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On Taxicab Circle Inverses of Lines

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Abstract — In this study, the inverses of lines with respect to the taxicab circle inversion are investigated. It is shown that the image of a line not passing through the inversion center is a closed curve consisting of two parabola arcs or a parabola arc and a line segment in taxicab plane. The properties of closed curves, which are taxicab circle inverses of lines are analytically determined according to vertical, horizontal, steep, gradual or separator line types. The distinctive properties of the taxicab circle inverses of lines are presented.

Subject Classification (2020): 51B20; 51F99; 51K99.

1. Introduction

The circle inversion is one of the most important and interesting geometric transformations. The inversion in a circle was introduced by Apollonius of Perga in his work "Plane Loci" and systematically studied by Steiner in 1830s. Since inversions have attracted attention of scientists from past to present, there are a lot of studies about them.

The circle inversions preserve angles and transform straight lines and circles into straight lines and/or circles. Many challenging problems in geometry become much more manageable when inversion is applied. Numerous scientists have studied and continue to study various aspects of this concept. Several generalizations of the inversion transformation have been introduced in the literature. In [7,9], the inversions with respect to the central conics were defined in Euclidean plane.

Non-Euclidean metric geometries have various applications in mathematics, physics, computer science, engineering and other fields, depending on the specific properties and distance functions they use. Among these geometries equipped with non-Euclidean metrics, taxicab geometry and maximum plane geometry have a rich literature [1-3,6,10,17-18,23]. The inversion with respect to taxicab circle has been defined and some properties such as cross ratio and harmonic conjugates have been given in [5]. Subsequently, the inversion in alpha plane [15], Chinese-Checker plane [21] and maximum plane [24] have been presented, and their corresponding features were examined. The circle inversion has been generalized to the spherical inversion in the three-dimensional taxicab space [20], Chinese-Checker space [19] and maximum space [8], utilizing a sphere. In [22], p-circle inversion which generalizes the

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classical inversion with respect to a circle (p = 2) and the taxicab inversion (p = 1), is defined, and new fractal patterns were obtained by applying this transformation to well-known fractals. A generalization of the alpha circle inversion fractal is also provided in [16].

In Euclidean geometry, the inverses of lines differ depending on whether they pass through the inversion center or not. In Euclidean circle inversion, inversion transforms the lines not passing through the inversion center into circles passing through the inversion center, circles passing through the inversion center into lines not passing through the center, and circles not passing through the center into circles not passing through the center. In some studies on circle inversion in non-Euclidean planes, the inverses of lines with this feature have been examined in the literature. However, it has been observed that this feature alone is not sufficient to classify the images of lines under the circle inversion in the taxicab plane and the maximum plane [4,11-17]. Therefore, in this study, it is aimed to determine the circle inversion of lines and their properties according to their types in the taxicab plane.

In this paper, the properties of images under the taxicab circle inversion have been analyzed analytically according to the types of lines. It is shown that the image of a line which does not pass through the center of the taxicab circle inversion is a closed curve different from a taxicab circle. The properties of these closed curves, which are taxicab circle inverses of lines, are determined according to vertical, horizontal, steep, gradual or separator line types. It is also demonstrated that the parallel line pencil forms a closed curve pencil passing through the inversion center under inversion with respect to the taxicab circle.

2. Preliminaries

We summarize below some definitions and theorems from the literature that are necessary for this study

The taxicab plane $\mathbb{R}_T^2$ is almost the same as the Euclidean plane $\mathbb{R}_T^2$. The points and the lines are the same, and the angles are measured in the same way. However, the distance function is different. In general, the taxicab distance between two points is measured as the sum of the change in horizontal and vertical directions between the two points, where Euclidean geometry is measured using the Pythagorean theorem.

Definition 2.1. Let P_1 and P_2 be two points whose coordinates are (x_1, y_1) and (x_2, y_2) in analytical plane, respectively. The taxicab distance between these points is $d_T(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$.

The isometry group of taxicab plane is the semi direct product of D(4) and T(2) where D(4) is the symmetry group of Euclidean square and T(2) is the group of all translations in the plane [23].

In [18], Krause classified lines depending on their slope as the following definition:

Definition 2.2 Let m be the slope of the line l in taxicab plane. The line l is called the steep line, the gradual line and the separator line in the cases of $| m |>1$, $| m |<1$ and $| m |=1$, respectively. In the special cases that the line l is parallel to x-axis or y-axis, l is named as the horizontal line or the vertical line, respectively [18].

Definition 2.3. The taxicab circle C_T with the center M and the radius r consists of the points X which satisfies the equation $d_T(M, X) = \text{r}$. The point M is called center of the taxicab circle, and r is called the length of the radius or simply the radius of the taxicab circle.

Every taxicab circle in the taxicab plane is an Euclidean square having sides with slopes ± 1. It is seen by definition 2.3 that the taxicab circle centered at the point $M = (m_1, m_2)$ with the radius r is the set $C_T = \{(x, y): |x_1 - m_1| + |y_1 - m_2| = \text{r}\}$. As particular case, the taxicab unit circle is the set $\{(x, y): |x| + |y| = 1\}$.

In the taxicab plane, the inversion with respect to the circle C_T with the center O and the radius $\mathrm{r_T}$ is denoted by $I_{(O,r)}$ and is defined as follows: For the collinear points O,the point P, and its image P'on the ray OP, $d_T(O,P).d_T(O,P') = \mathrm{r}^2$, where $d_T(O,P)$ represents the taxicab distance between O and P, [5].

Clearly, if P' is the inverse point of P, then P is the inverse point of the P'. Note that if P is in the interior of C_T, P' is exterior to C_T; and vice-versa. So, the interior of C_T except for O is mapped to the exterior and the exterior to the interior. C_T itself is left by the inversion pointwise fixed. O has no image, and no point of the plane is mapped to O. However, we can add to the taxicab plane a single point at infinite O_∞, which is the inverse of the center O of taxicab inversion circle C_T. So, the taxicab circle inversion $I_{(O,r)}$ is one-to-one map of extended taxicab plane.

Now in the extended taxicab plane $\mathbb{R}_T^2 \cup \{O_\infty\}$, the definition of inversion with respect to a taxicab circle C_T can be given as follows:

Definition 2.4. The transformation

$$I_{(O,r)}: \mathbb{R}_T^2 \cup \{O_\infty\} \to \mathbb{R}_T^2 \cup \{O_\infty\}$$
$$P \to I_{(O,r)}(P) = P'$$

defined by the circle C_T is called the taxicab circle inversion. The circle C_T is known as taxicab inversion circle, O is called the center of the taxicab inversion, r is called the taxicab inversion radius, and P' is called the taxicab circle inverse of the point P, [5].

For any point P on the taxicab inversion circle, the taxicab circle inversion map has the property $I_{(O,r)}(P) = P$.

Theorem 2.5. The taxicab circle inversion maps the point inside of the taxicab inversion circle to the point outside of it, and vice versa, [5].

Teorem 2.6. If the points $P = (x,y)$ and $P' = (x', y')$ are a pair of the inverse points in the taxicab circle inversion with the center $O = (0,0)$ and radius r, the following equality exists between the coordinates of P and P'

$$(x',y') = \left(\frac{r^2x}{(|x|+|y|)^2}, \frac{r^2y}{(|x|+|y|)^2}\right),$$

[5].

Corollary 2.7. If the points $P = (x,y)$ and $P' = (x', y')$ are a pair of the inverse points in the taxicab circle inversion with the center $O = (a,b)$ and radius r, the following equality exists between the coordinates of P and P'

$$(x',y') = \left(\frac{r^2(x-a)}{(|x-a|+|y-b|)^2}, \frac{r^2(y-b)}{(|x-a|+|y-b|)^2}\right),$$

[5].

Since the translation transformation preserves the taxicab distance in the taxicab plane, the center of the taxicab inversion circle can be taken as the origin without loss of generality. Therefore, throughout this paper, the taxicab inversion center will be considered as the origin unless otherwise stated.

3. Images of the lines under the taxicab circle inversion

In the Euclidean circle, inversion transforms the lines not passing through the inversion center into circles passing through the inversion center, circles passing through the inversion center into lines not passing through the center and circles not passing through the center into circles not passing through the center. In taxicab plane and maximum plane, lines passing through the inversion center are invariant under the inversion transformation, but the images of the lines not passing through the center have different shapes [4, 5, 11, 24].

In this section, the images of lines not passing through the inversion center in the extended taxicab plane are analytically considered and their properties are presented depending on their positions in the plane.

Theorem 3.1. Lines not passing through the center of the taxicab inversion circle do not remain invariant under taxicab circle inversion.

Proof. Let $O = (0,0)$ be the center of the taxicab inversion circle C_T with radius *r*, and let l be a line defined by the equation $ax + by + c = 0$, where at least one of a and b is non-zero, $c \neq 0$, $a, b, c \in \mathbb{R}$. The image of l under the taxicab inversion $I_{(0,r)}$ is given by the equation $ar^2x + br^2y + c(|x| + |y|)^2 = 0$. Since the coefficient c is not zero, the equation does not specify a line in the taxicab plane. Therefore, in the extented taxicab plane, lines that do not pass through the center of inversion do not remain invariant under the taxicab inversion transformation. This concludes the proof.

Theorem 3.2. The inverses of horizontal lines with respect to the taxicab circle are closed curves formed by the union of segments of two orthogonal parabolas passing through the inversion center."

Proof. Let $O = (0,0)$ be the center of the taxicab inversion circle with radius r and let l be a line defined by the equation $y = k, k \neq 0, k \in \mathbb{R}$. The inverse of the line l in C_T is the closed curve with equation $y = k(|x| + |y|)^2$. This means that the image is the closed curve consisting the union of two orthogonal parabola arcs passing through the origin and having the equations $y = k(x - y)^2$ and $y = k(-x + y)^2$. The axes of symmetry of the parabola segments forming this closed curve are $y = \frac{1}{4k} - |x|$ and directrices are $y = -\frac{1}{4k} + |x|$. So, the axes and directrices are perpendicular to each other since their slopes are 1 and -1. Hence, the inverses of the lines paralel to x-axis with respect to the taxicab circle are closed curves formed by the union of two parabola segments passing through the center of inversion and whose axes and directrices are perpendicular to each other.

In addition, when the slope of the symmetry axis is -1, the vertex and the focus of the parabola segment are obtained as $T_1 = (\frac{3}{16k}, \frac{1}{16k})$ and $O_1 = (\frac{1}{8k}, \frac{1}{8k})$, respectively. If the slope of the symmetry axis of the parabola is +1, the vertex and the focus of the parabola are obtained as $T_2 = (-\frac{3}{16k}, \frac{1}{16k})$ and $O_2 = (-\frac{1}{8k}, \frac{1}{8k})$, respectively.

Also, the following result are immediately obtained from the proof of Theorem 3.2.

Corollary 3.3. The taxicab inversion of a pencil of horizontal parallel lines not passing through the inversion center consists of a pencil of closed curves formed by the union of two parabola segments with symmetry axes are parallel to the separator lines. Also, each curve pencil in the taxicab inversion passes through the inversion center and is symmetric with respect to the perpendicular line passing through the inversion center.

Example 3.4. In Figure 1 (left), we show the taxicab inverse *I(l)* with the equation $y = k(|x| + |y|)^2$ in the taxicab unit circle centered at origin O of the line *l* with the equation $y = 1$; in Figure 1 (right), we illustrate the taxicab inversion with respect to the taxicab unit circle centered at the origin O for a pencil of horizontal parallel lines that do not intersect the inversion center.

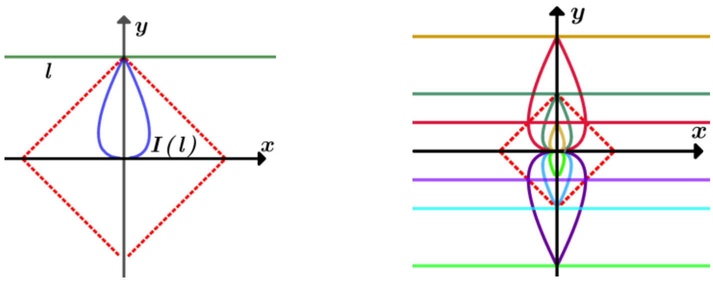

Figure 1. The taxicab circle inverses of parallel lines

Theorem 3.5. The symmetry axes and directrices of two parabola segments forming the inversion of a horizontal line with respect to a taxicab circle define a taxicab circle whose center is the center of inversion.

Proof. Inversion of the horizontal line with the equation $y = k,\ k \neq n, k \in \mathbb{R}$ with respect to a taxicab circle with center (m, n) and radius r is a closed curve with equation $(k-n)(|x-m|+|y-n|)^2 = r^2(y-n)$. This closed curve consists of the parabola segments with the symmetry axis $y = -x + m + n + \frac{r^2}{4(k-n)}$ and with the directrix $y = x + n - m - \frac{r^2}{4(k-n)}$ and the parabola segment with symmetry axis $y = x + n - m - \frac{r^2}{4(k-n)}$ and with the directrix $y = -x + n + m - \frac{r^2}{4(k-n)}$, respectively. The axes and directrices of these parabolas intersect at the points $\left(m, n + \frac{r^2}{4(k-n)}\right)$, $\left(m, n - \frac{r^2}{4(k-n)}\right)$, $\left(m + \frac{r^2}{4(k-n)}, n\right)$ and $\left(m - \frac{r^2}{4(k-n)}, n\right)$. Thus, the taxicab circle is formed, whose vertices are these points and whose edges are on the axes and directrices of the parabolas, with the equation $|x-m|+|y-n| = \frac{r^2}{4(k-n)}$. This completes the proof.

The reflection transformations with respect to the lines $y = x$ and $y = -x$ in the taxicab plane are isometries. Therefore, the theorems given for the taxicab inverses of horizontal lines can be given for taxicab inverses of vertical lines, too.

Theorem 3.6. The inverse of a vertical line with respect to the taxicab circle is closed curve formed by the union of two parabola arcs with axes and directrices perpendicular to each other and passing through inversion center.

Proof. Since the reflection transformation in the taxicab plane with respect to the line $y = x$ is an isometry, it can be easily proved by substituting the unknowns x and y in the proof of Theorem 3.2.

Corollary 3.7. The taxicab inversion of a pencil of vertical parallel lines with respect to the taxicab circle consists of a pencil of closed curves such that each closed curve in the pencil passes through the inversion center and is symmetric with respect to the horizontal line passing through the inversion center.

Proof. Since the inverse of each line in the vertical parallel line pencil with respect to the taxicab circle is closed curve formed by the union of two parabolas with axes and directrices perpendicular to each other and passing through the center of inversion, the proof is obvious.

Theorem 3.8. The axes and directrices of two parabola segments, which together compose the taxicab circle inverse of a vertical line with respect to a taxicab circle, determine a taxicab circle centered at the inversion center.

Proof. It can be easily proved by substituting the unknowns x and y in the proof of Theorem 3.4.

Theorem 3.9. The inverse of a separator line not passing through the center of the inversion circle is a closed figure consisting of a parabola arc with the vertex at the inversion center and a line segment.

Proof. Let $O = (0,0)$ be the center of the taxicab inversion circle C_T with radius r, and let l be a separator line. So, the line l can be defined by the equations $x + y + c = 0$ or $x - y + c = 0$, where $c \neq 0$, $c \in \mathbb{R}$. The image of the line l with equation $x + y + c = 0$ under taxicab circular inversion is a closed curve with equation the equation $r^2x + r^2y + c(|x| + |y|)^2 = 0$. This equation gives a line segment with the equation $x + y + \frac{r^2}{c} = 0$ parallel to the edge of the inversion circle when x and y coordinate values have the same sign, and a parabola segment with the equation $r^2x + r^2y + c(x - y)^2 = 0$ with the vertex at the origin and the symmetry axis the line $y = x$ when x and y have opposite signs. Similarly, the taxicab circular inverse of the line l with the equation x-y+c=0 is a closed curve with the equation $r^2x - r^2y + c(|x| + |y|)^2 = 0$. This equation represents a separator line segment with the equation $x - y + \frac{\mathrm{r}^2}{\mathrm{c}} = 0$ when x and y coordinate values have opposite signs, and a parabola segment with the equation $r^2x - r^2y + c(x - y)^2 = 0$ with its vertex at the origin, and its symmetry axis is $y = x$ when x and y have the same sign. So, the proof is completed.

Theorem 3.10. The inverse of a gradual line or a step line in the taxicab plane not passing through the inversion center in taxicab circle is a closed curve consisting of two parabola arcs with axes perpendicular to each other and passing through the inversion center.

Proof. Suppose l be a gradual line not passing through origin in the taxicab plane. Then the equation of l is $y = mx + n$, where $m, n \in \mathbb{R}$ and $m \neq 0, \pm 1, \infty$ and $n \neq 0$. The inverse of l in the taxicab circle centered at $O = (0,0)$ with the radius r has the equation $mr^2x - r^2y + n(|\mathrm{x}| + |\mathrm{y}|)^2 = 0$. This means that the image is the closed curve passing through the inversion center, formed by the union of two parabola arcs with equations $mr^2x - r^2y + n(x + y)^2 = 0$ for x and y coordinate values with the same sign, and $mr^2x - r^2y + n(x - y)^2 = 0$ for x and y coordinate values with the different signs. The symmetry axis of these two parabolas have the equations $x + y + \frac{(m-1)r^2}{4n} = 0$ and $x - y + \frac{(m+1)r^2}{4n} = 0$, respectively. Since the slopes of the symmetry axes of these parabolas are 1 and -1, they are perpendicular to each other. This completes the proof.

The vertices of these parabola arcs are $\left(\frac{(1-m)(3+m)}{16n(m+1)}r^2, \frac{(1-m)(1+3m)}{16n(m+1)}r^2\right)$ and $\left(\frac{(m+1)(3-m)}{(m-1)16n}r^2, \frac{(m+1)(3m-1)}{16n(m-1)}r^2\right)$, respectively.

Example 3.11. In Fig. 2 (left) we show the taxicab inversion in the taxicab unit circle of the separator line $x + y = 2$; In Fig. 2 (right) we show the taxicab inverse with respect to the taxicab unit circle of the gradual line l with $y = 0.5x - 0.25$.

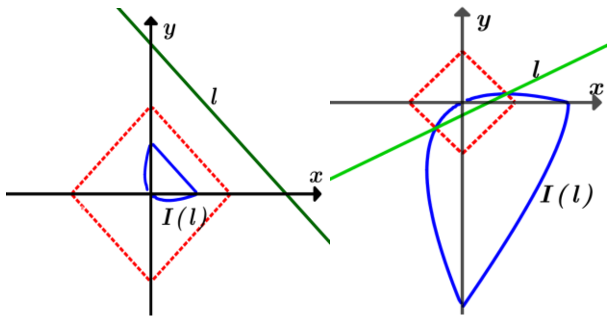

Figure 2. The taxicab circle inverses of separator line and gradual line

4. Conclusion

In In the present paper, we have explored the inverses of lines with respect to the taxicab circle inversion in the taxicab plane. We observed that the inverse of a line, different from the separator line and does not pass through the inversion center under the taxicab circle inversion, is the closed curve consisting of two parabola segments passing through the inversion center. On the other hand, the inverse of a separator line yields a closed curve comprising a line segment parallel to an edge of the inversion circle and a parabola segment. At the same time, it was seen that the axes and directrices of the parabola segments that form the inverse of a horizontal or vertical line determine a taxi circle whose center is the inversion center. It is also shown that the taxicab inversion of a pencil of parallel lines that do not pass through the center of inversion is a pencil of closed curves that are tangent at the center of inversion.
In conclusion, it is evident that taxicab circle inverses of lines in the analytic plane exhibit significantly different properties compared to Euclidean circle inverses.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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