i=0}^{\infty}v_i(x,t)=-1+e^x-2t-A_x^{-1}\mathbb{S}_t^{-1}\left[\frac{\mathfrak{u}}{s}A_x\mathbb{S}_t\left[\sum_{i=0}^{\infty}u_{ix}-\sum_{i=0}^{\infty}(u_i-v_i)\right]\right].\tag{3.14}$$

From (3.13) and (3.14) the recursive relations are

$$\begin{aligned}u_0(x,t)&=1+e^x-2t,\\ u_{i+1}(x,t)&=A_x^{-1}\mathbb{S}_t^{-1}\left[\frac{\mathfrak{u}}{s}A_x\mathbb{S}_t\left[\sum_{i=0}^{\infty}v_{ix}+\sum_{i=0}^{\infty}(u_i-v_i)\right]\right],\ \ i\geq 0,\\ v_0(x,t)&=-1+e^x-2t,\\ v_{i+1}(x,t)&=-A_x^{-1}\mathbb{S}_t^{-1}\left[\frac{\mathfrak{u}}{s}A_x\mathbb{S}_t\left[\sum_{i=0}^{\infty}u_{ix}-\sum_{i=0}^{\infty}(u_i-v_i)\right]\right],\ \ i\geq 0.\end{aligned}\tag{3.15}$$

In view of the recursive relations (3.15) we obtained other components as follows

$$\begin{aligned}
u_1(x,t) &= A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}}{s}A_x\mathbb{S}_t\Big[v_{0x}+(u_0-v_0)\Big]\Big]= A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}}{s}A_x\mathbb{S}_t[e^x+2]\Big]\\
&= A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}^2}{r(r-1)s^2}+\frac{2\mathfrak{u}^2}{r^2s^2}\Big]\\
&= te^x+2t,\\
v_1(x,t) &= A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}}{s}A_x\mathbb{S}_t\Big[u_{0x}-(u_0-v_0)\Big]\Big]= -A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}}{s}A_x\mathbb{S}_t[e^x-2]\Big]\\
&= -A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}^2}{r(r-1)s^2}-\frac{2\mathfrak{u}^2}{r^2s^2}\Big]\\
&= -te^x+2t,\\
u_2(x,t) &= A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}}{s}A_x\mathbb{S}_t\Big[v_{1x}+(u_1-v_1)\Big]\Big]= A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}}{s}A_x\mathbb{S}_t[te^x]\Big]\\
&= A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}^3}{r(r-1)s^3}\Big]\\
&= \frac{t^2}{2!}e^x,\\
v_2(x,t) &= A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}}{s}A_x\mathbb{S}_t\Big[u_{1x}-(u_1-v_1)\Big]\Big]= -A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}}{s}A_x\mathbb{S}_t[-te^x]\Big]\\
&= -A_x^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}^3}{r(r-1)s^3}\Big]\\
&= \frac{t^2}{2!}e^x,
\end{aligned}$$

and so on for other components. The series solutions are given by

$$\begin{aligned}
u(x,t) &= \sum_{i=0}^{\infty}u_i(x,t) = 1+e^x\Big(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+...\Big),\\
v(x,t) &= \sum_{i=0}^{\infty}v_i(x,t) = -1+e^x\Big(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+...\Big).
\end{aligned}$$

Then the solutions obtained by double Aboodh-Shehu decomposition method of (3.7) are given as follows:

$$\begin{aligned}
u(x,t) &= 1+e^{x+t},\\
v(x,t) &= -1+e^{x-t}.
\end{aligned}$$

Which is same as solution obtained by Sumudu decomposition method [6].

Example 3.3. Consider the system of nonlinear partial differential equations

$$\begin{aligned}
u_t+vu_y+u &= 1,\\
v_t-uv_y-v &= 1,
\end{aligned} \tag{3.16}$$

with initial conditions

$$\begin{aligned}
u(y,0) &= e^y,\\
v(y,0) &= e^{-y}.
\end{aligned} \tag{3.17}$$

Applying the double Aboodh-Shehu transform to both sides of equations (3.16), we have

$$\begin{aligned}
\tfrac{s}{\mathfrak{u}}U(r,s,\mathfrak{u})-A[u(y,0)] &= A_y\mathbb{S}_t[1]-A_y\mathbb{S}_t[vu_y+u],\\
\tfrac{s}{\mathfrak{u}}V(r,s,\mathfrak{u})-A[v(y,0)] &= A_y\mathbb{S}_t[1]+A_y\mathbb{S}_t[uv_y+v].
\end{aligned} \tag{3.18}$$

Application of single Aboodh transform to (3.17) and substitute in (3.18), we have

$$\begin{aligned}
U(r,s,\mathfrak{u}) &= \tfrac{\mathfrak{u}}{r(r-1)s}+\tfrac{\mathfrak{u}^2}{r^2s^2}-\tfrac{\mathfrak{u}}{s}A_y\mathbb{S}_t[vu_y+u],\\
V(r,s,\mathfrak{u}) &= \tfrac{\mathfrak{u}}{r(r+1)s}+\tfrac{\mathfrak{u}^2}{r^2s^2}+\tfrac{\mathfrak{u}}{s}A_y\mathbb{S}_t[uv_y+v].
\end{aligned} \tag{3.19}$$

By taking the inverse double Aboodh-Shehu transform in (3.19), we get

$$u(y,t) = e^{y} + t - A_y^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}}{s}A_y\mathbb{S}_t[vu_y + u]\Big],$$
$$v(y,t) = e^{-y} + t + A_y^{-1}\mathbb{S}_t^{-1}\Big[\frac{\mathfrak{u}}{s}A_y\mathbb{S}_t[uv_y + v]\Big].$$

The recursive relations are

$$\begin{aligned}
u_0(y,t) &= e^{y},\\
u_{i+1}(y,t) &= t - A_y^{-1}\mathbb{S}_t^{-1}\left[\frac{\mathfrak{u}}{s}A_y\mathbb{S}_t\left[\sum_{i=0}^{\infty} C_i(v,u) + \sum_{i=0}^{\infty} u_i\right]\right], \quad i \geq 0,\\
v_0(y,t) &= e^{-y},\\
v_{i+1}(y,t) &= t + A_y^{-1}\mathbb{S}_t^{-1}\left[\frac{\mathfrak{u}}{s}A_y\mathbb{S}_t\left[\sum_{i=0}^{\infty} D_i(u,v) + \sum_{i=0}^{\infty} v_i\right]\right], \quad i \geq 0,
\end{aligned} \tag{3.20}$$

where $u(y,t)$ and $v(y,t)$ are linear terms represented by the decomposition series and $C_i(v,u)$ and $D_i(u,v)$ are Adomian polynomials representing the nonlinear terms [15]. The few components of Adomian polynomials are given as follow

$$\begin{aligned}
C_0(v,u) &= v_0u_{0y},\\
C_1(v,u) &= v_0u_{1y} + v_1u_{0y},\\
C_2(v,u) &= v_0u_{2y} + v_1u_{1y} + v_2u_{0y},\\
C_3(v,u) &= v_0u_{3y} + v_1u_{2y} + v_2u_{1y} + v_3u_{0y},\\
&\ \vdots\\
C_i(v,u) &= \sum_{n=0}^{i} v_nu_{(i-n)y},\\
D_0(u,v) &= u_0v_{0y},\\
D_1(u,v) &= u_0v_{1y} + u_1v_{0y},\\
D_2(u,v) &= u_0v_{2y} + u_1v_{1y} + u_2v_{0y},\\
D_3(u,v) &= u_0v_{3y} + u_1v_{2y} + u_2v_{1y} + u_3v_{0y},\\
&\ \vdots\\
D_i(u,v) &= \sum_{n=0}^{i} u_nv_{(i-n)y}.
\end{aligned}$$

Using the derived Adomian polynomials into (3.20), we obtain

$$\begin{aligned}
u_0(y,t) &= e^{y},\\
v_0(y,t) &= e^{-y},\\
u_1(y,t) &= t - A_y^{-1}\mathbb{S}_t^{-1}[\frac{\mathfrak{u}}{s}A_y\mathbb{S}_t[C_0(v,u) + u_0]] = t - A_y^{-1}\mathbb{S}_t^{-1}[\frac{\mathfrak{u}}{s}A_y\mathbb{S}_t[v_0u_{0y} + u_0]]\\
&= t - A_y^{-1}\mathbb{S}_t^{-1}[\frac{\mathfrak{u}}{s}A_y\mathbb{S}_t[1 + e^{y}]] = t - A_y^{-1}\mathbb{S}_t^{-1}[\frac{\mathfrak{u}^2}{r^2s^2} + \frac{\mathfrak{u}^2}{r(r-1)s^2}]\\
&= -te^{y},\\
v_1(y,t) &= t + A_y^{-1}\mathbb{S}_t^{-1}[\frac{\mathfrak{u}}{s}A_y\mathbb{S}_t[D_0(u,v) + v_0]] = t + A_y^{-1}\mathbb{S}_t^{-1}[\frac{\mathfrak{u}}{s}A_y\mathbb{S}_t[u_0v_{0y} + v_0]]
\end{aligned}$$

$$
\begin{aligned}
&= t + A_y^{-1}\mathbb{S}_t^{-1}[\frac{\mathfrak{u}}{s}A_y\mathbb{S}_t[-1+e^{-y}]] = t + A_y^{-1}\mathbb{S}_t^{-1}[-\frac{\mathfrak{u}^2}{r^2s^2} + \frac{\mathfrak{u}^2}{r(r+1)s^2}]\\
&= te^{-y},\\
u_2(y,t) &= -A_y^{-1}\mathbb{S}_t^{-1}[\frac{\mathfrak{u}}{s}A_y\mathbb{S}_t[C_1(v,u)+u_1]] = -A_y^{-1}\mathbb{S}_t^{-1}[\frac{\mathfrak{u}}{s}A_y\mathbb{S}_t[v_0u_{1y}+v_1u_{0y}+u_1]]\\
&= -A_y^{-1}\mathbb{S}_t^{-1}[\frac{\mathfrak{u}}{s}A_y\mathbb{S}_t[-te^{y}]] = -A_y^{-1}\mathbb{S}_t^{-1}[-\frac{\mathfrak{u}^3}{r(r-1)s^3}]\\
&= \frac{t^2}{2!}e^{y},\\
v_2(y,t) &= A_y^{-1}\mathbb{S}_t^{-1}[\frac{\mathfrak{u}}{s}A_y\mathbb{S}_t[D_1(u,v)+v_0]] = A_y^{-1}\mathbb{S}_t^{-1}[\frac{\mathfrak{u}}{s}A_y\mathbb{S}_t[u_0v_{1y}+u_1v_{0y}+v_1]]\\
&= A_y^{-1}\mathbb{S}_t^{-1}[\frac{\mathfrak{u}}{s}A_y\mathbb{S}_t[te^{-y}]] = A_y^{-1}\mathbb{S}_t^{-1}[\frac{\mathfrak{u}^3}{r(r+1)s^3}]\\
&= \frac{t^2}{2!}e^{-y}.
\end{aligned}
$$

In the same way we can get

$$
\begin{aligned}
u_3(y,t) &= -\frac{t^3}{3!}e^{y},\\
v_3(y,t) &= \frac{t^3}{3!}e^{-y},
\end{aligned}
$$

and so on for other components. Therefore, the solutions obtained by double Aboodh-Shehu decomposition method are given by

$$
\begin{aligned}
u(y,t) &= \sum_{i=0}^{\infty} u_i(y,t) = e^{y}\Big(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + ...\Big) = e^{y-t},\\
v(y,t) &= \sum_{i=0}^{\infty} v_i(y,t) = e^{-y}\Big(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + ...\Big) = e^{-y+t}.
\end{aligned}
$$

Example 3.4. Consider the system of nonlinear partial differential equations

$$
\begin{aligned}
u_y(x,y,t) - v_xw_t &= -1,\\
v_y(x,y,t) - w_xu_t &= 1,\\
w_y(x,y,t) - u_xv_t &= -5,
\end{aligned}
\tag{3.21}
$$

with initial conditions

$$
\begin{aligned}
u(x,0,t) &= x + 3t,\\
v(x,0,t) &= x + 3t,\\
w(x,0,t) &= -x + 3t.
\end{aligned}
\tag{3.22}
$$

Taking the double Aboodh-Shehu transform to both sides of equations (3.21), we have

$$
\begin{aligned}
ru(x,r,s,\mathfrak{u}) - \frac{1}{r}\mathbb{S}[u(x,0,t)] &= -\frac{1}{r^2} + A_y\mathbb{S}_t[v_xw_t],\\
rv(x,r,s,\mathfrak{u}) - \frac{1}{r}\mathbb{S}[v(x,0,t)] &= \frac{1}{r^2} + A_y\mathbb{S}_t[w_xu_t],\\
rw(x,r,s,\mathfrak{u}) - \frac{1}{r}\mathbb{S}[w(x,0,t)] &= -\frac{5}{r^2} + A_y\mathbb{S}_t[u_xv_t].
\end{aligned}
\tag{3.23}
$$

Application of single Shehu transform to (3.22) then, substitute in (3.23) and rearranging the terms, we have

$$\begin{aligned} u(x,r,s,\mathrm{u}) &= \frac{x\mathrm{u}}{r^2 s} + \frac{3\mathrm{u}^2}{r^2 s^2} - \frac{1}{r^3} + \frac{1}{r} A_y \mathbb{S}_t [v_x w_t], \\ v(x,r,s,\mathrm{u}) &= \frac{x\mathrm{u}}{r^2 s} + \frac{3\mathrm{u}^2}{r^2 s^2} + \frac{1}{r^3} + \frac{1}{r} A_y \mathbb{S}_t [w_x u_t], \\ w(x,r,s,\mathrm{u}) &= -\frac{x\mathrm{u}}{r^2 s} + \frac{3\mathrm{u}^2}{r^2 s^2} - \frac{5}{r^3} + \frac{1}{r} A_y \mathbb{S}_t [u_x v_t]. \end{aligned} \tag{3.24}$$

By taking the inverse double Aboodh-Shehu transform in (3.24), we get

$$\begin{aligned} u(x,y,t) &= x + 3t - y + A_y^{-1}\mathbb{S}_t^{-1}\Big[\frac{1}{r} A_y \mathbb{S}_t [v_x w_t]\Big], \\ v(x,y,t) &= x + 3t + y + A_y^{-1}\mathbb{S}_t^{-1}\Big[\frac{1}{r} A_y \mathbb{S}_t [w_x u_t]\Big], \\ w(x,y,t) &= -x + 3t - 5y + A_y^{-1}\mathbb{S}_t^{-1}\Big[\frac{1}{r} A_y \mathbb{S}_t [u_x v_t]\Big]. \end{aligned}$$

The recursive relations are

$$\begin{aligned} u_0(x,y,t) &= x - y + 3t, \\ u_{i+1}(x,y,t) &= A_y^{-1}\mathbb{S}_t^{-1}\left[\frac{1}{r} A_y \mathbb{S}_t \left[\sum_{i=0}^{\infty} E_i(v,w)\right]\right], \quad i \geq 0, \\ v_0(x,y,t) &= x + y + 3t, \\ v_{i+1}(x,y,t) &= A_y^{-1}\mathbb{S}_t^{-1}\left[\frac{1}{r} A_y \mathbb{S}_t \left[\sum_{i=0}^{\infty} F_i(w,u)\right]\right], \quad i \geq 0, \\ w_0(x,y,t) &= -x - 5y + 3t, \\ w_{i+1}(x,y,t) &= A_y^{-1}\mathbb{S}_t^{-1}\left[\frac{1}{r} A_y \mathbb{S}_t \left[\sum_{i=0}^{\infty} G_i(u,v)\right]\right], \quad i \geq 0, \end{aligned}$$

where $E_i(v,w)$, $F_i(w,u)$, and $G_i(u,v)$ are Adomian polynomials representing the nonlinear terms [15] in above equations. The few components of Adomian polynomials are given as follow

$$\begin{aligned} E_0(v,w) &= v_{0x} w_{0t}, \\ E_1(v,w) &= v_{1x} w_{0t} + v_{0x} w_{1t}, \\ &\vdots \\ E_i(v,w) &= \sum_{n=0}^{i} v_{nx} w_{(i-n)t}, \\ F_0(w,u) &= w_{0x} u_{0t}, \\ F_1(w,u) &= w_{1x} u_{0t} + w_{0x} u_{1t}, \\ &\vdots \\ F_i(w,u) &= \sum_{n=0}^{i} w_{nx} u_{(i-n)t}, \\ G_0(u,v) &= u_{0x} v_{0t}, \\ G_1(u,v) &= u_{1x} v_{0t} + u_{0x} v_{1t}, \\ &\vdots \\ G_i(u,v) &= \sum_{n=0}^{i} u_{nx} v_{(i-n)t}. \end{aligned}$$

In view of this recursive relations we obtained other components of the solution as follows

$$
\begin{aligned}
u_1(x,y,t) &= A_y^{-1}\mathbb{S}_t^{-1}\Big[\frac{1}{r}A_y\mathbb{S}_t[E_0(v,w)]\Big] = A_y^{-1}\mathbb{S}_t^{-1}\Big[\frac{1}{r}A_y\mathbb{S}_t[v_{0x}w_{0t}]\Big] = A_y^{-1}\mathbb{S}_t^{-1}\Big[\frac{3}{r^3}\Big] = 3y,\\
v_1(x,y,t) &= A_y^{-1}\mathbb{S}_t^{-1}\Big[\frac{1}{r}A_y\mathbb{S}_t[F_0(w,u)]\Big] = A_y^{-1}\mathbb{S}_t^{-1}\Big[\frac{1}{r}A_y\mathbb{S}_t[w_{0x}u_{0t}]\Big] = A_y^{-1}\mathbb{S}_t^{-1}\Big[\frac{-3}{r^3}\Big] = -3y,\\
w_1(x,y,t) &= A_y^{-1}\mathbb{S}_t^{-1}\Big[\frac{1}{r}A_y\mathbb{S}_t[G_0(u,v)]\Big] = A_y^{-1}\mathbb{S}_t^{-1}\Big[\frac{1}{r}A_y\mathbb{S}_t[u_{0x}v_{0t}]\Big] = A_y^{-1}\mathbb{S}_t^{-1}\Big[\frac{3}{r^3}\Big] = 3y,\\
u_2(x,y,t) &= A_y^{-1}\mathbb{S}_t^{-1}\Big[\frac{1}{r}A_y\mathbb{S}_t[E_1(v,w)]\Big] = A_y^{-1}\mathbb{S}_t^{-1}\Big[\frac{1}{r}A_y\mathbb{S}_t[v_{1x}w_{0t}+v_{0x}w_{1t}]\Big] = 0,\\
v_2(x,y,t) &= A_y^{-1}\mathbb{S}_t^{-1}\Big[\frac{1}{r}A_y\mathbb{S}_t[F_1(w,u)]\Big] = A_y^{-1}\mathbb{S}_t^{-1}\Big[\frac{1}{r}A_y\mathbb{S}_t[w_{1x}u_{0t}+w_{0x}u_{1t}]\Big] = 0,\\
w_2(x,y,t) &= A_y^{-1}\mathbb{S}_t^{-1}\Big[\frac{1}{r}A_y\mathbb{S}_t[G_1(u,v)]\Big] = A_y^{-1}\mathbb{S}_t^{-1}\Big[\frac{1}{r}A_y\mathbb{S}_t[u_{1x}v_{0t}+u_{0x}v_{1t}]\Big] = 0.
\end{aligned}
$$

Similarly, $u_3(x,y,t) = v_3(x,y,t) = w_3(x,y,t) = 0$ and so on for rest terms.
Therefore, the solution of system (3.21) of nonlinear partial differential equations is given below

$$
\begin{aligned}
u(x,y,t) &= \sum_{i=0}^{\infty} u_i(x,y,t) = x + 2y + 3t,\\
v(x,y,t) &= \sum_{i=0}^{\infty} v_i(x,y,t) = x - 2y + 3t,\\
w(x,y,t) &= \sum_{i=0}^{\infty} w_i(x,y,t) = -x - 2y + 3t.
\end{aligned}
$$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have read and agreed to the published version of the manuscript.

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