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ABOUT VARIATIES OF G-SEQUENTAILLY METHODS, G-HULLS AND G-CLOSURES

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ABSTRACT. In the first countable spaces many topological concepts such as open and closed subsets; and continuous functions are defined for convergent sequences. The concept of limit defines a function from the set of all convergence sequences in X to X itself if X is a Hausdorff space. This is extended not only to topological spaces but also to sets. More specifically a *G-method* is defined to be a function defined on a subset of all sequences We say that a sequence $\mathbf{x} = (x_n)$ *G-convergences* to a if $G(\mathbf{x}) = a$. Then many topological objects such as open and closed subsets and many others including these sets have been extended in terms of G-convergence. G-continuity, G-compactness and G-connectedness have been studied by several authors ([1], [2], [3], [4]). On the other hand we know that in a topological space X, a sequence (x_n) converges to a point $a \in X$ if any open neighbourhood of a includes all terms except finite number. Similarly we define a sequence (x_n) to be G-sequentially converging to a if any G-open neighbourhood of a includes almost all terms. In this work provided some examples we indicate that G-convergence and G-sequentially convergence are different. We will prove that G-closed and G-sequentially closed subsets and therefore many others are different.ed.

1. INTRODUCTION

Useful tools for defining topological concepts in sequential terms are the convergences of the sequences.

Some authors explored A-continuity for methods of almost convergence and for related approaches, including Savaş and Das [5], Borsik and Salat [6].

The effects of substituting G-methods defined on a subspace of the real sequences for sequential convergence were examined by Connor and Grosse-Erdmann [7]. In order to apply this idea to topological groups, Çakallı extending this concept to

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topological groups, defined G- continuity in [1] (see also [8] for various additional forms of continuities). In [9] Mucuk and Şahan introduced the concepts of G-open sets and G-neighbourhoods in topological groups and looked into additional G-continuity features. Recently, Lin and Liu in [10] proposed the ideas of G-methods, G-submethods and G-topologies for arbitrary sets as well as topological spaces, and they also looked into the operations involving G-hulls, G-closures, G-kernels and G-interiors.

Yongxing and Fucai [11] expanded on several findings and discussed some G-connectedness, G-hull, and G-kernel properties. In [12] Brown and Mucuk studied the covering of disconnected topological groups. In their article [13] L. Liu and Z. Ping proposed the idea of the product G-method on sets, which results in a G-generalized topology. They also talked about the G-connectedness of the Cartesian product. We studied G-connectedness and G-sequential methods for product spaces in the works [14] and [15]. Authors explore the concepts of countably G- compact and sequentially GO-compact spaces in article [16]. The first countable spaces are sequential topological spaces and can be completely characterized by convergent sequences. A subset A of sequential space X is said to be closed, whenever any convergence sequence $\mathbf{x} = (x_n)$ in A has sequential limit in the same subset A. Open subsets in sequential spaces can be also defined in terms of sequences. Subset A is open if and only if any sequence converging to a point $a \in A$ is almost in A.

In [17] some counter examples of convergent G-methods are given; and G-open, G-closed subsets for these G-convergent methods are characterised. The main object of this paper is to define G-methods as G-sequential convergence and then to characterize a variety of G-open, G-closed subsets associated with these G-methods.

2. G-sequential convergence

Throughout the text, the letter X designates a topological space unless otherwise stated. The boldface letters x, y, z,... stand for the sequences of terms $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, $\mathbf{z} = (z_n)$,, whereas $s(X)$ and $c(X)$ stand for the sequences of all terms and the sequence of all convergent sequences of points in X, respectively. We define a G-method of sequential convergence for X as a map defined on a subset $c_G(X)$ of $s(X)$ into X. When for $\mathbf{x} \in c_G(X)$ and $G(\mathbf{x}) = \ell$, a sequence $\mathbf{x} = (x_n)$ is said to be *G-convergent* to ℓ. In particular, the G-method with $G = \lim$ is the lim function defined on $c(X)$. When a sequence $\mathbf{x}$ is G-convergent to ℓ, then any subsequence of $\mathbf{x}$ is likewise G-convergent to the same point ℓ, is referred to as the *preservation of the G-convergence of subsequences*. A sequence $\mathbf{x}$ is described as *regular* whenever any convergent sequence $\mathbf{x} = (x_n)$ is G-convergent with $G(\mathbf{x}) = \lim \mathbf{x}$. We remind that in a topological space X, a sequence $\mathbf{x} = (x_n)$ has limit a if and only if every open neighbourhood of a includes almost all terms of $\mathbf{x} = (x_n)$. Parallel to this, we can define a variety of G-convergence as follows:

For a set X, we say that a sequence $\mathbf{x} = (x_n)$ in X is *G-sequentially convergent* to a point $a \in X$, if every G-open neighbourhood U of a includes almost all terms of the sequence. Note that we here use additionally the word "sequentially" to distinguish from G-convergence. The notion of G-sequentially convergence defined in this manner enables us to obtain a variety of G-open, G-closed subsets and some others. We keep to use the word "sequentially" additionally for these varieties of the notions.

***G*-hull and *G*-closed subsets** : The point $\ell \in X$ is said to be in the *G-hull* of A if the subset A has a sequence $\mathbf{x} = (x_n)$ with $G(\mathbf{x}) = \ell$. A is said to be *G-closed* if $[A]^G \subseteq A$, which denotes the G-hull of A. A is G-closed if and only if $[A]^G = A$ since for a regular method G, one has $A \subseteq [A]^G$. Here it should be noted that $\emptyset$ is G-closed since $[\emptyset]^G = \emptyset$ and X is G-closed since $[X]^G \subseteq X$; and $[X]^G = X$ if G is regular. As seen in Example 2.1, even for a regular G-method, G-closure $[A]^G$ is not necessarily a G-closed subset. A subset A with $[A]^G = \emptyset$ is G-closed. The union of G-closed subsets of X is not always G-closed, but the intersection of G-closed subsets is also G-closed. The *G-closure* of A is defined to be the intersection of all G-closed subsets containing A, and denoted by $\overline{A}^G$ which is a G-closed subset. By the fact that $[A]^G \subseteq [K]^G \subseteq K$ whenever $A \subseteq K$ and K is a G-closed subset, we can deduce that $[A]^G \subseteq \overline{A}^G$.

A subset $A \subseteq X$ is called *G-open* if $X \setminus A$ is G-closed.

X and $\emptyset$ are G-open since they are both G-closed. Eventually the union of G-open subsets of X is G-open and the intersection of G-open subsets is not necessarily G-open. A subset $A \subseteq X$ is a *G-neighborhood* of a if there exists a G-open subset U of X such that $a \in U \subseteq A$. The union of G-open subsets of A is called *G-interior* of A and denoted by A^{0G} which is the largest G-open subset of A [9]. A is G-open if only if $A = A^{0G}$.

***G*-sequentially hull and *G*-sequentially closed subsets** : We say a point $l \in X$ is in the *G-sequentially hull* of a subset A if there exists a sequence $\mathbf{x} = (x_n)$ of the terms in A which G-sequentially converges to l and write $[A]_G$ for the set of G-sequentially hull points of A. Since for $a \in A$, the constant sequence $(x_n) = (a, a, \dots)$ is G-sequentially convergent to a we conclude that $A \subseteq [A]_G$. A is G-sequentially closed if $[A]_G \subseteq A$. Note that $[X]_G = X$ and $[\emptyset]_G = \emptyset$; and therefore $\emptyset$ and X are G-sequentially closed.

The *G-sequentially closure* of A, denoted by $\overline{A}_G$, is the intersection of all G-sequentially closed subsets containing A, which is also a G-sequentially closed subset. If $A \subseteq K$ and K is a G-sequentially closed subset, then $[A]_G \subseteq [K]_G \subseteq K$ and therefore $[A]_G \subseteq \overline{A}_G$.

We remark that a point a in a first countable space X is an interior point of the subset A if any sequence $\mathbf{x} = (x_n)$ converging to a is almost in A. Hence we can extend this notion to a G-method as follows: A point a is said to be a *G-sequentially interior point* of A and write $a \in A^0_G$ whenever any sequence $\mathbf{x} = (x_n)$ with G-sequentially convergence to a is almost in A or equivalently there is no any sequence $\mathbf{x} = (x_n)$ in $X \setminus A$ with G-sequentially convergence to a. By the fact that the constant sequence $(x_n) = (a, a, \dots)$ is G-sequentially convergent to a, one can see that $A^0_G \subseteq A$ and therefore A is G-sequentially open when $A \subseteq A^0_G$.

We can state following theorems to support the idea of G-sequential convergence.

Theorem 2.1. [3] *Let X be a set with a G-method. A subset A is G-sequentially open if and only if $X \setminus A$ is G-sequentially closed.*

Equivalently we can state the following theorem.

Theorem 2.2. *A is G-sequentially closed if and only if $X \setminus A$ is G-sequentially open.*

In the following examples, we shall define two G-methods and the compare G-convergence and G-sequential convergence together with associated features of G-sequentially closed subsets.

Example 2.1. Let G be a convergent method on $\mathbb{R}$ defined by $G(\mathbf{x}) = \lim \frac{x_n + x_{n+1}}{2}$ for some sequences $\mathbf{x} = (x_n)$. We can check the following properties for G-convergence.

(i) **G-closed and G-open subsets**. Since A is regular we have $A \subseteq \overline{A}^G$. Hence a subset A is G-closed if and only if $A = \overline{A}^G$. For the subset $A = \{0, 1\}$ one has $\overline{A}^G = \{0, \frac{1}{2}, 1\}$. Here note that since the sequence $(x_n) = (0, 1, 0, 1....)$ is G-converging to $1/2$ one has $1/2 \in \overline{A}^G$. Hence A is not G-closed.

If $A = \{x\}$ and $y \in \overline{A}^G$, then there exists a sequence $\mathbf{x} = (x_n)$ in A with $G(\mathbf{x}) = y$. But $\mathbf{x} = (x_n) = (x, x, \dots)$ and since G is regular $G(\mathbf{x}) = \lim(\mathbf{x}) = x$ and therefore $y = x$. Hence $A = \{x\}$ is G-closed and therefore G-open subsets are the complements $\mathbb{R} \setminus \{x\}$ for $x \in \mathbb{R}$.

(ii) **G-convergence and G-sequential convergence** This method is G converging for some sequences but it is not G-sequentially converging to any point. In below we give different types of examples for the G-sequentially convergence of sequences.

(a) For example the sequence $\mathbf{x} = (x_n) = (1, 3, 1, 3, \dots)$ is G-convergent to 2 but not G-sequentially converging to any point, because for any point $x \in \mathbb{R}$, we can choose a G-open neighbourhood $\mathbb{R} \setminus \{a\}$ of x, which does not include almost all terms of $\mathbf{x} = (x_n)$.

(b) For a constant $a \in X$ consider the sequence $\mathbf{x} = (x_n)$ defined by

$$x_n = \begin{cases} n, & \text{if } n \text{ is odd} \\ a, & \text{if } n \text{ is even} \end{cases}$$

Then $\mathbf{x} = (x_n)$ is not G-convergent to any point but G-sequentially convergent to the point a because any G-open neighbourhood $\mathbb{R} \setminus \{x\}$ of a includes almost all terms of the sequence. For any point x, which is different from a, the subset $\mathbb{R} \setminus \{a\}$ is a G-open neighbourhood of x but it does not include almost all terms and therefore $\mathbf{x} = (x_n)$ does not G-sequentially convergent to x.

(c) The sequence $\mathbf{x} = (x_n) = (\frac{1}{n})$ is G-convergent to 0 G-sequentially convergent to all points x's, because any G-open neighbourhood $\mathbb{R} \setminus \{a\}$ of x includes almost all terms of $\mathbf{x}$.

(iii) **G-sequentially closed and G-sequentially open subsets**

We can now characterize G-sequentially closure and hence G-sequentially closed subsets. Consider the following cases.

(a) If A is an infinite set, then we have a sequence $\mathbf{x} = (x_n) = (x_1, x_2, \dots)$ in A with different terms and $\mathbf{x} = (x_n)$ is G-sequentially convergent to every point $x \in \mathbb{R}$, since each G-open neighbourhood $\mathbb{R} \setminus \{a\}$ of x includes almost all terms of $\mathbf{x}$. Hence all points of $\mathbb{R}$ are in the G-sequentially hull of A and therefore $[A]_G = \mathbb{R}$.

(b) Let A be a finite set and $x \notin A$. If $\mathbf{x} = (x_n)$ is a sequence of the terms of A, then (x_n) is in the form $(x_n) = (.., x_{n_0}, \dots, x_{n_0}, ...)$ and G-open neighbourhood $\mathbb{R} \setminus \{x_{n_0}\}$ of x does not include almost all the terms. Hence $x \notin [A]_G$ for all $x \notin A$ and therefore $[A]_G = A$. We can write the generalization

$$[A]_G = \begin{cases} \mathbb{R}, & A \text{ is infinite} \\ A, & A \text{ is finite} \end{cases}$$

Hence we can conclude that finite subsets are G-sequentially closed, cofinite subsets are G-sequentially open and

$$A_G^0 = \begin{cases} A, & \text{if } A \text{ is cofinite} \\ \emptyset, & \text{otherwise} \end{cases}$$

Example 2.2. Let $c \in X$ be a constant element and G a method on the set X defined by $G(\mathbf{x}) = c$ for any sequence $\mathbf{x} = (x_n)$. Then we check the following.

(i) **G-closed and G-open subsets**. One can check that $[A]^G \subseteq A$ if and only if $c \in A$. Hence A is G-closed if and only if $c \in A$ If $(a_n) \subseteq A$ and $G(a_n) = c \in A$, then $[A]^G \subseteq A$. Thus $[A]^G = \{c\}$. and therefore we can state G-closed and G-open subsets as follows

$$\begin{cases} A \text{ is G-closed}, & \text{if } c \in A \text{ or } A = \emptyset \\ A \text{ is G-open}, & \text{if } c \notin A \text{ or } A = X \end{cases}$$

(ii) **G- convergence and G-sequential convergence**. For an $a \in X$ with $a \neq c$, the sequence $\mathbf{x} = (x_n)$ is G-sequentially convergent to a if and only if the terms of $\mathbf{x} = (x_n)$ is almost a since by (i) $\{a\}$ is a G-open neighbourhood of $a \in X$. Moreover by (i) the only G-open neighbourhood of c is $\mathbb{R}$ and therefore any sequence is also G-sequentially converging to c.

(iii) **G-sequentially closed and G-sequentially open subsets**.
Let $a \neq c$. Then by (i) $\{a\}$ is a G-open neighbourhood of a. Hence a sequence $\mathbf{x} = (x_n)$ in A is G-sequentially convergent to a if and only if the terms are almost a, i.e., $(x_n) = (a_1, a_2, ..., a_{n_0}, a, a, \dots)$. Hence $[A]_G \subseteq A$ and therefore all subsets are G- sequentially closed and also G-sequentially open.

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